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Beyond Consistency: A Closer Look at Free Formulas

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Abstract

In this paper, we introduce novel methods for drawing conclusions from inconsistent information in a highly cautious manner. While standard paraconsistent approaches typically rely on the formulas in the intersection of maximally consistent subsets, known as the *free formulas*, we argue that not all these formulas share the same degree of reliability. Our refined reasoning frameworks distinguish between free formulas, based on their actual involvement in the inconsistency, an so enabling inference only when conclusions are robustly supported. These methods are particularly valuable in high-stakes contexts where decisions carry irreversible or far-reaching consequences. We present several implementation techniques grounded in multi-valued semantics and syntactic independence, analyze their fundamental logical properties, and establish a hierarchy of their inferential strength.

1 Introduction and Motivation

A common approach for making rational conclusions from inconsistent data (a general approach called *paraconsistent reasoning*, 'beyond consistency' (da Costa 1974)), is to divide the information into different cases, each of which represents a consistent scenario, and to make inferences only by those parts of the data that are consistent across all these cases (Benferhat, Dubois, and Prade 1997; Benferhat, Dubois, and Prade 1999). This consistent subset of the data, consisting of what are usually referred to as the *free assertions* (Rescher and Manor 1970), is typically treated uniformly. Nevertheless, the following examples demonstrate that sometimes even the consistent part consists of assumptions that differ in their degree of certainty or usefulness.

Example 1. A 32-years-old patient complains about sudden headache and visual disturbances. There are two common explanations for these symptoms: a migraine aura or a minor stroke. Below is an encoding of the scenario:

1. The patient has zig-zag flashing lights in one eye. This is a classic sign of a migraine.
2. The patient has temporary weakness in the right hand. This is a hallmark of a stroke and not of a pure migraine.
3. The patient reports that she has had identical migraine auras since age 20. This strongly supports migraine and argues against a stroke.

Thus, any diagnosis conflicts with at least one symptom.

Suppose that the following two rules are also known:

4. In the case of a stroke, it is recommended to take statins to reduce vascular risk.
5. If the patient is over 15-years-old, parental supervision is not required.

Both of these rules are consistent with all the information provided in Items 1–3. Nevertheless, the second rule appears more useful in our case, because, among the two rules, only its applicability is assured. Indeed, it is unclear whether the condition of the other rule is satisfied.

Example 2. We have conflicting evidence on whether the museum is open today. We also know that the city is hosting events today. Here, accepting that an event is taking place yields a more robust conclusion than inferring that the museum is open or a nearby park is open, even though both conclusions remain consistent with the available data. Indeed, the existence of today's event is not challenged by any of the information we have, whereas the other conclusion still leaves uncertainty about which place is actually open.

In this paper, we propose several formalisms that support particularly cautious (and paraconsistent) inference mechanisms. According to these formalisms, drawing a conclusion requires more than mere consistency across all coherent scenarios; it must also be supported only by assertions whose reliability is fully warranted by the data. This type of inference is needed, for example, when decisions may have far-reaching or irreversible consequences for those affected by them.

The rest of the paper is organized as follows: In the next section, we review the basic definitions and notations underlying our framework. Section 3 presents several approaches to cautious paraconsistent reasoning, developed through a variety of syntactic and semantic considerations. In Section 4, we examine some common properties of these approaches and compare their respective inferential strengths. Finally, in Section 5 we conclude with some remarks and directions for future research.

2 Reasoning with Inconsistent Information

It is well-known that classical logic lacks a controlled mechanism for reasoning with contradictory information. This is due to the fact that, according to classical logic, *any* conclusion may be inferred from an inconsistent set of assumptions. To overcome this weakness, a wide range of alternative formalisms has been proposed in the literature, ranging from approaches that filter out the inconsistent part of the data and remove it from the knowledge base (such as belief-revision systems (Hansson 2022)), to methods of isolating and reasoning with consistent parts of the knowledge base and subsequently aggregating conclusions (Benferhat, Dubois, and Prade 1997; Benferhat, Dubois, and Prade 1999; Straßer 2025), to approaches that preserve inconsistent information while still supporting coherent inferences (Avron, Arieli, and Zamansky 2018). Non-classical formalisms of the latter kind are known as *paraconsistent* (Jaśkowski 1969; da Costa 1974). In this section, we review two major approaches to developing paraconsistent logics that will be used later on. The first one (Section 2.1) is fundamentally semantic, relying on extending the classical two-valued truth assignments to additional truth values. The second approach (Section 2.2) is syntactic in nature and examines (maximal) consistent subsets of the assumptions.

2.1 Multi-Valued Logics and Most Consistent Models

In the sequel, $\mathcal{L}$ denotes a propositional language. Sets of formulas are denoted by $\mathcal{S}, \mathcal{T}$, finite sets of formulas are denoted by Γ, Δ, formulas are denoted by ϕ, ψ, and atomic formulas are denoted by p, q, r (all possibly indexed). We denote by $\mathsf{WFF}(\mathcal{L})$ the set of (well-formed) formulas of $\mathcal{L}$, and by $\wp(\mathsf{WFF}(\mathcal{L}))$ the power set of $\mathsf{WFF}(\mathcal{L})$. The set of atomic formulas appearing in $\mathcal{S}$ is denoted $\mathsf{Atoms}(\mathcal{S})$.

Definition 1. A *propositional logic* (for $\mathcal{L}$) is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ (the consequence relation) is a relation on $\wp(\mathsf{WFF}(\mathcal{L})) \times \mathsf{WFF}(\mathcal{L})$, satisfying *reflexivity* (if $\phi \in \mathcal{S}$, then $\mathcal{S} \vdash \phi$), *monotonicity* (if $\mathcal{S}' \vdash \phi$ and $\mathcal{S}' \subseteq \mathcal{S}$, then $\mathcal{S} \vdash \phi$), and *transitivity* (if $\mathcal{S} \vdash \phi$ and $\mathcal{S}', \phi \vdash \psi$ then $\mathcal{S}, \mathcal{S}' \vdash \psi$).

In what follows, we assume that $\mathcal{L}$ contains at least a $\vdash$-negation operator $\neg$, satisfying $p \not\vdash \neg p$ and $\neg p \not\vdash p$ (for atomic p), a $\vdash$-conjunction operator $\wedge$, for which $\mathcal{S} \vdash \psi \wedge \phi$ iff $\mathcal{S} \vdash \psi$ and $\mathcal{S} \vdash \phi$, and a $\vdash$-disjunction operator $\vee$, for which $\mathcal{S}, \psi \vee \phi \vdash \sigma$ iff $\mathcal{S}, \psi \vdash \sigma$ and $\mathcal{S}, \phi \vdash \sigma$. Also, we denote by $\bigwedge \Gamma$ the conjunction of all the formulas in Γ.

The most standard semantic way of defining logics is by the following structures (see, e.g., (Urquhart 2001)).

Definition 2. A (multi-valued) *matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a non-empty set of truth values,
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, consisting of the *designated* elements of $\mathcal{V}$,
- $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to \mathcal{V}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.[1]

In what follows shall distinguish between truth values that represent the following semantic statuses:

- truth: $\mathcal{V}_t = \{x \in \mathcal{V} \mid x \in \mathcal{D}, \neg x \notin \mathcal{D}\}$,
- falsity: $\mathcal{V}_f = \{x \in \mathcal{V} \mid x \notin \mathcal{D}, \neg x \in \mathcal{D}\}$,
- inconsistency: $\mathcal{V}_\top = \{x \in \mathcal{V} \mid x \in \mathcal{D}, \neg x \in \mathcal{D}\}$,
- uncertainty $\mathcal{V}_\bot = \{x \in \mathcal{V} \mid x \notin \mathcal{D}, \neg x \notin \mathcal{D}\}$.

Definition 3. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- An $\mathcal{M}$-*valuation* for $\mathcal{L}$ is a function $\nu : \mathsf{WFF}(\mathcal{L}) \to \mathcal{V}$, such that for every n-ary connective $\diamond$ of $\mathcal{L}$,
$$\nu(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}_{\mathcal{M}}(\nu(\psi_1), \ldots, \nu(\psi_n)).$$
We denote the set of all the $\mathcal{M}$-valuations by $\Lambda_{\mathcal{M}}$.
- A valuation $\nu \in \Lambda_{\mathcal{M}}$ is an $\mathcal{M}$-*model* of a formula ψ if it is an element of
$$\mathsf{Mod}_{\mathcal{M}}(\psi) = \{\nu \in \Lambda_{\mathcal{M}} \mid \nu(\psi) \in \mathcal{D}\}.$$
- The $\mathcal{M}$-models of a set of formulas $\mathcal{S}$ are defined by
$$\mathsf{Mod}_{\mathcal{M}}(\mathcal{S}) = \bigcap_{\psi \in \mathcal{S}} \mathsf{Mod}_{\mathcal{M}}(\psi).$$

Definition 4. Given a set of formulas $\mathcal{S}$ and a matrix $\mathcal{M}$, the relation $\vdash_{\mathcal{M}}$ that is *induced by* $\mathcal{M}$, is defined by:
$$\mathcal{S} \vdash_{\mathcal{M}} \psi \text{ if } \mathsf{Mod}_{\mathcal{M}}(\mathcal{S}) \subseteq \mathsf{Mod}_{\mathcal{M}}(\psi).$$

We denote $\mathfrak{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, where $\mathcal{M}$ is a matrix for $\mathcal{L}$ and $\vdash_{\mathcal{M}}$ is the relation induced by $\mathcal{M}$. It is easy to verify that $\mathfrak{L}_{\mathcal{M}}$ is a propositional logic.

Example 3. Below are some well-known instances of matrices $\mathcal{M}$ and the induced logics $\mathfrak{L}_{\mathcal{M}}$.

- Propositional classical logic is induced by the matrix $\mathsf{CL} = \langle \{t, f\}, \{t\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$ where $\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}$ are the standard two-valued interpretations for $\vee$, $\wedge$ and $\neg$. Here, $\mathcal{V}_t = \{t\}$, $\mathcal{V}_f = \{f\}$ and $\mathcal{V}_\top = \mathcal{V}_\bot = \emptyset$.
- Priest's logic LP (Priest 1989) is induced by the 3-valued matrix $\mathsf{LP} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$. Thus, $\mathcal{V}_t = \{t\}$, $\mathcal{V}_f = \{f\}$, $\mathcal{V}_\top = \{\top\}$ and $\mathcal{V}_\bot = \emptyset$. The interpretations $\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}$ of the basic connectives are defined as follows:

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

	$\tilde{\neg}$
t	f
f	t
$\top$	$\top$

- Dunn-Belnap logic (Dunn 1976; Belnap 1977) is induced by $\mathsf{DB} = \langle \{t, f, \top, \bot\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$, where $\tilde{\vee}$, $\tilde{\wedge}$, $\tilde{\neg}$ are defined by the maximum, minimum, and the order reversing function on the partial order $f < \{\top, \bot\} < t$, namely: when t is the maximum, f is the minimum and the other values are incomparable. In this case, $\mathcal{V}_t = \{t\}$, $\mathcal{V}_f = \{f\}$, $\mathcal{V}_\top = \{\top\}$ and $\mathcal{V}_\bot = \{\bot\}$.

As we shall see in Section 3, it is sometimes useful to distinguish among the models in $\mathsf{Mod}_{\mathcal{M}}(\mathcal{S})$ and focus only on those that contain a minimal amount of inconsistent assignments. This idea is formalized next:

Definition 5. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- The *inconsistent (atomic) assignments* on $\mathcal{S}$ of an $\mathcal{M}$-valuation $\nu \in \Lambda_{\mathcal{M}}$ is the set
$$\mathsf{Inc}_{\mathcal{M}}(\nu, \mathcal{S}) = \{p \in \mathsf{Atoms}(\mathcal{S}) \mid \nu(p) \in \mathcal{V}_\top\}.$$

[1] While we restrict the focus to deterministic matrices, we note that the systems proposed in this paper can be adjusted also to non-deterministic matrices (Avron and Lev 2004).

- The $\mathcal{M}$-*most consistent models* of $\mathcal{S}$ (MCMs, for short) are the elements of the following set:

$$\mathsf{MCM}_{\mathcal{M}}(\mathcal{S}) = \{M \in \mathsf{Mod}_{\mathcal{M}}(\mathcal{S}) \mid \nexists M' \in \mathsf{Mod}_{\mathcal{M}}(\mathcal{S}) \text{ s.t. } \mathsf{Inc}_{\mathcal{M}}(M',\mathcal{S}) \subsetneq \mathsf{Inc}_{\mathcal{M}}(M,\mathcal{S})\}.$$

Example 4. Consider again Example 2. Denoting by p that the museum is open, by q that the park is open, and by r that there is a planned event, the state of affairs may be encoded by the set $\mathcal{S} = \{p,\ \neg p,\ p \vee q,\ r\}$. The models in $\mathsf{Mod}_{\mathsf{LP}}(\mathcal{S})$ and those in $\mathsf{MCM}_{\mathsf{LP}}(\mathcal{S})$ are listed in Table 1.

No.	p	q	r
M_1	$\top$	t	t
M_2	$\top$	f	t
M_3	$\top$	$\top$	t

No.	p	q	r
M_4	$\top$	t	$\top$
M_5	$\top$	f	$\top$
M_6	$\top$	$\top$	$\top$

Table 1: The LP-models and MCMs (highlighted) of $\mathcal{S}$ (Examp. 4)

As shown in the table, $\mathsf{MCM}_{\mathsf{LP}}(\mathcal{S}) = \{M_1, M_2\}$. Indeed, the most consistent LP-models assign an inconsistent value only to the atom at the 'core' of inconsistency of $\mathcal{S}$, which is the uncertainty about whether the museum is open (p): $\mathsf{Inc}_{\mathsf{LP}}(M_1,\mathcal{S}) = \mathsf{Inc}_{\mathsf{LP}}(M_2,\mathcal{S}) = \{p\}$.

2.2 Maximally Consistent Sets, Free Formulas, and Their Entailments

A common method of extracting meaningful information from inconsistent knowledge bases is by considering their maximally consistent subsets. This idea, which may be traced back to the seminal paper of Rescher and Manor (1970), is formalized below:

Definition 6. Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic. A set $\mathcal{S}$ of $\mathcal{L}$-formulas is $\vdash$-*inconsistent*, if there is some $\Gamma \subseteq \mathcal{S}$ such that $\mathcal{S} \vdash \neg \bigwedge \Gamma$, otherwise $\mathcal{S}$ is $\vdash$-*consistent*. We denote:

- $\mathsf{CS}_{\mathfrak{L}}(\mathcal{S})$: the set of the $\vdash$-consistent subsets of $\mathcal{S}$.
- $\mathsf{MCS}_{\mathfrak{L}}(\mathcal{S})$, the *maximally consistent subsets of* $\mathcal{S}$: the $\subseteq$-maximal sets in $\mathsf{CS}_{\mathfrak{L}}(\mathcal{S})$.
- $\mathsf{Free}_{\mathfrak{L}}(\mathcal{S})$, the *free formulas of* $\mathcal{S}$: the formulas in $\mathcal{S}$ that belong to every set in $\mathsf{MCS}_{\mathfrak{L}}(\mathcal{S})$.

When $\mathfrak{L} = \mathfrak{L}_{\mathcal{M}}$ we shall some times denote the set of free formulas by $\mathsf{Free}_{\mathcal{M}}(\mathcal{S})$ (instead of $\mathsf{Free}_{\mathfrak{L}}(\mathcal{S})$).

Example 5. Recall Examples 2 and 4, which are represented by the set $\mathcal{S} = \{p,\ \neg p,\ p \vee q,\ r\}$. It holds that

$$\mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}) = \big\{\{p,\ p \vee q,\ r\},\ \{\neg p,\ p \vee q,\ r\}\big\},$$

so the maximally consistent subsets of $\mathcal{S}$ correspond to the two coherent situations, where the museum is either open or closed. The free formulas are those that hold in both situations: $\mathsf{Free}_{\mathsf{CL}}(\mathcal{S}) = \{p \vee q,\ r\}$.

Example 6. Using p to denote migraine, q to denote stroke, and r to denote parental supervision, the situation described in Example 1 can be represented by the following set:

$$\mathcal{S} = \{p,\ \neg p \wedge q,\ p \wedge \neg q,\ \neg p \vee \neg q,\ \neg r\},$$

whose formulas represent, respectively, the 5 indications in the example. In this case,

$$\mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}) = \left\{ \begin{array}{l} \{p,\ p \wedge \neg q,\ \neg p \vee \neg q,\ \neg r\}, \\ \{\neg p \wedge q,\ \neg p \vee \neg q,\ \neg r\} \end{array} \right\}$$

and thus $\mathsf{Free}_{\mathsf{CL}}(\mathcal{S}) = \{\neg p \vee \neg q,\ \neg r\}$.

The free formulas are considered the most trustworthy assertions in inconsistent knowledge bases. Formalisms for reasoning with such knowledge bases treat all the free formulas equally reliable, making inferences by their transitive closure: For $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, we define:

$$\mathcal{S} \mid\!\sim^{\mathsf{fr}}_{\mathfrak{L}} \psi \text{ iff } \mathsf{Free}_{\mathfrak{L}}(\mathcal{S}) \vdash \psi \qquad (1)$$

This entailment relation has been shown useful in several areas, e.g., in the context of the WIDTIO principle (When in Doubt, Throw it Out) in belief revision (Winslett 1990; Liberatore and Schaerf 2004), or for making inferences from the grounded semantics of logic-based argumentation frameworks with undercut as the attack rule (see (Arieli, Borg, and Straßer 2018; Arieli, Borg, and Heyninck 2019)).

As stated in the introduction, the aim of this paper is to examine cases in which the free formulas do not all share the same importance or utility (as some remain genuinely unaffected by the inconsistency), and to derive suitable consequence relations that reflect these differences. Before suggesting alternative entailment relations, in the remainder of this section we consider several properties and characteristics of inference based on all the free formulas.

Proposition 1. *(Arieli et al. 2021) For every propositional logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, the entailment $\mid\!\sim^{\mathsf{fr}}_{\mathfrak{L}}$ is $\vdash$-cumulative in the sense of Kraus, Lehmann, and Magidor (1990), namely, the following properties are satisfied for $\mid\!\sim\, = \,\mid\!\sim^{\mathsf{fr}}_{\mathfrak{L}}$:*

Cautious reflexivity: If ϕ is $\vdash$-consistent, then $\phi \mid\!\sim \phi$
Right weakening: If $\mathcal{S} \mid\!\sim \phi$ and $\phi \vdash \psi$, then $\mathcal{S} \mid\!\sim \psi$
Cautious monotonicity: If $\mathcal{S} \mid\!\sim \phi$ and $\mathcal{S} \mid\!\sim \psi$, then $\mathcal{S}, \phi \mid\!\sim \psi$
Cautious cut: If $\mathcal{S} \mid\!\sim \psi$ and $\mathcal{S}, \psi \mid\!\sim \phi$, then $\mathcal{S} \mid\!\sim \phi$
Logical equivalence: If $\phi \vdash \phi$ and $\psi \vdash \phi$ and $\mathcal{S}, \phi \mid\!\sim \sigma$, then $\mathcal{S}, \psi \mid\!\sim \sigma$

While being cumulative, $\mid\!\sim^{\mathsf{fr}}_{\mathfrak{L}}$ is *not* $\vdash$-preferential in the sense of (Kraus, Lehmann, and Magidor 1990), since the following property is not satisfied when $\mid\!\sim\, = \,\mid\!\sim^{\mathsf{fr}}_{\mathfrak{L}}$:

Or: If $\mathcal{S}, \phi \mid\!\sim \sigma$ and $\mathcal{S}, \psi \mid\!\sim \sigma$ then $\mathcal{S}, \phi \vee \psi \mid\!\sim \sigma$.

To see this, consider the following counterexample w.r.t. classical logic and $\mathcal{S} = \{\neg p,\ \neg q,\ p \vee r,\ q \vee r\}$. We have:

- $\mathcal{S}, p \mid\!\sim^{\mathsf{fr}}_{\mathsf{CL}} r$, since $\mathsf{Free}_{\mathsf{CL}}(\mathcal{S} \cup \{p\}) = \{\neg q,\ p \vee r,\ q \vee r\}$,
- $\mathcal{S}, q \mid\!\sim^{\mathsf{fr}}_{\mathsf{CL}} r$, since $\mathsf{Free}_{\mathsf{CL}}(\mathcal{S} \cup \{q\}) = \{\neg p,\ p \vee r,\ q \vee r\}$,
- $\mathcal{S}, p \vee q \not\mid\!\sim^{\mathsf{fr}}_{\mathsf{CL}} r$, since $\mathsf{Free}_{\mathsf{CL}}(\mathcal{S} \cup \{p \vee q\}) = \{p \vee r, q \vee r\}$.

Note 1. Some further remarks on reasoning with the free formulas are in-order here (unless otherwise stated, the base logic in the examples below is classical logic, CL).

1. *Reasoning with the free formulas is context dependent.* Let $\mathcal{S}_1 = \{p, \neg p, q\}$ and $\mathcal{S}_2 = \{p, \neg p \wedge q\}$. Although the two sets are logically equivalent, $q \in \mathsf{Free}(\mathcal{S}_1)$ while $q \notin \mathsf{Free}(\mathcal{S}_2)$, and so $\mathcal{S}_1 \mid\!\sim^{\mathsf{fr}} q$ while $\mathcal{S}_2 \not\mid\!\sim^{\mathsf{fr}} q$.

2. *Reasoning only with the free formula might be too restrictive.* Let $\mathcal{S} = \{p \wedge q, \neg p \wedge q\}$. Then $\mathsf{MCS}(\mathcal{S}) = \{\{p \wedge q\}, \{\neg p \wedge q\}\}$, and so $\mathsf{Free}(\mathcal{S}) = \emptyset$, although one might still want to conclude here q.

   A possible way to overcome this issue is to strengthen the entailment in (1) for a propositional logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ as follows:

$$\mathcal{S} \mid\!\sim^{\mathsf{mcs}}_{\mathfrak{L}} \psi \text{ iff } \mathcal{T} \vdash \psi \text{ for every } \mathcal{T} \in \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S}) \quad (2)$$

   The next proposition easily follows from the fact that, by definition, $\mathsf{Free}_{\mathfrak{L}}(\mathcal{S}) = \bigcap \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S})$:

   Proposition 2. *If $\mathcal{S} \mid\!\sim^{\mathsf{fr}}_{\mathfrak{L}} \psi$, then $\mathcal{S} \mid\!\sim^{\mathsf{mcs}}_{\mathfrak{L}} \psi$.*

   As the example above shows, the converse of Proposition 2 does not hold.

We refer, for instance, to (Benferhat, Dubois, and Prade 1997; Benferhat, Dubois, and Prade 1999) for further discussion and results on free formulas in inconsistent knowledge bases, to (Straßer 2025) for a generalization of Proposition 1 and related results, and to (Arieli, Borg, and Straßer 2018) for a related study in the context of argumentation theory.

3 Cautious Paraconsistent Reasoning

As noted earlier, when one must be particularly cautious in drawing inferences, relying on all the free formulas may be problematic. In such situations, a cautious reasoner may prefer to refine the entailment $\mid\!\sim^{\mathsf{fr}}_{\mathfrak{L}}$ in (1) so that only conclusions supported by the most reliable free formulas will be accepted. Accordingly, and contrary to Item (2) in Note 1, $\mid\!\sim^{\mathsf{fr}}_{\mathfrak{L}}$ should be weakened rather than strengthened. In the subsections below, we explore several ways to achieve this by further refining the set of free formulas.

3.1 Entailments Based on Reduction Methods

One way to distinguish among the free formulas is to shift from the underlying logic to a more refined one, where differences between them become more apparent. Below, we formalize this idea and illustrate how it works.

Definition 7. Let $\mathfrak{L}_1 = \langle \mathcal{L}, \vdash_{\mathcal{M}_1} \rangle$ and $\mathfrak{L}_2 = \langle \mathcal{L}, \vdash_{\mathcal{M}_2} \rangle$ be two multi-valued logics (Definition 4) where $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$, such that $\mathfrak{L}_2$ is contained in $\mathfrak{L}_1$ (i.e., $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$).[2] The set of $\mathfrak{L}_1$-free formulas of $\mathcal{S}$, *reduced to* $\mathfrak{L}_2$, is defined as:

$$\mathsf{Free}^{\forall}_{\mathfrak{L}_1 \downarrow \mathfrak{L}_2}(\mathcal{S}) = \{\psi \in \mathsf{Free}_{\mathcal{M}_1}(\mathcal{S}) \mid \forall M \in \mathsf{MCM}_{\mathcal{M}_2}(\mathcal{S}), M(\psi) \in (\mathcal{V}_2)_t\},$$

$$\mathsf{Free}^{\exists}_{\mathfrak{L}_1 \downarrow \mathfrak{L}_2}(\mathcal{S}) = \{\psi \in \mathsf{Free}_{\mathcal{M}_1}(\mathcal{S}) \mid \exists M \in \mathsf{MCM}_{\mathcal{M}_2}(\mathcal{S}), M(\psi) \in (\mathcal{V}_2)_t\}.$$

Accordingly, we define:

$$\mathcal{S} \mid\!\sim^{\forall\mathsf{fr}}_{\mathfrak{L}_1 \downarrow \mathfrak{L}_2} \psi \text{ iff } \mathsf{Free}^{\forall}_{\mathfrak{L}_1 \downarrow \mathfrak{L}_2}(\mathcal{S}) \vdash_{\mathfrak{L}_1} \psi \quad (3)$$

$$\mathcal{S} \mid\!\sim^{\exists\mathsf{fr}}_{\mathfrak{L}_1 \downarrow \mathfrak{L}_2} \psi \text{ iff } \mathsf{Free}^{\exists}_{\mathfrak{L}_1 \downarrow \mathfrak{L}_2}(\mathcal{S}) \vdash_{\mathfrak{L}_1} \psi \quad (4)$$

[2]For instance, it is well-known that $\vdash_{\mathcal{M}_{\mathsf{LP}}} \subset \vdash_{\mathcal{M}_{\mathsf{CL}}}$ (see, e.g., (Avron, Arieli, and Zamansky 2018)).

The idea behind the entailments in (3) and (4) is that although consequences are still determined by the core logic $\mathfrak{L}_1$, the logic $\mathfrak{L}_2$ is incorporated for making a discrimination between the formulas in $\mathsf{Free}_{\mathfrak{L}_1}(\mathcal{S})$. The following example demonstrates this.

Example 7. Consider the following set of formulas, formulated within classical logic, CL.

$$\mathcal{S} = \{p \wedge \neg q,\ \neg p \wedge q,\ p \vee q,\ p \vee s,\ r\}.$$

There are three free formulas here: r, $p \vee s$, and $p \vee q$. Yet, these formulas intuitively reflect different levels of involvement in the inconsistency of $\mathcal{S}$: r is the most reliable, as it is entirely unrelated to the inconsistency, $p \vee s$ is at an intermediate level, since one of its disjuncts (p) belongs to a minimally inconsistent subset of $\mathcal{S}$ (i.e., $\{p \wedge \neg q,\ \neg p \wedge q\}$), and $p \vee q$ is the least reliable formula among the three, as each of its disjuncts appears in a (different) minimally inconsistent subset of $\mathcal{S}$.

To discriminate between the free formulas, we again incorporate Priest 3-valued logic LP (cf. Example 4). The LP-models of $\mathcal{S}$ (those in $\mathsf{Mod}_{\mathsf{LP}}(\mathcal{S})$)) and the most consistent ones (in $\mathsf{MCM}_{\mathsf{LP}}(\mathcal{S})$) are listed in Table 2.

No.	p	q	s	r
M_1	$\top$	$\top$	t	t
M_2	$\top$	$\top$	f	t
M_3	$\top$	$\top$	$\top$	t

No.	p	q	s	r
M_4	$\top$	$\top$	t	$\top$
M_5	$\top$	$\top$	f	$\top$
M_6	$\top$	$\top$	$\top$	$\top$

Table 2: The LP-models and MCMs (highlighted) of $\mathcal{S}$ (Examp. 7)

Again, the most reliable cases are represented by the models whose $\top$-assignments are $\subseteq$-minimized. Here, these are $\mathsf{MCM}_{\mathsf{LP}}(\mathcal{S}) = \{M_1, M_2\}$, thus $\mathsf{Free}^{\forall}_{\mathsf{CL}\downarrow\mathsf{LP}}(\mathcal{S}) = \{r\}$. It follows that while

$$\mathcal{S} \mid\!\sim^{\mathsf{fr}}_{\mathsf{CL}} \psi \text{ for every } \psi \in \{r,\ p \vee s,\ p \vee q\},$$

we have:

$$\mathcal{S} \mid\!\sim^{\forall\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}} r, \text{ but } \mathcal{S} \not\mid\!\sim^{\forall\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}} p \vee s \text{ and } \mathcal{S} \not\mid\!\sim^{\forall\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}} p \vee q.$$

Intuitively, this occurs because each of the latter disjunctions contains at least one unreliable disjunct.

For an intermediate level of cautiousness, one may apply $\mid\!\sim^{\exists\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}}$. Since $\mathsf{Free}^{\exists}_{\mathsf{CL}\downarrow\mathsf{LP}}(\mathcal{S}) = \{r, p \vee s\}$, we now have:

$$\mathcal{S} \mid\!\sim^{\exists\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}} r \text{ and } \mathcal{S} \mid\!\sim^{\exists\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}} p \vee s, \text{ while } \mathcal{S} \not\mid\!\sim^{\exists\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}} p \vee q.$$

Here, inferences are blocked only when both disjuncts are deemed unreliable.

Example 8. A process similar to the one in Example 7 can be applied to the encoding of Example 2 given in Example 4. Indeed, to distinguish between the free formulas r and $p \vee q$ of $\mathcal{S} = \{p,\ \neg p,\ p \vee q,\ r\}$, it is again possible to consider $\mathsf{Mod}_{\mathsf{LP}}(\mathcal{S})$ and $\mathsf{MCM}_{\mathsf{LP}}(\mathcal{S})$, shown in Table 1. By this, we get that $\mathcal{S} \mid\!\sim^{\forall\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}} r$, whereas $\mathcal{S} \not\mid\!\sim^{\forall\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}} p \vee q$, as suggested in Example 2.

3.2 Formula-Based Inconsistency Minimization

A variant of the entailment relations in (3) and (4) gives rise to additional ways of reasoning with free formulas. For example, inconsistent assignments may be evaluated relative to formulas rather than to atomic assignments. In other words, instead of using the sets defined in the first item of Definition 5, we may consider the following revised sets:

$$\mathsf{Inc}^{\star}_{\mathcal{M}}(\nu, \mathcal{S}) = \{\psi \in \mathcal{S} \mid \nu(\psi) \in \mathcal{V}_{\top}\}.$$

The MCMs of $\mathcal{S}$ are then defined exactly as in the second item of Definition 5, but now with respect to this revised notion of Inc:

$$\mathsf{MCM}^{\star}_{\mathcal{M}}(\mathcal{S}) = \{M \in \mathsf{Mod}_{\mathcal{M}}(\mathcal{S}) \mid \nexists M' \in \mathsf{Mod}_{\mathcal{M}}(\mathcal{S}) \text{ s.t. } \mathsf{Inc}^{\star}_{\mathcal{M}}(M', \mathcal{S}) \subsetneq \mathsf{Inc}^{\star}_{\mathcal{M}}(M, \mathcal{S})\}.$$

This yields the following revised definitions of the 'reliable' free formulas w.r.t. $\mathfrak{L}_i = \langle \mathcal{L}, \vdash_{\mathcal{M}_i} \rangle$ $(i = 1, 2)$:

$$\mathsf{Free}^{\star\forall}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S}) = \{\psi \in \mathsf{Free}_{\mathcal{M}_1}(\mathcal{S}) \mid \forall M \in \mathsf{MCM}^{\star}_{\mathcal{M}_2}(\mathcal{S}), M(\psi) \in (\mathcal{V}_2)_t\},$$

$$\mathsf{Free}^{\star\exists}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S}) = \{\psi \in \mathsf{Free}_{\mathcal{M}_1}(\mathcal{S}) \mid \exists M \in \mathsf{MCM}^{\star}_{\mathcal{M}_2}(\mathcal{S}), M(\psi) \in (\mathcal{V}_2)_t\}.$$

Consequently, two further families of entailment relations can be defined as follows:

$$\mathcal{S} \mid\!\sim^{\star\forall\mathsf{fr}}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2} \psi \text{ iff } \mathsf{Free}^{\star\forall}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S}) \vdash_{\mathfrak{L}_1} \psi \tag{5}$$

$$\mathcal{S} \mid\!\sim^{\star\exists\mathsf{fr}}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2} \psi \text{ iff } \mathsf{Free}^{\star\exists}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S}) \vdash_{\mathfrak{L}_1} \psi \tag{6}$$

Example 9. Let $\mathcal{S} = \{p \wedge \neg q,\ \neg p \wedge q,\ p \vee q,\ p \vee s,\ r\}$ be the set of formulas from Example 7. To obtain $\mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S})$ from $\mathsf{Mod}_{\mathsf{LP}}(\mathcal{S})$, we now need to take all the formulas in $\mathcal{S}$ into consideration:

No.	p	q	s	r	$p \wedge \neg q$	$\neg p \wedge q$	$p \vee q$	$p \vee s$
M_1	$\top$	$\top$	t	t	$\top$	$\top$	$\top$	t
M_2	$\top$	$\top$	f	t	$\top$	$\top$	$\top$	$\top$
M_3	$\top$	$\top$	$\top$	t	$\top$	$\top$	$\top$	$\top$
M_4	$\top$	$\top$	t	$\top$	$\top$	$\top$	$\top$	t
M_5	$\top$	$\top$	f	$\top$	$\top$	$\top$	$\top$	$\top$
M_6	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

Table 3: $\mathsf{Mod}_{\mathsf{LP}}(\mathcal{S})$ and $\mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S})$ (highlighted) (Example 9)

Note that while $\mathsf{MCM}_{\mathsf{LP}}(\mathcal{S}) = \{M_1, M_2\}$, we now obtain $\mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S}) = \{M_1\}$. Consequently, this time we have $\mathcal{S} \mid\!\sim^{\star\forall\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}} p \vee s$ (whereas $\mathcal{S} \not\mid\!\sim^{\forall\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}} p \vee s$).

3.3 Entailments By Syntactic Considerations

Notably, all methods discussed so far treat all formulas in a consistent set uniformly. Indeed, if $\mathcal{S}$ is an $\mathfrak{L}$-consistent set of assertions, then:

$$\mathsf{Free}_{\mathfrak{L}_1}(\mathcal{S}) = \mathsf{Free}^{\forall}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S}) = \mathsf{Free}^{\exists}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S}) = \mathsf{Free}^{\star\forall}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S}) = \mathsf{Free}^{\star\exists}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S}) = \mathcal{S}.$$

Yet, as noted earlier, consistency by itself may be insufficient for highly cautious reasoning, even when the entire set of assertions is consistent. This is demonstrated next.

Example 10. Consider the sets $\mathcal{S}_1 = \{\neg p \vee \neg q,\ p\}$ and $\mathcal{S}_2 = \{\neg p \vee \neg q,\ r\}$. Both of these sets are (classically) consistent. Yet, for an especially cautious reasoner, it will be harder to infer p from $\mathcal{S}_1$ than to infer r from $\mathcal{S}_2$, since the additional information in $\mathcal{S}_1$ still leaves some room for the possibility that $\neg p$ holds, whereas in $\mathcal{S}_2$ there is no indication whatsoever that $\neg r$ could hold.

To demonstrate the possibility of such a situation, think of a military radar system whose purpose is to detect airborne objects and decide whether they should be intercepted. Let:

- p**:** The object is an enemy aircraft.
- q**:** A partial or ambiguous identification signal has been detected.
- r**:** A serious technical malfunction has been detected in the main radar sensor, requiring an immediate switch to the backup system.

In this setting $\neg p \vee q$ may be interpreted by the system as "Either the object is not an enemy aircraft, or there is an indication that it may be a friendly one". Thus, although the set of assertions is not strictly inconsistent, a cautious decision system would first switch to a backup system (that is, it accepts r) *before* concluding that the aircraft is hostile (that is, it accepts p).

One way to address this is to refine the definition of free formulas in Definition 6 more carefully, using syntactic considerations, as we do next.

Definition 8. Let $\mathsf{Lit}(\psi)$ be the set of literals (i.e., atomic formulas or their negations) that appear in the formula ψ, and let $\mathsf{Lit}(\mathcal{S}) = \bigcup_{\psi\in\mathcal{S}} \mathsf{Lit}(\psi)$. We denote:

$$\mathsf{Free}^{!}_{\mathfrak{L}}(\mathcal{S}) = \{\psi \in \mathsf{Free}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Lit}(\psi) \subseteq \mathsf{Free}_{\mathfrak{L}}(\mathsf{Lit}(\mathcal{S}))\}.$$

Now, similarly to (1), we define:

$$\mathcal{S} \mid\!\sim^{\mathsf{fr!}}_{\mathfrak{L}} \psi \text{ iff } \mathsf{Free}^{!}_{\mathfrak{L}}(\mathcal{S}) \vdash \psi \tag{7}$$

Example 11. Consider again the sets $\mathcal{S}_1$ and $\mathcal{S}_2$ from Example 10. Then, $\mathsf{Free}^{!}_{\mathsf{CL}}(\mathcal{S}_2) = \mathcal{S}_2$, while $\mathsf{Free}^{!}_{\mathsf{CL}}(\mathcal{S}_1) = \emptyset$. Thus, for instance, $\mathcal{S}_2 \mid\!\sim^{\mathsf{fr!}}_{\mathsf{CL}} r$, while $\mathcal{S}_1 \not\mid\!\sim^{\mathsf{fr!}}_{\mathsf{CL}} p$.

An even more radical solution to the issue raised in Example 10 would be to consider the *safe formulas* of $\mathcal{S}$ (Grant and Hunter 2017), namely: those that are syntactically independent of the rest of the formulas in $\mathcal{S}$:

Definition 9. Let $\mathsf{Atoms}(\psi)$ be the set of atoms that appear in ψ, and let $\mathsf{Atoms}(\mathcal{S}) = \bigcup_{\psi\in\mathcal{S}} \mathsf{Atoms}(\psi)$. We denote:

$$\mathsf{Safe}_{\mathfrak{L}}(\mathcal{S}) = \{\psi \in \mathsf{Free}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Atoms}(\psi) \cap \mathsf{Atoms}(\mathcal{S} \setminus \{\psi\}) = \emptyset\}.$$

Then, we define:

$$\mathcal{S} \mid\!\sim^{\mathsf{sf}}_{\mathfrak{L}} \psi \text{ iff } \mathsf{Safe}_{\mathfrak{L}}(\mathcal{S}) \vdash \psi \tag{8}$$

Example 12. Consider again the sets in Example 10. Then $\mathsf{Safe}_{\mathsf{CL}}(\mathcal{S}_2) = \mathcal{S}_2$, while $\mathsf{Safe}_{\mathsf{CL}}(\mathcal{S}_1) = \emptyset$. Thus, again, we have that $\mathcal{S}_2 \mid\!\sim^{\mathsf{sf}}_{\mathsf{CL}} r$, while $\mathcal{S}_1 \not\mid\!\sim^{\mathsf{sf}}_{\mathsf{CL}} p$.

4 Analysis of the Entailment Relations

In this section, we take a closer look at the entailment relations introduced in the previous section, and investigate some of their basic properties and their relative strength.

4.1 Basic Properties

Some basic properties of the entailment relations are considered below.

Proposition 3. *All the entailments introduced in Section 3 (Equations (3)–(8)) are paraconsistent.*

Proof. Let $\mathrel{|\!\sim}$ be any of the entailment relations in Section 3. Since $\mathsf{Free}(\{p, \neg p\}) = \emptyset$, we have that $p, \neg p \mathrel{|\!\not\sim} q$ for any atomic p, q. $\square$

Proposition 4. *All the entailments introduced in Section 3 (Equations (3)–(8)) are non-monotonic.*

Proof. Again, let $\mathrel{|\!\sim}$ be any of the entailment relations in Section 3. Then $p \mathrel{|\!\sim} p$ but $p, \neg p \mathrel{|\!\not\sim} p$. $\square$

Proposition 5. *If $\mathcal{S}$ is $\vdash_{\mathsf{CL}}$-consistent, then for every $\circ \in \{\forall, \exists\}$ it holds that $\mathcal{S} \mathrel{|\!\sim}^{\circ\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\circ\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{DB}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\star\circ\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{LP}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}^{\star\circ\mathsf{fr}}_{\mathsf{CL}\downarrow\mathsf{DB}} \psi$ iff $\mathcal{S} \vdash_{\mathsf{CL}} \psi$.*

Proof. Follows from the fact that if $\mathcal{S}$ is classically consistent, then the most consistent 3-valued models of $\mathcal{S}$ in LP are the same as the most consistent 4-valued models of $\mathcal{S}$ in DB, and both are the same as the 2-valued models of $\mathcal{S}$ in CL. Thus, for every $\circ \in \{\forall, \exists\}$, we get: $\mathsf{Free}^{\circ}_{\mathsf{CL}\downarrow\mathsf{LP}}(\mathcal{S}) = \mathsf{Free}^{\circ}_{\mathsf{CL}\downarrow\mathsf{DB}}(\mathcal{S}) = \mathsf{Free}^{\star\circ}_{\mathsf{CL}\downarrow\mathsf{LP}}(\mathcal{S}) = \mathsf{Free}^{\star\circ}_{\mathsf{CL}\downarrow\mathsf{DB}}(\mathcal{S}) = \mathsf{Free}_{\mathsf{CL}}(\mathcal{S}) = \mathcal{S}$. $\square$

In the remaining of this section we consider the behavior of the free formulas with respect to conjunctive and disjunctive information. We do so for classical logic and its reduction to LP, thus in what follows, for $\circ \in \{\forall, \exists\}$, we abbreviate $\mathsf{MCS}_{\mathsf{CL}}(\mathcal{S})$ (respectively, $\mathsf{Free}_{\mathsf{CL}}(\mathcal{S})$, $\mathsf{Free}^{\circ}_{\mathsf{CL}\downarrow\mathsf{LP}}(\mathcal{S})$, $\mathsf{Free}^{\star\circ}_{\mathsf{CL}\downarrow\mathsf{LP}}(\mathcal{S})$, $\mathsf{Free}^{!}_{\mathsf{CL}}(\mathcal{S})$, $\mathsf{Safe}_{\mathsf{CL}}(\mathcal{S})$) by $\mathsf{MCS}(\mathcal{S})$ (respectively, $\mathsf{Free}(\mathcal{S})$, $\mathsf{Free}^{\circ}(\mathcal{S})$, $\mathsf{Free}^{\star\circ}(\mathcal{S})$, $\mathsf{Free}^{!}(\mathcal{S})$, $\mathsf{Safe}(\mathcal{S})$)), and write $\vdash$ instead of $\vdash_{\mathsf{CL}}$.

Definition 10. For a set $\mathsf{Fr} \in \{\mathsf{Free}, \mathsf{Free}^{\exists}, \mathsf{Free}^{\star\exists}, \mathsf{Free}^{\forall}, \mathsf{Free}^{\star\forall}, \mathsf{Free}^{!}, \mathsf{Safe}\}$ we consider the following properties, defined with respect to an arbitrary set $\mathcal{S}$ of formulas:

- *Conjunction elimination* ($\wedge$E):
  if $\phi_1 \wedge \phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$ then $\phi_1 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1\})$ and $\phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_2\})$.
- *Conjunction introduction* ($\wedge$I):
  if $\phi_1 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1\})$ and $\phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_2\})$ then $\phi_1 \wedge \phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$.
- *Disjunction elimination* ($\vee$E):
  if $\phi_1 \vee \phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$ then $\phi_1 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1\})$ or $\phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_2\})$.
- *Disjunction introduction* ($\vee$I):
  if $\phi_1 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1\})$ or $\phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_2\})$ then $\phi_1 \vee \phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$.
- *Strong disjunction elimination* (s-$\vee$E):
  if $\phi_1 \vee \phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$ then $\phi_1 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1\})$ and $\phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_2\})$.
- *Weak disjunction introduction* (w-$\vee$I):
  if $\phi_1 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1\})$ and $\phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_2\})$ then $\phi_1 \vee \phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$.

Proposition 6. A summary of the properties in Definition 10 with respect to the different variants of free-formula sets is presented in Table 4.

	$\wedge$E	$\wedge$I	$\vee$E	$\vee$I	s-$\vee$E	w-$\vee$I
Free	✓	×	×	✓	×	✓
$\mathsf{Free}^{\exists}$	×	×	×	×	×	×
$\mathsf{Free}^{\star\exists}$	✓	×	×	✓	×	✓
$\mathsf{Free}^{\forall}$	×	×	×	×	×	✓
$\mathsf{Free}^{\star\forall}$	✓	×	×	✓	×	✓
$\mathsf{Free}^{!}$	✓	✓	✓	✓	✓	✓
Safe	✓	✓	✓	✓	✓	✓

Table 4: Summary of results (Proposition 6)

Note 2. Clearly, with respect to reduction-based entailment relations, the results shown in Table 4 may change when the underlying logic is non-classical, or when the target logic of the reduction is different from LP. We chose classical logic as the base logic since it is the most widely accepted logic, and LP as the logic to which the reduction is performed, since LP is a very common paraconsistent logic and it is widely accepted in related areas, such as belief revision and inconsistency measurement, where inconsistent information must be handled. Moreover, LP is closely related to CL: It has the same set of tautologies, and its $\{t, f\}$-fragment coincides with CL. This makes LP a natural candidate for our reduction methods for CL.

In the rest of this section (4.1), we provide a series of comments, lemmas, and counterexamples regarding the content of Table 4 .

Note 3. The satisfaction of the properties in Definition 10 by the syntactically defined free-formulas sets (namely, $\mathsf{Free}^{!}$ and Safe) follows straightforwardly. For instance, to see $\wedge$E and s-$\vee$E for Safe, let $\circ \in \{\wedge, \vee\}$ and assume that $\phi_1 \circ \phi_2 \in \mathsf{Safe}(\mathcal{S} \cup \{\phi_1 \circ \phi_2\})$. This means that $\mathsf{Atoms}(\phi_1 \circ \phi_2) \cap \mathsf{Atoms}(\mathcal{S}) = \emptyset$. Thus, $\mathsf{Atoms}(\phi_1) \cap \mathsf{Atoms}(\mathcal{S}) = \emptyset$ and $\mathsf{Atoms}(\phi_2) \cap \mathsf{Atoms}(\mathcal{S}) = \emptyset$, and so $\phi_1 \in \mathsf{Safe}(\mathcal{S} \cup \{\phi_1\})$ and $\phi_2 \in \mathsf{Safe}(\mathcal{S} \cup \{\phi_2\})$.

Note 4. In contrast to the result in the previous note, all the free formulas sets that are not syntactically defined violate $\wedge$I, $\vee$E, and s-$\vee$E. To see this, let $\mathsf{Fr} \in \{\mathsf{Free}, \mathsf{Free}^{\exists}, \mathsf{Free}^{\star\exists}, \mathsf{Free}^{\forall}, \mathsf{Free}^{\star\forall}\}$. Then:

- For the violation of $\wedge$I, let $\mathcal{S} = \{\neg p \vee \neg q\}$, $\phi_1 = p$ and $\phi_2 = q$. Then $\phi_1 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1\})$ and $\phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_2\})$, while $\phi_1 \wedge \phi_2 \notin \mathsf{Free}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$. Thus, $\phi_1 \wedge \phi_2 \notin \mathsf{Fr}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$.
- For the violation of $\vee$E, let $\mathcal{S} = \{\neg p \wedge q,\ p \wedge \neg q\}$, $\phi_1 = p$ and $\phi_2 = q$. Then $\mathsf{MCS}(\mathcal{S} \cup \{\psi_1 \vee \psi_2\}) = \{\{\neg p \wedge q,\ p \vee q\}, \{p \wedge \neg q, p \vee q\}\}$, and so $\psi_1 \vee \psi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\psi_1 \vee \psi_2\})$. On the other hand, $\mathsf{MCS}(\mathcal{S} \cup \{\psi_1\}) = \{\{\neg p \wedge q\}, \{p \wedge \neg q,\ p\}\}$ thus $\psi_1 \notin \mathsf{Free}(\mathcal{S} \cup \{\psi_1\})$. From similar reasons $\psi_2 \notin \mathsf{Free}(\mathcal{S} \cup \{\psi_2\})$.
- The violation $\vee$E implies the violation of s-$\vee$E, but not the

other way around. For an example in which $\vee$E is preserved while s-$\vee$E is not satisfied, let $\mathcal{S} = \{\neg p \wedge (\neg q \vee r)\}$, $\phi_1 = p$ and $\phi_2 = q$. Then $\phi_1 \vee \phi_2 \in \mathsf{Fr}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$ but $\phi_1 \notin \mathsf{Free}(\mathcal{S} \cup \{\phi_1\})$, thus $\phi_1 \notin \mathsf{Fr}(\mathcal{S} \cup \{\phi_1\})$.

Next, we consider the conjunction elimination property $\wedge$E:

Lemma 1 ($\wedge$E for Free)**.** *If* $\phi_1 \wedge \phi_2 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$ *then* $\phi_1 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_1\})$ *and* $\phi_2 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_2\})$.

Proof. Assume that $\phi_1 \notin \mathsf{Free}(\mathcal{S} \cup \{\phi_1\})$. Thus, there is some $\mathcal{T} \in \mathsf{MCS}(\mathcal{S} \cup \{\phi_1\})$ for which $\mathcal{T} \vdash \neg\phi_1$. Then $\mathcal{T} \subseteq \mathcal{S}$, and clearly $\mathcal{T} \in \mathsf{MCS}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$. But then $\phi_1 \wedge \phi_2 \notin \mathsf{Free}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$. □

To show $\wedge$E for $\mathsf{Free}^{\star\exists}$, we need the following lemma:

Notation 1. We denote by $!\phi$ the formula $\phi \wedge \neg\phi$.

Lemma 2. *If there is no* $M \in \mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi\})$ *for which* $M(!\phi) = f$*, then* $\mathcal{S} \cup \{\phi\} \vdash_{\mathsf{LP}} !\phi$.

Proof. Suppose that $\mathcal{S} \cup \{\phi\} \nvdash_{\mathsf{LP}} !\phi$. Then there is some $M \in \mathsf{Mod}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi\})$ for which $M(!\phi) = f$, and so $M(\phi) = t$. Either $M \in \mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi\})$ or there is an $M' \in \mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi\})$ s.t. $\mathsf{Inc}^{\star}_{\mathsf{LP}}(M', \mathcal{S}) \subsetneq \mathsf{Inc}^{\star}_{\mathsf{LP}}(M, \mathcal{S})$. In either case, there is a model $N \in \mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi\})$ for which $N(\phi) \in \{t, f\}$, thus $N(!\phi) = f$. This contradicts the assumption of the lemma. □

Lemma 3 ($\wedge$E for $\mathsf{Free}^{\star\exists}$)**.** *If* $\phi_1 \wedge \phi_2 \in \mathsf{Free}^{\star\exists}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$ *then* $\phi_1 \in \mathsf{Free}^{\star\exists}(\mathcal{S} \cup \{\phi_1\})$ *and* $\phi_2 \in \mathsf{Free}^{\star\exists}(\mathcal{S} \cup \{\phi_2\})$.

Proof. Suppose that $\phi_1 \wedge \phi_2 \in \mathsf{Free}^{\star\exists}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$ but $\phi_1 \notin \mathsf{Free}^{\star\exists}(\mathcal{S} \cup \{\phi_1\})$. Then $\phi_1 \wedge \phi_2 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$ and by Lemma 1, also $\phi_1 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_1\})$. Thus, there is no $M \in \mathsf{MCM}^{\star}(\mathcal{S} \cup \{\phi_1\})$ for which $M(\phi_1) = f$. Thus, by Lemma 2, $\mathcal{S} \cup \{\phi_1\} \vdash_{\mathsf{LP}} !\phi_1$. By the monotonicity of $\vdash_{\mathsf{LP}}$, also $\mathcal{S} \cup \{\phi_1, \phi_2\} \vdash_{\mathsf{LP}} !\phi_1$ and so $\mathcal{S} \cup \{\phi_1 \wedge \phi_2\} \vdash_{\mathsf{LP}} !\phi_1$. It follows that $\mathcal{S} \cup \{\phi_1 \wedge \phi_2\} \vdash_{\mathsf{LP}} !(\phi_1 \wedge \phi_2)$, but then $\phi_1 \wedge \phi_2 \notin \mathsf{Free}^{\star\exists}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$, a contradiction. □

The proof of Lemma 3 relies on Lemma 2. We now show that this is indeed necessary: Lemma 2 does not hold when $\mathsf{MCM}^{\star}$ is replaced by MCM (Note 5) and consequently $\wedge$E is not satisfied when $\mathsf{Free}^{\star\exists}$ is replaced by $\mathsf{Free}^{\exists}$ (Note 6).

Note 5. To see that Lemma 2 does not apply to MCM, let $\mathcal{S} = \{!p \vee !q,\ \neg s \vee !r\}$ and $\phi = !p \vee !q \vee s$. There are two types of models in $\mathsf{MCM}(\mathcal{S} \cup \{\phi\})$: M_1 with $M_1(!p) = \top$, $M_1(!q) = f$ and $M_1(s) = f$, and M_2 with $M_2(!q) = \top$, $M_2(!p) = f$ and $M_2(s) = f$. Thus, there is no $M \in \mathsf{MCM}(\mathcal{S} \cup \{\phi\})$ such that $M(\phi) = f$. However, there is a model $M \in \mathsf{Mod}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi\})$ in which $M(!p) = \top$, $M(!q) = \top$, $M(!r) = \top$ and $M(s) = t$. Thus, $M(\phi) = t$, $M(!\phi) = f$, and so $\mathcal{S} \cup \{\phi\} \nvdash_{\mathsf{LP}} !\phi$.

Note 6 (no $\wedge$E for $\mathsf{Free}^{\exists}$)**.** Let $\mathcal{S} = \{!p \vee !q,\ (\neg s \vee !r) \wedge !u\}$, $\phi_1 = !p \vee !q \vee s$, and $\phi_2 = s$.

• We show that $\phi_1 \notin \mathsf{Free}^{\exists}(\mathcal{S} \cup \{\phi_1\})$: Note, first, that $\phi_1 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_1)$. Now, there are two types of models in $\mathsf{MCM}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1\})$: M_1 that renders only p and u as $\top$, and M_2 that renders only q and u as $\top$. In each of these models s is assigned f. (Note that if s is assigned t then the value of r must be $\top$ and therefore the model is not most consistent). Thus, in every most consistent model, ϕ_1 is interpreted as $\top$, and so $\phi_1 \notin \mathsf{Free}^{\exists}(\mathcal{S} \cup \{\phi_1\})$.

• We show that $\phi_1 \wedge \phi_2 \in \mathsf{Free}^{\exists}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$. First, note that $\phi_1 \wedge \phi_2 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$. Now, there are four types of models in $\mathsf{MCM}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$, including the following:

– M_1 with $\mathsf{Inc}_{\mathsf{LP}}(M_1, \mathcal{S}) = \{p, r, u\}$ and $M_1(s) = t$,
– M_1' with $\mathsf{Inc}_{\mathsf{LP}}(M_1', \mathcal{S}) = \{q, r, u\}$ and $M_1'(s) = t$,
– M_2 with $\mathsf{Inc}_{\mathsf{LP}}(M_2, \mathcal{S}) = \{p, s, u\}$,
– M_2' with $\mathsf{Inc}_{\mathsf{LP}}(M_2, \mathcal{S}) = \{q, s, u\}$.

For $M \in \{M_1, M_1'\}$ we have $M(\phi_1 \wedge \phi_2) = t$, therefore $\phi_1 \wedge \phi_2 \in \mathsf{Free}^{\exists}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$.

• Altogether, $\mathsf{Free}^{\exists}$ does not satisfy $\wedge$E.

A similar situation holds w.r.t. the universal variations: $\wedge$E is satisfied by $\mathsf{Free}^{\star\forall}$ (Lemma 4) but it is violated by $\mathsf{Free}^{\forall}$ (Note 7).

Lemma 4 ($\wedge$E for $\mathsf{Free}^{\star\forall}$)**.** $\phi_1 \wedge \phi_2 \in \mathsf{Free}^{\star\forall}(\mathcal{S} \cup \{\phi_1 \wedge \phi_2\})$ *implies* $\phi_1 \in \mathsf{Free}^{\star\forall}(\mathcal{S} \cup \{\phi_1\})$ *and* $\phi_2 \in \mathsf{Free}^{\star\forall}(\mathcal{S} \cup \{\phi_2\})$.

Proof. See the supplementary material. □

Note 7 (no $\wedge$E for $\mathsf{Free}^{\forall}$)**.** Let $\mathcal{S} = \{!p \vee !q\}$, $\phi_1 = s \vee !p$, $\phi_2 = s \vee !q$, and $\phi = \phi_1 \wedge \phi_2$. Note that every $M \in \mathsf{MCM}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi\})$ it holds that $M(s) = t$ and either $M(p) = \top$ or $M(q) = \top$. Thus, $\phi \in \mathsf{Free}^{\forall}(\mathcal{S} \cup \{\phi\})$. However, there is an $M \in \mathsf{MCM}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1\})$ s.t. $M(p) = \top$ and $M(s) = f$. Therefore, $M(\phi_1) = \top$, and so $\phi_1 \notin \mathsf{Free}^{\forall}(\mathcal{S} \cup \{\phi_1\})$. This shows that $\wedge$E does not hold for $\mathsf{Free}^{\forall}$.

We now turn to the properties involving disjunction.

Lemma 5 ($\vee$I for Free)**.** *If* $\phi_1 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_1\})$ *or* $\phi_2 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_2\})$*, then* $\phi_1 \vee \phi_2 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$.

Proof. Suppose that $\phi_1 \vee \phi_2 \notin \mathsf{Free}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$. Then, there is a set $\mathcal{T} \subseteq \mathcal{S}$ that is maximally consistent and for which $\mathcal{T} \vdash \neg(\phi_1 \vee \phi_2)$. Thus, $\mathcal{T} \vdash \neg\phi_1$ and $\mathcal{T} \vdash \neg\phi_2$, and so $\phi_1 \notin \mathsf{Free}(\mathcal{S} \cup \{\phi_1\})$ and $\phi_2 \notin \mathsf{Free}(\mathcal{S} \cup \{\phi_2\})$. □

Note 8 (no $\vee$I and w-$\vee$I for $\mathsf{Free}^{\exists}$)**.** Consider the set $\mathcal{S} = \{(p \wedge !q \wedge !r) \vee (\neg p \wedge !q) \vee (!p \wedge \neg q) \vee (!p \wedge q \wedge !r)\}$ and the following (non-exhaustive list of) LP-models of $\mathcal{S}$:

No.	p	q	r
M_1	t	$\top$	$\top$
M_2	f	$\top$	t
M_2'	f	$\top$	f

No.	p	q	r
M_3	$\top$	f	t
M_3'	$\top$	f	f
M_4	$\top$	t	$\top$

Since $p \in \mathsf{Free}(\mathcal{S} \cup \{p\})$ and $M_1 \in \mathsf{MCM}_{\mathsf{LP}}(\mathcal{S} \cup \{p\})$ we get that $p \in \mathsf{Free}^{\exists}(\mathcal{S} \cup \{p\})$. Similarly, since $q \in \mathsf{Free}(\mathcal{S} \cup \{q\})$ and $M_4 \in \mathsf{MCM}_{\mathsf{LP}}(\mathcal{S} \cup \{q\})$ we get that $q \in \mathsf{Free}^{\exists}(\mathcal{S} \cup \{q\})$. However, $p \vee q \notin \mathsf{Free}^{\exists}(\mathcal{S} \cup \{p \vee q\})$. To see this, note that $\mathsf{MCM}_{\mathsf{LP}}(\mathcal{S} \cup \{p \vee q\}) = \{M_2, M_2', M_3, M_3'\}$. In these models, $p \vee q$ is rendered inconsistent.

Lemma 6 ($\vee$I for $\mathsf{Free}^{\star\exists}$). *If $\phi_1 \in \mathsf{Free}^{\star\exists}(\mathcal{S}\cup\{\phi_1\})$ or $\phi_2 \in \mathsf{Free}^{\star\exists}(\mathcal{S} \cup \{\phi_2\})$, then $\phi_1 \vee \phi_2 \in \mathsf{Free}^{\star\exists}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$.*

Proof. If $\phi_1 \in \mathsf{Free}^{\star\exists}(\mathcal{S}\cup\{\phi_1\})$, then $\phi_1 \in \mathsf{Free}(\mathcal{S}\cup\{\phi_1\})$. By $\vee$I for Free, $\phi_1 \vee \phi_2 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$.

Now, since $\phi_1 \in \mathsf{Free}^{\star\exists}(\mathcal{S} \cup \{\phi_1\})$, there is an $M_1 \in \mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S}\cup\{\phi_1\})$ s.t. $M_1(\phi_1) = t$. Thus, $M_1(\phi_1\vee\phi_2) = t$. Hence, M_1 is an LP- model of $\mathcal{S} \cup \{\phi_1 \vee \phi_2\}$, and $M_1(\phi_1 \vee \phi_2) = t$. This immediately implies that there is an $M_1' \in \mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S}\cup\{\phi_1 \vee \phi_2\})$ s.t. $M_1'(\phi_1 \vee \phi_2) = t$. We therefore conclude that $\phi_1 \vee \phi_2 \in \mathsf{Free}^{\star\exists}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$. □

Note 9 (no $\vee$I for $\mathsf{Free}^{\forall}$). Consider the set $\mathcal{S} = \{!p \vee !q\}$, and let $\phi_1 = r$ and $\phi_2 = !p$. Then, clearly, $\phi_1 \in \mathsf{Free}^{\forall}(\mathcal{S} \cup \{\phi_1\})$. Consider now the set $\mathcal{S} \cup \{\phi_1 \vee \phi_2\}$. There is an $M \in \mathsf{MCM}_{\mathsf{LP}}(\mathcal{S}\cup\{\phi_1\vee\phi_2\})$ s.t. $M(p) = \top$ and $M(r) = f$. Since $M(\phi_1 \vee \phi_2) = \top$, we have that $\phi_1 \vee \phi_2 \notin \mathsf{Free}^{\forall}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$. This shows that $\vee$I fails for $\mathsf{Free}^{\forall}$.

Lemma 7 ($\vee$I for $\mathsf{Free}^{\star\forall}$). *If $\phi_1 \in \mathsf{Free}^{\star\forall}(\mathcal{S}\cup\{\phi_1\})$ or $\phi_2 \in \mathsf{Free}^{\star\forall}(\mathcal{S} \cup \{\phi_2\})$, then $\phi_1 \vee \phi_2 \in \mathsf{Free}^{\star\forall}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$.*

Proof. Suppose that $\phi_1 \in \mathsf{Free}^{\star\forall}(\mathcal{S} \cup \{\phi_1\})$. Then $\phi_1 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_1\})$ and by $\vee$I for Free, $\phi_1 \vee \phi_2 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$. Assume for a contradiction that there is some $M \in \mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$ such that $M(\phi_1 \vee \phi_2) = \top$. Then, $M \in \mathsf{Mod}_{\mathsf{LP}}(\mathcal{S}\cup\{\phi_1\})\cup\mathsf{Mod}_{\mathsf{LP}}(\mathcal{S}\cup\{\phi_2\})$. Without loss of generality, $M \in \mathsf{Mod}_{\mathsf{LP}}(\mathcal{S}\cup\{\phi_1\})$. This in particular means that $M(\phi_1) \in \{t, \top\}$, but $M(\phi_1) \neq t$ (Otherwise, $M(\phi_1 \vee \phi_2) = t$ which contradicts our assumption on M). Therefore, $M(\phi_1) = \top$.

Now, since $\phi_1 \in \mathsf{Free}^{\star\forall}(\mathcal{S} \cup \{\phi_1\})$, for every $M' \in \mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S}\cup\{\phi_1\})$, $M'(\phi_1) = t$, and so also $M'(\phi_1\vee\phi_2) = t$. Since $M(\phi_1) = \top$, $M \notin \mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1\})$. Therefore, there is an $M_1 \in \mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1\})$ s.t. $\mathsf{Inc}^{\star}_{\mathsf{LP}}(M_1, \mathcal{S} \cup \{\phi_1\}) \subset \mathsf{Inc}^{\star}_{\mathsf{LP}}(M, \mathcal{S} \cup \{\phi_1\})$. Hence, $M_1(\phi_1) = t$, and so also $M_1(\phi_1 \vee \phi_2) = t$.

It follows that $M \notin \mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S}\cup\{\phi_1\vee\phi_2\})$, since $M_1 \in \mathsf{Mod}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$ and $\mathsf{Inc}^{\star}_{\mathsf{LP}}(M_1, \mathcal{S} \cup \{\phi_1 \vee \phi_2\}) \subset \mathsf{Inc}^{\star}_{\mathsf{LP}}(M, \mathcal{S} \cup \{\phi_1 \vee \phi_2\})$. This contradicts the assumption that $M \in \mathsf{MCM}^{\star}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$. □

Lemma 8 (w-$\vee$I for $\mathsf{Free}^{\forall}$). *If $\phi_1 \in \mathsf{Free}^{\forall}(\mathcal{S} \cup \{\phi_1\})$ and $\phi_2 \in \mathsf{Free}^{\forall}(\mathcal{S}\cup\{\phi_2\})$, then $\phi_1\vee\phi_2 \in \mathsf{Free}^{\forall}(\mathcal{S}\cup\{\phi_1\vee\phi_2\})$.*

Proof. Suppose for a contradiction that $\phi_1 \in \mathsf{Free}^{\forall}(\mathcal{S} \cup \{\phi_1\})$ and $\phi_2 \in \mathsf{Free}^{\forall}(\mathcal{S} \cup \{\phi_2\})$, but $\phi_1 \vee \phi_2 \notin \mathsf{Free}^{\forall}(\mathcal{S} \cup \{\phi_1\vee\phi_2\})$. Then, there is an $M \in \mathsf{MCM}_{\mathsf{LP}}(\mathcal{S}\cup\{\phi_1\vee\phi_2\})$, for which $M(\phi_1 \vee \phi_2) = \top$. Clearly, $M \in \mathsf{Mod}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1\}) \cup \mathsf{Mod}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_2\})$. Without loss of generality, let $M \in \mathsf{Mod}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1\})$. Since $\phi_1 \in \mathsf{Free}^{\forall}(\mathcal{S} \cup \{\phi_1\})$, $\phi_1 \in \mathsf{Free}(\mathcal{S} \cup \{\phi_1\})$ and thus by $\vee$I for Free, $\phi_1 \vee \phi_2 \in \mathsf{Free}(\mathcal{S}\cup\{\phi_1\vee\phi_2\})$. Also, every $M' \in \mathsf{MCM}_{\mathsf{LP}}(\mathcal{S}\cup\{\phi_1\})$, $M'(\phi_1) = t$. Hence, $M \notin \mathsf{MCM}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1\})$ (Indeed, $M(\psi_1) \neq t$, otherwise $M(\psi_1 \vee \psi_2) = t$, contradicting our assumption). But then there is an $M' \in \mathsf{MCM}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1\})$ for which $\mathsf{Inc}_{\mathsf{LP}}(M', \mathcal{S} \cup \{\phi_1\}) \subset \mathsf{Inc}_{\mathsf{LP}}(M, \mathcal{S} \cup \{\phi_1\})$. Now, let M'' be the same valuation as M', except that all atoms in $\mathsf{Atom}(\phi_2) \setminus \mathsf{Atom}(\mathcal{S} \cup \{\phi_1\})$ are mapped to t (if there are any, otherwise we let $M'' = M'$). Then, $\mathsf{Inc}_{\mathsf{LP}}(M'', \mathcal{S} \cup \{\phi_1 \vee \phi_2\}) \subset \mathsf{Inc}_{\mathsf{LP}}(M', \mathcal{S} \cup \{\phi_1 \vee \phi_2\})$, and $M'' \in \mathsf{Mod}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$. This contradicts the assumption that $M \in \mathsf{MCM}_{\mathsf{LP}}(\mathcal{S} \cup \{\phi_1 \vee \phi_2\})$. □

Note 10 (w-$\vee$I for Free, $\mathsf{Free}^{\star\exists}$, and $\mathsf{Free}^{\star\forall}$). Since w-$\vee$I is implied by $\vee$I, and since by Lemma 5 (respectively, by Lemma 6, Lemma 7) $\vee$I holds for Free (respectively, for $\mathsf{Free}^{\star\exists}$, $\mathsf{Free}^{\star\forall}$), it follows that also w-$\vee$I holds for Free (respectively, for $\mathsf{Free}^{\star\exists}$, $\mathsf{Free}^{\star\forall}$).

4.2 Relative Strengths

The next result shows the relations among the entailments considered in this paper for reasoning with free formulas.

Proposition 7. *Let $\mathfrak{L}_1 = \langle \mathcal{L}, \vdash_{\mathcal{M}_1} \rangle$ and $\mathfrak{L}_2 = \langle \mathcal{L}, \vdash_{\mathcal{M}_2} \rangle$ be two multi-valued logics for $\mathcal{M}_1 = \langle \mathcal{V}_1, \mathcal{D}_1, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}_2, \mathcal{D}_2, \mathcal{O}_2 \rangle$ (respectively), such that $\mathfrak{L}_2$ is contained in $\mathfrak{L}_1$. Then:*

1. *if $\mathcal{S} \mid\!\sim^{\mathsf{fr}}_{\mathfrak{L}_1} \psi$ then $\mathcal{S} \mid\!\sim^{\mathsf{mcs}}_{\mathfrak{L}_1} \psi$,*
2. *if $\mathcal{S} \mid\!\sim^{\forall\mathsf{fr}}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2} \psi$ then $\mathcal{S} \mid\!\sim^{\exists\mathsf{fr}}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2} \psi$,*
3. *if $\mathcal{S} \mid\!\sim^{\star\forall\mathsf{fr}}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2} \psi$ then $\mathcal{S} \mid\!\sim^{\star\exists\mathsf{fr}}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2} \psi$,*
4. *if $\mathcal{S} \mid\!\sim^{\exists\mathsf{fr}}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2} \psi$ then $\mathcal{S} \mid\!\sim^{\mathsf{fr}}_{\mathfrak{L}_1} \psi$,*
5. *if $\mathcal{S} \mid\!\sim^{\star\exists\mathsf{fr}}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2} \psi$ then $\mathcal{S} \mid\!\sim^{\mathsf{fr}}_{\mathfrak{L}_1} \psi$,*
6. *if $\mathcal{S} \mid\!\sim^{\mathsf{fr!}}_{\mathfrak{L}_1} \psi$ then $\mathcal{S} \mid\!\sim^{\mathsf{fr}}_{\mathfrak{L}_1} \psi$,*
7. *if $\mathcal{S} \mid\!\sim^{\mathsf{fr!}}_{\mathfrak{L}_1} \psi$ then $\mathcal{S} \mid\!\sim^{\forall\mathsf{fr}}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2} \psi$,*
8. *if $\mathcal{S} \mid\!\sim^{\mathsf{sf}}_{\mathfrak{L}_1} \psi$ then $\mathcal{S} \mid\!\sim^{\forall\mathsf{fr}}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2} \psi$,*
9. *if $\mathcal{S} \mid\!\sim^{\mathsf{sf}}_{\mathfrak{L}_1} \psi$ then $\mathcal{S} \mid\!\sim^{\star\forall\mathsf{fr}}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2} \psi$,*
10. *if $\mathcal{S} \mid\!\sim^{\mathsf{sf}}_{\mathfrak{L}_1} \psi$ then $\mathcal{S} \mid\!\sim^{\mathsf{fr!}}_{\mathfrak{L}_1} \psi$.*

Proof. Item 1 is Proposition 2. Items 2 and 3 follow, respectively, from the facts that $\mathsf{Free}^{\forall}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S}) \subseteq \mathsf{Free}^{\exists}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S})$ and $\mathsf{Free}^{\star\forall}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S}) \subseteq \mathsf{Free}^{\star\exists}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S})$.

Items 4–6 follow, respectively, from the facts that the sets $\mathsf{Free}^{\exists}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S})$, $\mathsf{Free}^{\star\exists}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S})$, and $\mathsf{Free}^{!}_{\mathfrak{L}_1}(\mathcal{S})$, are all included in $\mathsf{Free}_{\mathfrak{L}_1}(\mathcal{S})$.

For Item 7, we show that $\mathsf{Free}^{!}_{\mathfrak{L}}(\mathcal{S}) \subseteq \mathsf{Free}^{\forall}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S})$. Indeed, let $\psi \in \mathsf{Free}^{!}_{\mathfrak{L}}(\mathcal{S})$. In particular, there is no literal in ψ that has a complementary literal in a formula in $\mathcal{S}$. Thus, there is an $\mathfrak{L}_2$-model M_ψ of $\mathcal{S}$ that does not assign a value in $(\mathcal{V}_2)_\top$ to any literal in ψ, and so M_ψ does not assign a value in $(\mathcal{V}_2)_\top$ to any atom in ψ. In other words, $\mathsf{Inc}(M_\psi, \psi) = \emptyset$. It thus follows that for *every* most consistent model M of $\mathcal{S}$, $\mathsf{Inc}(M, \psi) = \emptyset$ (Otherwise, if $\mathsf{Inc}(M, \psi) \neq \emptyset$, let $M'(p) = M_\psi(p)$ if $M(p) \in (\mathcal{V}_2)_\top$ and $M'(p) = M(p)$ if $M(p) \notin (\mathcal{V}_2)_\top$. It is easy to see that M' is a model of $\mathcal{S}$ that is strictly more consistent than M, contradicting the assumption that M is a most consistent model of $\mathcal{S}$). Now, since M is an $\mathfrak{L}_2$-model of ψ, necessarily $M(\psi) \in (\mathcal{V}_2)_t$. Therefore, $\psi \in \mathsf{Free}^{\forall}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S})$.

Items 8–10 follow, respectively, from the fact that the set $\mathsf{Safe}_{\mathfrak{L}_1}(\mathcal{S})$ is included in each of the sets $\mathsf{Free}^{\forall}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S})$, $\mathsf{Free}^{\star\forall}_{\mathfrak{L}_1\downarrow\mathfrak{L}_2}(\mathcal{S})$, and $\mathsf{Free}^{!}_{\mathfrak{L}_1}(\mathcal{S})$. (To see this, take note that if a consistent formula is syntactically disjoint from the other formulas in $\mathcal{S}$, namely: its does not share atoms with the other formulas in $\mathcal{S}$, it can be assigned a value in $\mathcal{V}_t$ by any most consistent model of $\mathcal{S}$). $\square$

The examples in the paper show that the converses of some of the items in Proposition 7 do not hold. Counterexamples for the other reversed conditions in Proposition 7 can be easily constructed.

Next, we study further connections among the entailment relations under additional assumptions on the base logics.

Definition 11. A multi-valued logic $\mathfrak{L} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ induced by a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is called *$\mathcal{V}_\top$-free*, if for every valuation ν and formula ψ, it holds that if $\nu(p) \notin \mathcal{V}_\top$ for every $p \in \mathsf{Atoms}(\psi)$, then $\nu(\psi) \notin \mathcal{V}_\top$ as well.

Lemma 9. *All the logics in Example 3 are $\mathcal{V}_\top$-free.*

Proof. For classical logic the claim is trivial, since $\mathcal{V}_\top = \emptyset$. For any other logic in Example 3 the proof is by a simple induction on the structure of ψ. $\square$

Note 11. Lemma 9 holds also for some extensions of the logics in Example 3. For instance, when the language of LP or of Dunn-Belnap logic is extended by the deductive implication connective $\supset$, defined for every $a, b \in \mathcal{V}$ by $a \supset b = t$ if $a \notin \mathcal{D}$ and $a \supset b = b$ otherwise (see (D'Ottaviano and da Costa 1970)). However, not every extension of these logic is $\mathcal{V}_\top$-free. One such case is the extension of the language of Dunn-Belnap logic with the consensus operator $\oplus$ (see (Fitting 2006)), for which $t \oplus f = \top$.

Proposition 8. *Let $\mathfrak{L}_1 = \langle \mathcal{L}, \vdash_{\mathcal{M}_1} \rangle$ and $\mathfrak{L}_2 = \langle \mathcal{L}, \vdash_{\mathcal{M}_2} \rangle$ be two multi-valued logics as in Proposition 7. If $\mathfrak{L}_2$ is $\mathcal{V}_\top$-free, then $\mathcal{S} \mid\!\sim^{\mathsf{fr!}}_{\mathfrak{L}_1} \psi$ implies that $\mathcal{S} \mid\!\sim^{\star\forall\mathsf{fr}}_{\mathfrak{L}_1 \downarrow \mathfrak{L}_2} \psi$.*

Proof. Let $\psi \in \mathsf{Free}^{!}_{\mathfrak{L}_1}(\mathcal{S})$. The proof of Item 7 in Proposition 7 shows that $\mathsf{Free}^{!}_{\mathfrak{L}_1}(\mathcal{S}) \subseteq \mathsf{Free}^{\forall}_{\mathfrak{L}_1 \downarrow \mathfrak{L}_2}(\mathcal{S})$ and that for every $M \in \mathsf{MCM}_{\mathfrak{L}_2}(\mathcal{S})$, $\mathsf{Inc}(M, \psi) = \emptyset$. Now, since $\mathfrak{L}_2$ is $\mathcal{V}_\top$-free, we conclude that also for every $M \in \mathsf{MCM}^{\star}_{\mathfrak{L}_2}(\mathcal{S})$ it holds that $\mathsf{Inc}(M, \psi) = \emptyset$. Since M is an $\mathfrak{L}_2$-model of ψ, necessarily $M(\psi) \in (\mathcal{V}_2)_t$. Therefore, $\psi \in \mathsf{Free}^{\star\forall}_{\mathfrak{L}_1 \downarrow \mathfrak{L}_2}(\mathcal{S})$. Altogether, we have shown that $\mathsf{Free}^{!}_{\mathfrak{L}_1}(\mathcal{S}) \subseteq \mathsf{Free}^{\star\forall}_{\mathfrak{L}_1 \downarrow \mathfrak{L}_2}(\mathcal{S})$, and so the proposition follows. $\square$

The relations among the entailments considered in this paper are illustrated in Figure 1. A solid arrow from $\vdash_1$ to $\vdash_2$ indicates that $\vdash_1 \subset \vdash_2$. A dashed arrow has a similar meaning, except that the corresponding containment holds only under additional conditions.

5 Conclusion and Related Work

The free formulas of an inconsistent knowledge base are usually regarded as representing the reliable information in the base. It is thus common to treat these formulas in a uniform manner. In this paper, we argued that such a homogeneous treatment is sometimes too coarse, and accordingly we introduced and examined several entailment relations that capture more refined distinctions.

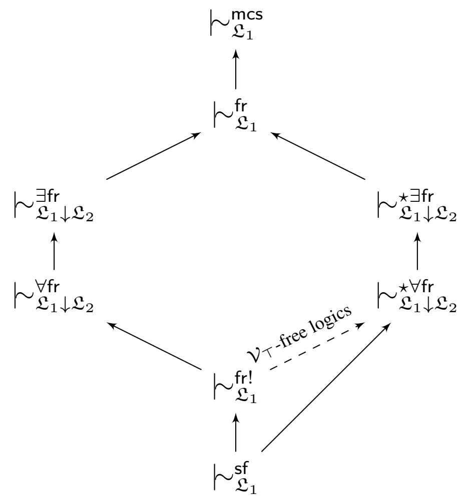

Figure 1: Relations among the entailments

Our approach builds on established techniques for handling inconsistency. For instance, the use of minimally inconsistent models, primarily for defining general entailment relations in nonmonotonic reasoning (instead of making a distinction among the free formulas, as in our case), can be traced back to Batens (1986) and Shoham (1988) and has since been widely adopted (see, e.g., (Kraus, Lehmann, and Magidor 1990; Dubois and Prade 1991; Priest 1991) for some early influential follow-up papers).

A similar process of refining the treatment of consistent information by distinguishing between free formulas may also be beneficial in other formal frameworks. For example, the standard postulates in belief revision (such as the AGM postulates, (Alchourrón, Gärdenfors, and Makinson 1985)) do not distinguish between different pieces of information within consistent knowledge bases. Likewise, quantitative methods for measuring the degree of inconsistency in inconsistent knowledge bases usually uniformly treat consistent knowledge bases as maximally consistent (see, e.g., (Salhi 2021; Liu, Besnard, and Doutre 2023)). In this context, it is customary to adopt the *free-formula independence property* (Hunter and Konieczny (2010), see also Mu (2020) and Raddaoui, Straßer, and Jabbour (2024)), according to which an inconsistency measure $\mathcal{I}$ should satisfy the condition that

$$\forall \varphi \in \mathsf{Free}(\mathcal{S}),\ \mathcal{I}(\mathcal{S}) = \mathcal{I}(\mathcal{S} \setminus \{\psi\}).$$

Consequently, no formula in $\mathsf{Free}(\mathcal{S})$ affects $\mathcal{I}$. Clearly, if one wishes to distinguish between the different levels of certainty that various free formulas embody, the above criterion must be relaxed (or abandoned). Indeed, the observation that not all the free formulas are independent of the inconsistency, when the inconsistency is characterized by models of paraconsistent logics, has also been made in the context of consistency measuring, leading to several works involving the safe formulas of a knowledge base, such as (Thimm 2009) and (Hunter and Konieczny 2010). In that context, inconsistency measures that take into account the $\top$-free MCMs[3], and the related notion of Bi-free formulas, have been considered by Mu (2019; 2020). We note, however, that these works focus exclusively on Priest's LP and address inconsistency measures rather than inference (entailment) relations.

Note 12. An example for an inconsistency measure that avoids the free-formula independence property is the following: Given a logic $\mathfrak{L}$ that is induced by a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, we define, for every $\psi \in \mathcal{S}$,

$$I^{\mathcal{S}}_{\mathfrak{L}}(\psi) = \frac{|\{M \in \mathsf{MCM}_{\mathfrak{L}}(\mathcal{S}) \mid M(\psi) \in \mathcal{V}_\top\}|}{|\mathsf{MCM}_{\mathfrak{L}}(\mathcal{S})|}. \quad (9)$$

That is, the relative inconsistency measure of ψ is determined by checking the most consistent models (witnesses) of $\mathcal{S}$, and computing the proportion among these witnesses of those who testify for the uncertainty of ψ.

Revisiting Example 7, $I^{\mathcal{S}}_{\mathsf{LP}}$ as defined in (9) w.r.t. Priest logic LP, is capable of making a distinction between the three formulas in $\mathsf{Free}_{\mathsf{CL}}(\mathcal{S})$. Indeed, using Table 2, it is easy to verify that $I^{\mathcal{S}}_{\mathsf{LP}}(r) = 0$, $I^{\mathcal{S}}_{\mathsf{LP}}(p \vee s) = \frac{1}{2}$, and $I^{\mathcal{S}}_{\mathsf{LP}}(p \vee q) = 1$, thus r is the most reliable formula, while $p \vee q$ is the least reliable formula among the free formulas in this case, as intuitively expected (recall the discussion in Example 7).

A further study of inconsistency measures such as the one introduced in the last note is a subject for future work. Another key issue for future exploration is the establishment of computational complexity bounds for the entailment relations introduced in this paper. Finally, it would be interesting to generalize our methods to more expressive formalisms involving first-order languages and/or priorities among formulas (including uncertainty-based frameworks involving probabilistic, possibilistic, or fuzzy logics).

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AI Declaration

Generative AI tools were used only to a limited extent and solely for verifying the factual details appearing in Example 1. No other use of generative AI tools was made in the preparation of the paper.

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[3]In our notation, an $\mathcal{M}$-most consistent model ν of $\mathcal{S}$ is $\top$-free, if $\mathsf{Inc}_{\mathcal{M}}(\nu, \mathcal{S}) = \emptyset$ (recall Definition 5). This property should not be confused with the $\mathcal{V}_\top$-freeness of logics (Definition 11).

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