Proceedings of the 23rd International Conference on Principles of Knowledge Representation and Reasoning
Main Track

Proof-Search for Normative and Doxastic Reasoning and Its Use in Logical Argumentation

Kees van Berkel[1] , **Andrea Sabatini**[2]
[1]Institute of Logic and Computation, TU Wien, Austria
[2]Classe di Lettere e Filosofia, Scuola Normale Superiore di Pisa, Italy
kees.van.berkel@tuwien.ac.at, andrea.sabatini@sns.it

Abstract

Logical argumentation uses calculi to generate arguments and counter-arguments to capture defeasible reasoning. In this paper, we introduce modular proof-search procedures for a large class of such argument calculi developed to capture two core forms of defeasible reasoning: normative reasoning, formalized *via* input/output logics, and doxastic reasoning, formalized *via* normal default logic. Our approach relies on modular, rule-based, and terminating decomposition trees via step-by-step decomposition of norms and defaults. When successful, a terminated decomposition tree certifies derivability of a given argument in its corresponding calculus, including arguments for obligations and beliefs, as well as defeating arguments concluding inapplicability of norms and defaults. We show how rule-based decomposition of norms and defaults is used to determine nonmonotonic inference for credulous reasoning with maximal consistent sets of norms and defaults as well as stable sets of arguments in formal argumentation.

1 Introduction

Everyday reasoning is inherently defeasible: for agents to plan, decide, comply, and, in general, move around in this world, provisional inferences need to be made under incomplete, uncertain, and often inconsistent information (Pollock 1987; Toulmin 1958). Two types of reasoning stand out: normative and doxastic reasoning. The former addresses agents needing to comply with conflict-sensitive norm systems, whereas the latter serves agents updating their beliefs under, possibly, conflicting information. To deal with defeasible reasoning in computer science, a wide variety of nonmonotonic (nm) logics has been developed (see (Strasser 2025) for an overview). In particular, input/output logics (Makinson and van der Torre 2001) and default logic (Reiter 1980) are paradigmatic formal approaches to defeasible normative, respectively, doxastic reasoning using maximal consistent sets. Over the past decades, logical argumentation (Dung 1995) rose to prominence as a unifying framework for nm logics (Arieli et al. 2021), opening the door to comparative studies of nm logics and more modular approaches by combining formalisms. Various results have been obtained for input/output and default style logics (van Berkel and Straßer 2022; van Berkel, Straßer, and Zhou 2024; Liao et al. 2018; Straßer and Pardo 2021).

The use of sequent-style calculi for logical argumentation is particularly promising for defeasible reasoning, as it provides context of how arguments are derived, yields Toulmin-style arguments, and introduces nuances in attack and defeat relations between arguments (Arieli and Straßer 2019; Arieli and Straßer 2015). Logical argumentation approaches instantiate argumentation frameworks with arguments derived from a knowledge base using a calculus. However, actual derivation methods are often missing: (i) the relevant arguments are assumed to be given for the construction of argumentation frameworks, and (ii) since these calculi lack proof-search methods, they cannot be harnessed to check for nm inference. These are strong assumptions that we aim to loosen, by giving proof-search methods for these calculi.

Contributions This paper seeks to combine and harness the benefits of various approaches: logical argumentation, nonmonotonic reasoning through maximal consistent sets, and proof-theory. We do this by defining *decomposition trees* for a large class of argument calculi defined for normative and doxastic reasoning (van Berkel and Straßer 2022; van Berkel, Straßer, and Zhou 2024). Such trees decompose the norms/defaults of a given argument, whose derivability is checked, until the argument is reduced to elementary base logic inferences. As a consequence, decomposition trees enjoy (a generalized version of) the subformula property. This yields finite terminating proof-search procedures. Our decomposition approach is highly modular, which means that proof-search methods can be straightforwardly obtained for a large class of nm logics. Key contributions are as follows:

- We obtain terminating proof-search methods for arguments and counterarguments (attackers) of argument calculi for normative and doxastic reasoning (van Berkel and Straßer 2022; van Berkel, Straßer, and Zhou 2024);
- Decomposition trees determine membership of norms and defaults, in maximal consistent set construction for constraint input/output logics (Makinson and van der Torre 2001), respectively normal default logic (Reiter 1980);
- Terminating heuristics apply decomposition trees to capture credulous reasoning, establishing a close relation to semantics in logical argumentation (Dung 1995).

Outline Sect. 2 recalls argument calculi and we embed this approach in logical argumentation in Sect. 3. Section 4 is devoted to decomposition trees and Sect. 5 presents our decision procedure. Related work is discussed in Sect. 6.

2 Argument Calculi

We first define argument calculi. Since our methods hold for normative and doxastic reasoning, we use a general language referring to 'input' (i) and 'output' (o), through application of 'rules' (r), instead of facts, obligations, and norms for normative reasoning, and beliefs and defaults for doxastic reasoning. We use 'constraints' (c) to control the output.

Let $\mathcal{L}$ be a propositional language and $x \in \{i, o, c\}$ be a set of labels. We define a labelled language $\mathcal{L}^x = \{\phi^x \mid \phi \in \mathcal{L}\}$ and say $\mathcal{L}^i$, $\mathcal{L}^o$, and $\mathcal{L}^c$ are the input, output, respectively constraint language. Let $\mathcal{L}^r = \{(\phi, \psi) \mid \phi^x, \psi^x \in \mathcal{L}^x\}$ be the language of rules and $\overline{\mathcal{L}^r} = \{\neg(\phi, \psi) \mid (\phi, \psi) \in \mathcal{L}^r\}$ of inapplicable rules. In a normative context, we read (ϕ, ψ) as "if ϕ, then ψ is obligatory" and, within a doxastic context, as "if ϕ, then ψ is believed." A formula $\neg(\phi, \psi)$ states that the rule (ϕ, ψ) is inapplicable. The complete language $\mathcal{L}^{ac}$ is defined as $\mathcal{L}^i \cup \mathcal{L}^o \cup \mathcal{L}^c \cup \mathcal{L}^r \cup \overline{\mathcal{L}^r}$. We use $\Delta, \Gamma, \dots \subseteq \mathcal{L}^{ac}$ for arbitrary finite sets and write $\Delta^x \subseteq \mathcal{L}^x$ for $x \in \{i, o, c, r\}$.

We work with knowledge bases of the type $\mathbb{K} = \langle \mathcal{I}, \mathcal{R}, \mathcal{C} \rangle$ where $\mathcal{I} \subseteq \mathcal{L}^i$, $\mathcal{R} \subseteq \mathcal{L}^r$, and $\mathcal{C} \subseteq \mathcal{L}^c$. The idea behind input/output-style reasoning (Makinson and van der Torre 2001) is that input triggers rules from which output is detached, where the output is restricted by a set of constraints. Variations on input/output style logics are due to conditions on how rules are triggered, output is detached, and constraints are enforced. Normal default logic can be seen as an input/output style formalism with logical consistency as the only constraint (Horty 2012; Reiter 1980).

An Argument Calculus (AC for short) (van Berkel and Straßer 2022) is a sequent-style proof system in which arguments, called sequents, are derived through the application of rules. An argument is of the form $\Delta \Rightarrow \Gamma$ where Δ constitute the reasons for concluding Γ. We are interested in two types of argument. The first type contains arguments of the form

$$\phi^i, (\phi, \psi) \Rightarrow \psi^o$$

providing input ϕ^i and rules (ϕ, ψ) as reasons for concluding output ψ^o. The second type of argument, such as

$$\phi^i, \neg\psi^c \Rightarrow \neg(\phi, \psi)$$

expresses the defeasibility of arguments of the first type by giving reasons, i.e., ϕ^i and $\neg\psi^c$, for the *inapplicability* of rules, that is, applying (ϕ, ψ) would yield the constraint-inconsistent output ψ^o. Different Argument Calculi enable the construction of complex arguments of both types. The results in this paper extend to all calculi under a classical base logic defined in (van Berkel and Straßer 2022; van Berkel, Straßer, and Zhou 2024), which characterize all constraint input/output logics and a class of normal default logics (Proposition 1). Argument calculi are sets of derivation rules, stipulating the conditions under which arguments can be derived from other arguments.

Definition 1 (Argument Calculi)**.** *Let* L *be classical logic for* $\mathcal{L}^x$ *(*$x \in \{i, o, c\}$*) and* $\vdash_{\mathsf{L}}$ *be derivability in* L*. We define* $\Delta \Rightarrow \Gamma$ *as an* AC*-argument if* $\Delta \subseteq \mathcal{L}^{ac}$ *is a regular finite set and* $\Gamma \subseteq \mathcal{L}^{ac}$ *is restricted to at most one formula from* $\mathcal{L}^{ac}$*.*

The minimal system $\mathsf{AC}_\varnothing$ *is defined by the rules* ***Ax****,* ***BDet****, and* ***Cut****, (Fig. 1). We write* $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$ *for the input/output style extension of* $\mathsf{AC}_\varnothing$ *with* $\{$***Taut****,* ***IO1****,* ***IO2***$\} \subseteq \mathcal{S} \subseteq \{$***Taut****,* ***IO1****,* ***IO2****,* ***TP****,* ***SDet****,* ***Cases***$\}$ *and* $\mathsf{AC}^{\mathsf{dl}}_{\mathcal{S}}$ *for the default style extension of* $\mathsf{AC}_\varnothing$ *with* ***DL1*** *and* $\mathcal{S} = \{$***TP****,* ***SDet***$\}$*.*

An $\mathsf{AC}^{l}_{\mathcal{S}}$*-derivation (*$l \in \{\mathsf{io}, \mathsf{dl}\}$*) of* $\Delta \Rightarrow \Gamma$ *is a tree whose leaves are initial sequents of* $\mathsf{AC}^{l}_{\mathcal{S}}$*, whose root is* $\Delta \Rightarrow \Gamma$*, and whose rule-applications are instances of the rules of* $\mathsf{AC}^{l}_{\mathcal{S}}$*. We write* $\vdash^{l}_{\mathcal{S}} \Delta \Rightarrow \Gamma$ *if* $\Delta \Rightarrow \Gamma$ *is* $\mathsf{AC}^{l}_{\mathcal{S}}$*-derivable.*

$$\dfrac{}{\vdash_{\mathsf{L}} \Delta^x \Rightarrow \Gamma^x}\ \textbf{Ax}\text{, for } x \in \{i, o, c\} \qquad \dfrac{}{\Rightarrow (\top, \top)}\ \textbf{Taut}$$

$$\dfrac{}{\phi^i, (\phi, \psi) \Rightarrow \psi^o}\ \textbf{BDet} \qquad \dfrac{}{\phi^i \Rightarrow \phi^o}\ \textbf{TP}$$

$$\dfrac{\Delta, \phi^i \Rightarrow \Gamma \qquad \Delta', \psi^i \Rightarrow \Gamma}{\Delta, \Delta', (\phi \vee \psi)^i \Rightarrow \Gamma}\ \textbf{Cases}^b \qquad \dfrac{\phi^i, \Delta \Rightarrow \Gamma}{\phi^o, \Delta \Rightarrow \Gamma}\ \textbf{SDet}^a$$

$$\dfrac{\Delta \Rightarrow \phi \qquad \phi, \Delta' \Rightarrow \Gamma}{\Delta, \Delta' \Rightarrow \Gamma}\ \textbf{Cut}^c \qquad \dfrac{\Delta \Rightarrow \phi^o}{\Delta, \neg\phi^c \Rightarrow}\ \textbf{IO1}$$

$$\dfrac{\Delta, (\phi, \psi) \Rightarrow}{\Delta \Rightarrow \neg(\phi, \psi)}\ \textbf{IO2} \qquad \dfrac{}{\phi^i, \neg\psi^o \Rightarrow \neg(\phi, \psi)}\ \textbf{DL1}$$

Figure 1: Rules for constructing $\mathsf{AC}^{l}_{\mathcal{S}}$ (Def. 1). The upper level contains initial rules, the lower logical, structural and defeasibility rules. Condition (a) denotes $\Delta \cap \mathcal{L}^r \neq \varnothing$ if **TP** $\notin \mathcal{S}$; (b) that $\Delta \cap \mathcal{L}^r \neq \varnothing$ and $\Delta' \cap \mathcal{L}^r \neq \varnothing$, when **TP** $\notin \mathcal{S}$; (c) indicates $\phi \in \mathcal{L}^{ac}$.

We briefly discuss the AC-rules of Figure 1. The rule **Ax** introduces initial arguments that express tautologies in the base logic for input, output, and constraints. **Taut** ensures that tautologies are included among input and output. **BDet** expresses the basic notion of detachment: i.e., from input ϕ^i and rule (ϕ, ψ) detach output ψ^o. **TP** ensures that all input will be considered among the output. **SDet** captures successive detachment where the detached output from one rule may serve as input for detaching output from another. **Cut** is the usual structural rule for sequent-style calculi.

The following rules capture defeasibility: **IO1** and **IO2** together capture defeasibility in input/output style reasoning. The former tells us that if Δ gives reasons for output ϕ^o, then Δ is inconsistent with the constraint $\neg\phi^c$. **IO2** tells us that from an inconsistent left-hand side (lhs), we conclude that at least one of its rules is inapplicable $\neg(\phi, \psi)$. The rule **DL1** captures defeasibility in default style reasoning: if $\neg\psi^o$ is among the detached output, and (ϕ, ψ) can be triggered (by ϕ^i and possibly ϕ^o since **SDet** $\in \mathcal{S}$ of $\mathsf{AC}^{\mathsf{dl}}_{\mathcal{S}}$), then the rule is inapplicable $\neg(\phi, \psi)$. **DL1** yields a more restricted form of **IO1** and **IO2**. Example 1 shows their difference.

Example 1. *Let* $\mathbb{K}_1 = \langle \mathcal{I}, \mathcal{R}, \mathcal{C} \rangle$*,* $\mathcal{I} = \{r^i\}$*,* $\mathcal{R} = \{(\top, p), (\top, q), (p, \neg q)\}$*,* $\mathcal{C} = \varnothing$ *and* $\mathsf{AC}^{l}_{\mathcal{S}}$ *with* ***SDet*** $\in \mathcal{S}$ *and* $l \in \{\mathsf{io}, \mathsf{dl}\}$*. We adopt an abstract example to cover both normative and doxastic reasoning but refer, e.g., to (van Berkel, Straßer, and Zhou 2024) for concrete scenarios. Using* ***BDet*** *we can derive the arguments* $a = \top^i, (\top, q) \Rightarrow q^o$ *and* $b = \top^i, (\top, p) \Rightarrow p^o$*. Using* ***SDet*** $\in \mathcal{S}$ *(and applications of* ***Cut****), we derive* $c = \top^i, (\top, p), (p, \neg q) \Rightarrow \neg q^o$*, and* $d = \top^i, (\top, p), (\top, q), (p, \neg q) \Rightarrow$ *and, when* ***TP*** $\in \mathcal{S}$*, we have* $h = r^i \Rightarrow r^o$*. Applying* ***IO1*** *and* ***IO2*** *for* $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$*, we derive arguments for rule inapplicability, expressing a* *threefold conflict:* $e = \top^i, (\top, p), (p, \neg q) \Rightarrow \neg(\top, q)$, $f = \top^i, (\top, p), (\top, q) \Rightarrow \neg(p, \neg q)$, *and* $g = \top^i, (\top, q), (p, \neg q) \Rightarrow \neg(\top, p)$. *Arguments* e *and* f *are also* $\mathsf{AC}^{\mathsf{dl}}_{\mathcal{S}}$ *derivable using **DL1** and **Cut**, but* g *is not. Note for this that* $(p, \neg q)$ *is not triggered by the input and detachable output from* $(\top, q)$*. In default logic only rules that are triggered may serve as reasons, this is not the case for input/output logic. See (van Berkel and Straßer 2022) for example derivations.*

3 Logical Argumentation and Properties

The two types of AC-derivable arguments reflect the two mechanisms of defeasible reasoning: logical inference and defeasibility. Arguments of the second type express *attacks*, where $\Delta \Rightarrow \neg(\phi, \psi)$ attacks all $\Gamma \Rightarrow \Sigma$ that use $(\phi, \psi) \in \Gamma$ as a reason. Interaction between these argument types is fully harnessed when argumentation frameworks are constructed from AC-derivable arguments to characterize conflicts between arguments derivable from a knowledge base $\mathbb{K}$.

Definition 2 (Argumentation Frameworks)**.** *Let* $\mathsf{AC}^l_{\mathcal{S}}$ *be a calculus and* $\mathbb{K} = \langle \mathcal{I}, \mathcal{R}, \mathcal{C} \rangle$ *a knowledge base. An* $\mathsf{AC}^l_{\mathcal{S}}$*-induced* argumentation framework $\mathcal{AF}^l_{\mathcal{S}}(\mathbb{K}) = \langle \mathsf{arg}, \mathsf{att} \rangle$ *consists of arguments* arg *and an attack relation* att *defined:*

- $\mathsf{arg} = \{ \vdash^l_{\mathcal{S}} \Delta \Rightarrow \Gamma \mid \Delta \subseteq \mathcal{I} \cup \mathcal{R} \cup \mathcal{C} \}$;
- $\mathsf{att} = \{ (a, b) \mid a{=}\Delta \Rightarrow \neg(\phi, \psi), b{=}(\phi, \psi), \Gamma \Rightarrow \Sigma \in \mathsf{arg} \}$.

In short, $\mathcal{AF}^l_{\mathcal{S}}(\mathbb{K})$ captures the conflict interaction between all $\mathbb{K}$-based $\mathsf{AC}^l_{\mathcal{S}}$-arguments. Procedures, called *semantics*, are then defined for selecting sets of conflict-free arguments from $\mathcal{AF}^l_{\mathcal{S}}(\mathbb{K})$, which can be justified as defensible (Baroni, Caminada, and Giacomin 2011; Dung 1995).

Definition 3 (Semantics and nonmonotonic inference)**.** *Let* $\mathcal{AF}^l_{\mathcal{S}}(\mathbb{K}){=}\langle \mathsf{arg}, \mathsf{att} \rangle$*, and* $\mathcal{E} \subseteq \mathsf{arg}$*:*

- $\mathcal{E}$ *is* conflict-free *whenever for each* $a, b \in \mathcal{E}$, $(a, b) \notin \mathsf{att}$*;*
- $\mathcal{E}$ *is* stable *if* $\mathcal{E}$ *is conflict-free and for each* $a \in \mathsf{arg} \setminus \mathcal{E}$*, there is a* $b \in \mathcal{E}$ *with* $(b, a) \in \mathsf{att}$.

Credulous ($\mid\!\sim^{l,c}_{\mathcal{S}}$) *and skeptical inference* ($\mid\!\sim^{l,s}_{\mathcal{S}}$) *are defined:*

- $\mathbb{K} \mid\!\sim^{l,c}_{\mathcal{S}} \phi^o$ *iff* $a \in \bigcup \mathtt{Stable}$ *with* $a = \Delta \Rightarrow \phi^o$*;*
- $\mathbb{K} \mid\!\sim^{l,s}_{\mathcal{S}} \phi^o$ *iff there is an* $a \in \bigcap \mathtt{Stable}$ *with* $a = \Delta \Rightarrow \phi^o$.

Example 2. *Fig. 2 shows the* $\mathcal{AF}^l_{\mathcal{S}}(\mathbb{K}_1)$*s for Ex. 1, where (i) and (ii) contain the frameworks for* $l = \mathsf{io}$ *with* $\mathcal{S} = \{\textbf{SDet}\}$*, respectively,* $l = \mathsf{dl}$ *(the frameworks are partial for the sake of illustration). These frameworks yield different stable extensions (some of which are shaded grey): For Fig 2-(i), we have* $\mathtt{Stable} = \{\{g, a\}, \{e, b, c\}, \{f, a, b\}\}$*, and for Fig 2-(ii) we get* $\mathtt{Stable} = \{\{a, b, f, h\}, \{b, c, e, h\}\}$*. We have* $\mathbb{K}_1 \mid\!\sim^{\mathsf{io},c}_{\mathcal{S}} q^o, \neg q^o, p^o$ *but* $\mathbb{K}_1 \not\mid\!\sim^{\mathsf{io},c}_{\mathcal{S}} (q \wedge \neg q \wedge p)^o$ *in Fig 2-(i). We have* $\mathbb{K}_1 \mid\!\sim^{\mathsf{dl},c}_{\mathcal{S}} q^o, \neg q^o, p^o$*, and* $\mathbb{K}_1 \mid\!\sim^{\mathsf{dl},s}_{\mathcal{S}} p^o$ *but* $\mathbb{K}_2 \not\mid\!\sim^{\mathsf{dl},s}_{\mathcal{S}} q^o, \neg q^o$ *in Fig 2-(ii).*

Example 3. *Let* $\mathbb{K}_2$ *be* $\mathbb{K}_1$ *with the constraint set* $\mathcal{C} = \{\neg(r \wedge q)\}$*. Clearly, this constraint will only play a role for systems* $\mathsf{AC}^l_{\mathcal{S}}$ *with* $l = \mathsf{io}$ *and* ***TP*** $\in \mathcal{S}$*, where* r *is considered among the output. Using **BDet**, **Ax**, **IO1**, **IO2**, and **Cut**, we derive* $i = \neg(r \wedge q)^c, r^i \Rightarrow \neg(\top, q)$*. Argument* i *expresses that the rule* $(\top, q)$ *should not be applied in light of this new constraint. The resulting argumentation framework is presented in Fig. 2-(iii) and it has due to the unassailable argument* i*, only one extension:* $\{b, c, e, h, i\}$*. Hence, credulous and skeptical inference colapse and we have* $\mathbb{K}_2 \mid\!\sim^{\mathsf{io},s}_{\mathcal{S}} p^o, r^o, \neg q^o$.

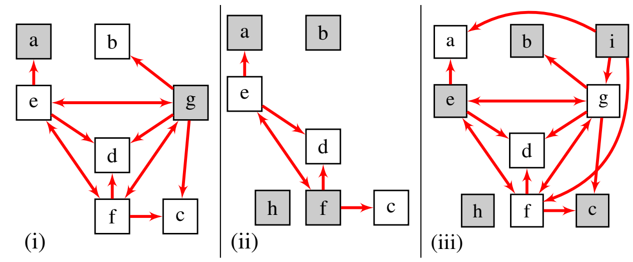

Figure 2: Partial argumentation framework for Ex. 1 and 3. Figure (i) corresponds to $\mathcal{AF}^{io}_{\mathcal{S}}(\mathbb{K}_1)$ with $\mathcal{S} = \{\textbf{SDet}\}$, (ii) to $\mathcal{AF}^{dl}_{\mathcal{S}}(\mathbb{K}_1)$, and (iii) to $\mathcal{AF}^{io}_{\mathcal{S}}(\mathbb{K}_2)$ with $\mathcal{S} = \{\textbf{TP}, \textbf{SDet}\}$. Arrows denote attacks and some stable extensions are shaded grey.

3.1 Some Properties of Argument Calculi

In the next section, we provide proof-search methods for argument calculi by means of decomposition trees but, first, we show some useful properties of AC. For instance, recall that the $\mathsf{AC}^l_{\mathcal{S}}$-instantiated argumentation frameworks introduced in this section characterize the entire class of constraint input/output logics and normal default logic (Prop. 1).

Proposition 1 (Soundness and Completeness (van Berkel and Straßer 2022; van Berkel, Straßer, and Zhou 2024))**.** *Let* $\mathcal{AF}^l_{\mathcal{S}}(\mathbb{K})$ *be* $\mathsf{AC}^l_{\mathcal{S}}$*-induced argumentation framework based on* $\mathbb{K}$*. Then,* $\mathcal{E} \subseteq \mathsf{arg}$ *is a stable extension iff* $\{(\phi, \psi) \mid (\phi, \psi) \in \Delta$ *for* $\Delta \Rightarrow \Gamma \in \mathcal{E}\}$ *is a maximal consistent set (MCS) in the corresponding input/output or normal default logic.*

We recall for $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$ and prove for $\mathsf{AC}^{\mathsf{dl}}_{\mathcal{S}}$ that we can reduce attacking arguments $\Delta \Rightarrow \neg(\phi, \psi)$ in $\mathsf{AC}^l_{\mathcal{S}}$ to obligation concluding arguments $\Delta' \Rightarrow \phi^o$. The upshot of this result is that we only need proof-search for the latter type of argument, making our approach more unified.

Proposition 2 (Lem. 6.3 in (van Berkel 2023))**.** *If* $\vdash^{\mathsf{io}}_{\mathcal{S}} \Delta \Rightarrow \neg(\phi, \psi)$*, then* $\vdash_{\mathcal{S}} \Delta, (\phi, \psi) \Rightarrow$.

Proposition 3 (Lem. 6.7 in (van Berkel 2023))**.** *If* $\vdash^{\mathsf{io}}_{\mathcal{S}} \Delta \Rightarrow$*, then* $\vdash^{\mathsf{io}}_{\mathcal{S}} \Delta \setminus \mathcal{L}^c \Rightarrow \phi^o$ *such that* $\Delta \cap \mathcal{L}^c \vdash_{\mathsf{L}^c} \neg\phi^c$.

Corollary 1. $\vdash^{\mathsf{io}}_{\mathcal{S}} \Delta \Rightarrow \neg(\phi, \psi)$ *iff* $\vdash^{\mathsf{io}}_{\mathcal{S}} \Gamma \setminus \mathcal{L}^c, (\phi, \psi) \Rightarrow \theta^o$ *and* $\Delta \cap \mathcal{L}^c \vdash_{\mathsf{L}^c} \neg\theta^c$.

Proposition 4. $\vdash^{\mathsf{dl}}_{\mathcal{S}} \Delta \Rightarrow \neg(\phi, \psi)$ *iff* $\vdash^{\mathsf{dl}}_{\mathcal{S}} \Delta \Rightarrow (\phi \wedge \neg\psi)^o$.

We conclude this section with some useful properties of $\mathsf{AC}^l_{\mathcal{S}}$, which will be used in subsequent sections.

Lemma 1. *Let* $l \in \{\mathsf{io}, \mathsf{dl}\}$*. (i) If* $\vdash^l_{\mathcal{S}} \Delta \Rightarrow \theta$ *and* $\theta \in \mathcal{L}^x (x \in \{c, i\})$*, then* $\Delta \subseteq \mathcal{L}^x$ *and* $\Delta \vdash_{\mathsf{L}} \theta$*. If* $l = \mathsf{dl}$*,* $\vdash^l_{\mathcal{S}} \Delta \Rightarrow \Gamma$ *and* $(\Delta \cup \Gamma) \cap \mathcal{L}^c \neq \varnothing$*, then* $\Delta, \Gamma \subseteq \mathcal{L}^c$ *and* $\Delta \vdash_{\mathsf{L}} \Gamma$*. (ii) If* $\vdash^l_{\mathcal{S}} \Delta \Rightarrow \theta$*,* $\Delta \subseteq \mathcal{L}^i \cup \{(\top, \top)\}$*,* $\theta \in \mathcal{L}^o$ *and* ***TP*** $\notin \mathcal{S}$*,* $\vdash_{\mathsf{L}} \theta$.

$\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$	i/o logic	$\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$	i/o logic
$\mathcal{S} = \{$**Taut**$\}$	out_1	$\mathcal{S} = \{$**Taut, Cases**$\}$	out_2
$\mathcal{S} = \{$**Taut,SDet**$\}$	out_3	$\mathcal{S} = \{$**Taut,SDet,Cases**$\}$	out_4

Table 1: Correspondence of argument calculi $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$ and input/output logics, with out^+_i for $i \in \{1,2,3,4\}$ when adding **TP**.

Lemma 2. *Let $\Gamma \subseteq \mathcal{L}^{ac}$, $x \in \{i,o\}$ and $y \in \{o,c\}$ and let $h(\pi)$ be the height of a derivation π defined as the number of nodes in the longest branch of π. If there exists an $\mathsf{AC}^{l}_{\mathcal{S}}$-proof π of $\phi^y, \Gamma \Rightarrow \theta^x$ with $h(\pi) \leq n$ and $\vdash_{\mathsf{L}} \phi$, there exists an $\mathsf{AC}^{l}_{\mathcal{S}}$-proof ρ of $\Gamma \Rightarrow \theta^x$ with $h(\rho) \leq n$.*

Lemma 3. *Let $\Gamma^i \neq \varnothing$ and $\Gamma^o \neq \varnothing$. If $\vdash^{l}_{\mathcal{S}} \Gamma^i, \Gamma^o \Rightarrow \phi^o$, $\vdash_{\mathsf{L}} \phi$.*

Lemma 4. *Let $\Gamma \subseteq \mathcal{L}^i \cup \mathcal{L}^c$ and $\Delta^r \neq \varnothing$. If there is a $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$-derivation π of $\psi^o, \Gamma, \Delta^r \Rightarrow \phi^o$ with $h(\pi) \leq n$, there is an $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$-derivation ρ of $\psi^i, \Gamma, \Delta^r \Rightarrow \phi^o$ with $h(\rho) \leq n$.*

4 Decomposition Trees

The central idea underlying all these decomposition methods is that we take an arbitrary argument $s = \Delta \Rightarrow \Gamma$ and begin decomposing its norms or defaults $(\phi,\psi) \in \Delta \cap \mathcal{L}^r$, carefully distinguishing the roles of the antecedent ϕ and the consequent ψ. Henceforth, for readability, we will just refer to (ϕ,ψ) as norms. By splitting such norms, the decomposition produces a tree-like structure. This process continues until all norms have been fully decomposed, resulting in a number of leaves containing only propositional formulas with mixed input 'i' and output 'o' labels. We then check each leaf for propositional derivability. Different input/output-style systems come with their own decomposition rules.

In this section, we prove the correspondence between $\mathsf{AC}^{l}_{\mathcal{S}}$ derivability and decomposition trees. We know the sound and complete correspondence between $\mathsf{AC}^{l}_{\mathcal{S}}$ and input/output systems (Prop. 1). Table 1 recalls the relationship between the specific argument calculi and these input/output systems. In what follows, we name these decomposition procedures after the state of the art input/output logics for which $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$ are sound and complete. We begin the presentation of our proof-search method by examining decomposition for input/output logic out_3 (cf. Table 1). We then show how the method extends to out^+_3 and all other input/output logics. We provide intuitions and examples as we go. We note that the only difference between out^+_3 and default logic is how attackers are generated. Since we know from Proposition 4 that attacking arguments can be reduced to arguments concluding obligation, we also obtain decomposition procedures for default logic.

4.1 out_3-Decomposition Trees

Our first method simply decomposes each norm. Intuitively, we check whether the antecedent of the norm *follows* from the information present in the argument (left branch in Def. 4) and whether its consequent *contributes* to deriving the conclusion of the argument (right branch in Def. 4).

Definition 4 (Decomposition trees for out_3)**.** *Let $x \in \{i,o\}$. We generate a terminated out_3-decomposition tree for $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \theta^x$ by applying the following procedure, starting from $\Gamma^r_0 = \Gamma^r$:*

1. *if $\Gamma^r_k = \Theta^r, (\phi,\psi)$ and $\Theta^r \neq \varnothing$, apply the rule*
$$\frac{\Delta^i, \Pi^o, \Theta^r, (\phi,\psi) \Rightarrow \xi^x}{\Delta^i, \Pi^o, \Theta^r \Rightarrow \phi^i \qquad \psi^x, \Delta^i, \Pi^o, \Theta^r \Rightarrow \xi^x}$$
and take $\Gamma^r_{k+1} = \Theta^r$;
2. *if $\Gamma^r_k = \{(\phi,\psi)\}$, apply one of the following rules:*
$$\frac{\Delta^i, \Pi^o, (\phi,\psi) \Rightarrow \xi^i}{\Delta^i \Rightarrow \phi^i \qquad \psi^i, \Delta^i, \Pi^i \Rightarrow \xi^i} \qquad \frac{\Delta^i, \Pi^o, (\phi,\psi) \Rightarrow \xi^o}{\Delta^i \Rightarrow \phi^i \qquad \psi^o, \Pi^o \Rightarrow \xi^o}$$
and take $\Gamma^r_{k+1} = \varnothing$.

We say that a terminated out_3-decomposition for $\Gamma^i, \Gamma^r \Rightarrow \theta^x$ is successful *whenever any leaf $\Pi^x \Rightarrow \xi^x$ is such that $\Pi \vdash_{\mathsf{L}} \xi$, and* unsuccessful *otherwise.*

Example 4. *The following is a terminated out_3-decomposition tree for $p^i, (p,q), (p \wedge q, r) \Rightarrow (r \wedge q)^o$:*

$$\dfrac{p^i, (p,q), (p\wedge q, r) \Rightarrow (r\wedge q)^o}{\dfrac{p^i, (p,q) \Rightarrow (p\wedge q)^i}{p^i \Rightarrow p^i \qquad p^i, q^i \Rightarrow (p \wedge q)^i}\,2 \qquad \dfrac{r^o, p^i, (p,q) \Rightarrow (r\wedge q)^o}{p^i \Rightarrow p^i \qquad r^o, q^o \Rightarrow (r\wedge q)^o}\,2}\,1$$

Example 5. *The following are terminated out_3-decomposition trees for $p^i, (p\wedge q, r), (r,q) \Rightarrow q^o$:*

$$\dfrac{p^i, (p\wedge q, r), (r,q) \Rightarrow q^o}{\dfrac{p^i, (p\wedge q, r) \Rightarrow r^i}{p^i \Rightarrow p\wedge q^i \qquad p^i, r^i \Rightarrow r^i}\,2 \qquad \dfrac{q^o, p^i, (p\wedge q, r) \Rightarrow q^o}{p^i \Rightarrow p \wedge q^i \qquad q^o \Rightarrow q^o}\,2}\,1$$

$$\dfrac{p^i, (p\wedge q, r), (r,q) \Rightarrow q^o}{\dfrac{p^i, (r,q) \Rightarrow p\wedge q^i}{p^i \Rightarrow r^i \qquad p^i, q^i \Rightarrow p\wedge q^i}\,2 \qquad \dfrac{r^o, p^i, (r,q) \Rightarrow q^o}{p^i \Rightarrow r^i \qquad r^o, q^o \Rightarrow q^o}\,2}\,1$$

Example 4 displays a successful out_3-decomposition tree for $p^i, (p,q), (p \wedge q, r) \Rightarrow (r \wedge q)^o$, whereas Example 5 shows (the only) two unsuccessful out_3-decomposition trees for $p^i, (p\wedge q, r), (r,q) \Rightarrow q^o$.

We state a number of preliminary lemmas, which underpin our main result (Thm. 1): there exists a successful out_3-decomposition trees for a sequent $\Gamma^i, \Gamma^r \Rightarrow \xi^o$ exactly when this sequent is derivable in $\mathsf{AC}^{io}_{\mathcal{S}}$, with $\mathcal{S} = \{$**SDet**$\}$. The lemmas are needed in order to show how out_3-decomposition trees can be composed so as to internalize **Cut** applications.

Lemma 5. *Let $\Delta^r = (\phi_1,\psi_1), \ldots, (\phi_m,\psi_m)$, with $m \geq 1$.*

(i) *If there exists a successful out_3-decomposition tree for $\Gamma^i, \Delta^r \Rightarrow \phi^x$, there exists a successful out_3-decomposition tree for $\Pi^i, \Gamma^i, \Delta^r \Rightarrow \phi^x$ for any Π^i.*

(ii) *If $\Gamma \vdash_{\mathsf{L}} \phi$ and there exists a successful out_3-decomposition tree for $\phi^i, \Delta^i, \Delta^r \Rightarrow \theta^x$, there exists a successful out_3-decomposition tree for $\Gamma^i, \Delta^i, \Delta^r \Rightarrow \theta^x$.*

(iii) *If there exists a successful out_3-decomposition tree for $\Gamma^i, \Delta^r \Rightarrow \phi^x$ and $\psi_1, \ldots, \psi_m \vdash_{\mathsf{L}} \psi$, there exists a successful out_3-decomposition tree for $\Gamma^i, \Delta^r \Rightarrow \psi^x$.*

Lemma 6. *Let $\Gamma^r \neq \varnothing$.*

(i) *If there exists a successful* out_3*-decomposition tree for* $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \phi^o$*, then there exists a successful* out_3*-decomposition tree for* $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \phi^i$.

(ii) *If there exists a successful* out_3*-decomposition tree for* $\Gamma^i, \Gamma^r \Rightarrow \phi^o$*, then there exists a successful* out_3*-decomposition tree for* $\psi^o, \Gamma^i, \Gamma^r \Rightarrow \psi^o$.

(iii) *If there exists a successful* out_3*-decomposition tree for* $\Delta^i, \Gamma^r \Rightarrow \phi^i$ *and* $\phi, \Delta \vdash_{\mathsf{L}} \psi$*, then there exists a successful* out_3*-decomposition tree for* $\Delta^i, \Gamma^r \Rightarrow \psi^i$.

Lemma 7. *Let* $\Delta_1^r \neq \varnothing$ *and* $\Delta_2^r \neq \varnothing$.

(i) *If there are successful* out_3*-decomposition trees for* $\Gamma^i, \Delta_1^r \Rightarrow \phi^i$ *and* $\phi^i, \Gamma^i, \Delta_2^r \Rightarrow \psi^i$*, then there is a successful* out_3*-decomposition tree for* $\Gamma^i, \Delta_1^r, \Delta_2^r \Rightarrow \psi^i$.

(ii) *If there are successful* out_3*-decomposition trees for* $\Gamma^i, \Delta^r \Rightarrow \phi^i$ *and* $\phi^i, \Gamma^i, \Delta^o, \Delta_2^r \Rightarrow \psi^i$*, there is a successful* out_3*-decomposition tree for* $\Gamma^i, \Delta^o, \Delta_1^r, \Delta_2^r \Rightarrow \psi^i$.

(iii) *If there are successful* out_3*-decomposition trees for* $\Gamma^i, \Delta^r \Rightarrow \phi^i$ *and* $\phi^i, \Gamma^i, \Delta^o, \Delta_2^r \Rightarrow \psi^o$*, there is a successful* out_3*-decomposition tree for* $\Gamma^i, \Delta^o, \Delta_1^r, \Delta_2^r \Rightarrow \psi^o$.

We now prove the main theorem.

Theorem 1. *Let* $\Gamma^r \neq \varnothing$ *and* $\mathcal{S} = \{\textbf{\textit{SDet}}\}$*. Then* $\vdash_{\mathcal{S}}^{\mathsf{io}} \Gamma^i, \Gamma^r \Rightarrow \phi^o$ *iff there exists (at least) one successful* out_3*-decomposition tree for* $\Gamma^i, \Gamma^r \Rightarrow \phi^o$.

Proof. ($\Rightarrow$) If $\vdash_{\mathcal{S}} \Gamma^i, \Gamma^r \Rightarrow \phi^o$, there exists (at least) one $\mathsf{AC}_{\mathcal{S}}^{\mathsf{io}}$-derivation π of $\Gamma^i, \Gamma^r \Rightarrow \phi^o$: we reason by induction on the height of π. If $h(\pi) = 1$, the last rule applied is **BDet** and the conclusion is immediate. If $h(\pi) > 1$, the last rule applied is **Cut**: we reason by cases over the Cut formula. If the latter is θ^x, we distinguish the three cases (a) $x = i$, (b) $x = o$ and (c) $x = c$ as follows.

a The $\mathsf{AC}_{\mathcal{S}}^{\mathsf{io}}$-derivation π has the following form:

$$\dfrac{\begin{array}{c}\vdots\\ \Delta^i, \Gamma_1^r \Rightarrow \theta^i\end{array} \qquad \begin{array}{c}\vdots\\ \theta^i, \Pi^i, \Gamma_2^r \Rightarrow \phi^o\end{array}}{\Gamma^i, \Gamma^r \Rightarrow \phi^o}\ \textbf{Cut}$$

By Lem. 11-(i), $\Gamma_1^r = \varnothing$ and $\Delta \vdash_{\mathsf{L}} \theta$. By inductive hypothesis, there exists a successful out_3-decomposition tree for $\theta^i, \Pi^i, \Gamma_2^r \Rightarrow \phi^o$. Lem. 5-$(ii)$ ensures the existence of a successful out_3-decomposition tree for $\Delta^i, \Pi^i, \Gamma_2^r \Rightarrow \phi^o$, and thus for $\Gamma^i, \Gamma^r \Rightarrow \phi^o$.

b The $\mathsf{AC}_{\mathcal{S}}^{\mathsf{io}}$-derivation π has the following form:

$$\dfrac{\begin{array}{c}\vdots\\ \Delta^i, \Gamma_1^r \Rightarrow \theta^o\end{array} \qquad \begin{array}{c}\vdots\\ \theta^o, \Pi^i, \Gamma_2^r \Rightarrow \phi^o\end{array}}{\Gamma^i, \Gamma^r \Rightarrow \phi^o}$$

If $\Gamma_1^r = \varnothing$, we leverage Lem. 11-(ii) to infer that $\vdash_{\mathsf{L}} \theta$. Lem. 2 ensures the existence of an $\mathsf{AC}_{\mathcal{S}}$-derivation of $\Pi^i, \Gamma_2^r \Rightarrow \phi^o$ whose height does not surpass the height of the $\mathsf{AC}_{\mathcal{S}}$-derivation of $\theta^o, \Pi^i, \Gamma_2^r \Rightarrow \phi^o$. We apply the inductive hypothesis to infer the existence of a successful out_3-decomposition tree for $\Pi^i, \Gamma_2^r \Rightarrow \phi^o$. By Lem. 5-$(i)$, this yields the existence of a successful out_3-decomposition tree for $\Delta^i, \Pi^i, \Gamma_2^r \Rightarrow \phi^o$, and thus for $\Gamma^i, \Gamma^r \Rightarrow \phi^o$.

If $\Gamma_1^r \neq \varnothing$, we apply the inductive hypothesis to infer the existence of a successful out_3-decomposition tree for $\Delta^i, \Gamma_1^r \Rightarrow \theta^o$. Lem. 5-$(i)$ ensures there is a successful out_3-decomposition tree $\mathcal{T}$ for $\Gamma^i, \Gamma_1^r \Rightarrow \theta^o$. We reason by cases over Γ_2^r and Π^i to reach the conclusion.

If $\Gamma_2^r = \varnothing$ and $\Pi^i = \varnothing$, then $\vdash_{\mathcal{S}} \theta^o \Rightarrow \phi^o$: Lem. 1-$(i)$ ensures that $\theta \vdash_{\mathsf{L}} \phi$. Lem. 5-$(iii)$ entails the existence of a successful out_3-decomposition tree for $\Gamma^i, \Gamma^r \Rightarrow \phi^o$. On the other hand, if $\Gamma_2^r = \varnothing$ and $\Pi^i \neq \varnothing$, Lem. 3 guarantees that $\vdash_{\mathsf{L}} \phi$: hence, Lem. 5-(iii) ensures the conclusion as in the previous subcase.

If $\Gamma_2^r \neq \varnothing$ and π is the $\mathsf{AC}_{\mathcal{S}}^{\mathsf{io}}$-proof of $\theta^o, \Pi^i, \Gamma_2^r \Rightarrow \phi^o$, Lem. 4 guarantees the existence of an $\mathsf{AC}_{\mathcal{S}}^{\mathsf{io}}$-proof ρ of $\theta^i, \Pi^i, \Gamma_2^r \Rightarrow \phi^o$ with $h(\rho) \leq h(\pi)$: the inductive hypothesis ensures the existence of a successful out_3-decomposition tree for $\theta^i, \Pi^i, \Gamma_2^r \Rightarrow \phi^o$. We distinguish two subcases:

(1) If $\Gamma_2^r = \{(\psi, \xi)\}$, the existence of a successful out_3-decomposition tree for $\theta^i, \Pi^i, \Gamma_2^r \Rightarrow \phi^o$ entails that $\theta^i, \Pi^i \vdash_{\mathsf{L}} \psi^i$, and thus that $\theta^i, \Gamma^i \vdash_{\mathsf{L}} \psi^i$. By Lem. 6-(i), the existence of a successful out_3-decomposition tree for $\Gamma^i, \Gamma_1^r \Rightarrow \theta^o$ ensures the existence of a successful out_3-decomposition tree for $\Gamma^i, \Gamma_1^r \Rightarrow \theta^i$. By Lem. 6-(iii), this entails the existence of a successful out_3-decomposition tree $\mathcal{U}_1$ for $\Gamma^i, \Gamma_1^r \Rightarrow \psi^i$. On the other hand, Lem. 6-(ii) guarantees the existence of a successful out_3-decomposition tree $\mathcal{U}_2$ for $\xi^o, \Gamma^i, \Gamma_1^r \Rightarrow \xi^o$. If we plug $\mathcal{U}_1$ and $\mathcal{U}_2$ into a single out_3-decomposition tree, we obtain a successful decomposition tree for $\Gamma^i, \Gamma^r \Rightarrow \xi^o$: since $\xi^o \vdash_{\mathsf{L}} \phi^o$, Lem. 5–$(iii)$ ensures the existence of a successful decomposition tree for $\Gamma^i, \Gamma^r \Rightarrow \phi^o$ – as desired.

(2) Suppose that $\Gamma_2^r = \{(\psi_1, \theta_1), \ldots, (\psi_m, \theta_m)\}$ with $m > 1$, and (without loss of generality) that the enumeration of the rules in Γ_2^r mirrors the order of decomposition in a successful out_3-decomposition tree for the sequent $\theta^i, \Pi^i, (\psi_1, \theta_1), \ldots, (\psi_m, \theta_m) \Rightarrow \phi^o$. Hence, there exists a successful out_3-decomposition tree for $\theta^i, \Pi^i, (\psi_2, \theta_2), \ldots, (\psi_m, \theta_m) \Rightarrow \psi_1^i$. By Lem. 5-$(i)$, this entails the existence a successful out_3-decomposition tree for the sequent $\theta^i, \Gamma^i, (\psi_2, \theta_2), \ldots, (\psi_m, \theta_m) \Rightarrow \psi_1^i$. On the other hand, Lem. 6(i) ensures the existence of a successful out_3-decomposition tree for $\Gamma^i, \Gamma_1^r \Rightarrow \theta^i$. Lem. 7(i) guarantees the existence of a successful out_3-decomposition tree $\mathcal{U}_1$ for $\Gamma^i, \Gamma_1^r, (\psi_2, \theta_2), \ldots, (\psi_m, \theta_m) \Rightarrow \psi_1^i$. By Lem. 5-$(i)$, the existence of a successful out_3-decomposition tree for $\theta_1^o, \theta^i, \Pi^i, (\psi_2, \theta_2), \ldots, (\psi_m, \theta_m) \Rightarrow \phi^o$ entails the existence of a successful out_3-decomposition tree for the sequent $\theta_1^o, \theta^i, \Gamma^i, (\psi_2, \theta_2), \ldots, (\psi_m, \theta_m) \Rightarrow \phi^o$: hence, Lem. 7-(iii) ensures the existence of a successful out_3-decomposition tree $\mathcal{U}_2$ for the sequent $\theta_1^o, \Gamma^i, \Gamma_1^r(\psi_2, \theta_2), \ldots, (\psi_m, \theta_m) \Rightarrow \phi^o$. To complete the proof, we plug $\mathcal{U}_1$ and $\mathcal{U}_2$ into a single, successful out_3-decomposition tree for $\Gamma^i, \Gamma^r \Rightarrow \phi^o$.

c We argue as in the case where $x = i$.

If the Cut formula is $(\top, \top)$, we leverage the inductive hypothesis to infer the existence of a successful out_3-

decomposition tree of $\Gamma^i, \Gamma^r, (\top,\top) \Rightarrow \phi^o$ where the first decomposed rule is $(\top,\top)$ (easy to verify). The immediate subtree with root $\top^o, \Gamma^i, \Gamma^r \Rightarrow \phi^o$ is a successful out_3-decomposition tree: if we remove $\top^o$ from each of its nodes, we obtain a successful out_3-decomposition tree for $\Gamma^i, \Gamma^r \Rightarrow \phi^o$ – as desired.

($\Leftarrow$) We reason by induction over the height of the successful out_3-decomposition tree for $\Gamma^i, \Gamma^r \Rightarrow \phi^o$. The base case is immediate. Suppose that $\Gamma^r = \{(\psi_1,\theta_1),\dots,(\psi_m,\theta_m)\}$ with $m > 1$, and (without loss of generality) that the enumeration of the rules in Γ^r mirrors the order of decomposition in a successful out_3-decomposition tree for $\Gamma^i, (\psi_1,\theta_1),\dots,(\psi_m,\theta_m) \Rightarrow \phi^o$. Such decomposition tree has a leaf of the form $\Gamma^i \Rightarrow \psi^i_m$ and one like $\Gamma^i, \theta^i_m \Rightarrow \psi^i_{m-1}$: hence, we can take the following derivation π_{m-1}, and label π_m its immediate subderivation on the left.

$$\dfrac{\delta \qquad \dfrac{\dfrac{\dfrac{}{\Gamma^i, \theta^i_m \Rightarrow \psi^i_{m-1}}\,\mathbf{Ax} \qquad \dfrac{}{\psi^i_{m-1}, (\psi_{m-1},\theta_{m-1}) \Rightarrow \theta^o_{m-1}}\,\mathbf{BDet}}{\Gamma^i, \theta^i_m, (\psi_{m-1},\theta_{m-1}) \Rightarrow \theta^o_{m-1}}}{\Gamma^i, \theta^o_m, (\psi_{m-1},\theta_{m-1}) \Rightarrow \theta^o_{m-1}}\,\mathbf{SDet}}{\Gamma^i, (\psi_m,\theta_m), (\psi_{m-1},\theta_{m-1}) \Rightarrow \theta^o_{m-1}}\,\mathbf{Cut}$$

with δ:

$$\dfrac{\mathbf{Ax}\,\dfrac{}{\Gamma^i \Rightarrow \psi^i_m} \qquad \mathbf{BDet}\,\dfrac{}{\psi^i_m, (\psi_m,\theta_m) \Rightarrow \theta^o_m}}{\Gamma^i, (\psi_m,\theta_m) \Rightarrow \theta^o_m}$$

Moreover, the decomposition tree has a leaf of the form $\Gamma^i, \theta^i_{m-1}, \theta^i_m \Rightarrow \psi^i_{m-2}$. It is easy to infer there exists a $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$-proof ρ_{m-2} of $\Gamma^i, \theta^o_{m-1}, \theta^o_m, (\psi_{m-2},\theta_{m-2}) \Rightarrow \theta^o_{m-2}$: a **Cut** application between π_m and ρ_{m-2}, followed by a **Cut** application with π_{m-1}, yields an $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$-proof π_{m-2} of the sequent

$$\Gamma^i, (\psi_m,\theta_m), (\psi_{m-1},\theta_{m-1}), (\psi_{m-2},\theta_{m-2}) \Rightarrow \theta^o_{m-2}$$

Since the decomposition tree has leaves of the form $\Gamma^i, \theta^i_k, \dots, \theta^i_m \Rightarrow \psi^i_{k-1}$ for each $2 \le k \le m-1$, we can iterate the procedure presented to generate an $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$-proof π_h of $\Gamma^i, (\psi_m,\theta_m), \dots, (\psi_h,\theta_h) \Rightarrow \theta^o_h$ for each $m-3 \le h \le (m-(m-1))$ (*via* **Cut** applications). On the other hand, our decomposition tree has a leaf of the form $\theta^o_m, \dots, \theta^o_1 \Rightarrow \phi^o$: we apply **Cut** to plug the proofs $\pi_m, \dots, \pi_1$ into an $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$-proof of $\Gamma^i, \Gamma^r \Rightarrow \phi^o$ – as desired. ☐

4.2 out^+_3-Decomposition Trees

The difference between out_3 and out^+_3 lies in the fact that the latter includes all inputs among its outputs (cf. **TP**). Intuitively, this means that, in order to characterize this logic, i-labeled formulas can be treated as o-labeled formulas. The following definition formalizes this idea.

Definition 5 (Decomposition trees for out^+_3)**.** *Let $x \in \{i,o\}$. We generate a terminated out^+_3-decomposition tree for $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \theta^x$ by applying the following procedure, starting from $\Gamma^r_0 = \Gamma^r$:*

1. *if $\Gamma^r_k = \Theta^r, (\phi,\psi)$ and $\Theta^r \neq \varnothing$, apply the rule*

$$\dfrac{\Delta^i, \Pi^o, \Theta^r, (\phi,\psi) \Rightarrow \xi^x}{\Delta^i, \Pi^o, \Theta^r \Rightarrow \phi^i \qquad \psi^x, \Delta^i, \Pi^o, \Theta^r \Rightarrow \xi^x}$$

and take $\Gamma^r_{k+1} = \Theta^r$;

2. *if $\Gamma^r_k = \{(\phi,\psi)\}$, apply the following rule:*

$$\dfrac{\Delta^i, \Pi^o, (\phi,\psi) \Rightarrow \xi^x}{\Delta^i \Rightarrow \phi^i \qquad \psi^x, \Delta^x, \Pi^x \Rightarrow \xi^x}$$

and take $\Gamma^r_{k+1} = \varnothing$.

We say that a terminated out_3-decomposition for $\Gamma^i, \Gamma^r \Rightarrow \theta^x$ is successful *whenever any leaf $\Pi^x \Rightarrow \xi^x$ is such that $\Pi \vdash_{\mathsf{L}} \xi$, and* unsuccessful *otherwise.*

We gather useful lemmas as preliminary steps towards the adequacy result of this subsection, i.e., Thm. 2. The role of these lemmas is to ensure that out^+_3-decomposition trees can be composed so as to internalize **Cut** applications.

Lemma 8. *Let $\Delta^r = (\phi_1,\psi_1),\dots,(\phi_m,\psi_m)$, with $m \ge 1$.*

(i) *If there exists a successful out_3-decomposition tree for $\Gamma^i, \Delta^r \Rightarrow \phi^x$, there exists a successful out^+_3-decomposition tree for $\Delta^i, \Delta^i, \Gamma^r \Rightarrow \phi^x$ for any Δ^i.*

(ii) *If $\Gamma \vdash_{\mathsf{L}} \phi$ and there exists a successful out^+_3-decomposition tree for $\phi^i, \Delta^i, \Delta^r \Rightarrow \theta^x$, there exists a successful out_3-decomposition tree for $\Gamma^i, \Delta^i, \Delta^r \Rightarrow \theta^x$.*

(iii) *If there exists a successful out_3-decomposition tree for $\Gamma^i, \Delta^r \Rightarrow \phi^x$ and $\psi_1,\dots,\psi_m \vdash_{\mathsf{L}} \psi$, there exists a successful out^+_3-decomposition tree for $\Gamma^i, \Delta^r \Rightarrow \psi^x$.*

Lemma 9. *Let $\Delta^r \neq \varnothing$. The following statements hold:*

(i) *If there exists a successful out^+_3-decomposition tree for $\Gamma^i, \Delta^r \Rightarrow \phi^o$, then there exists a successful out^+_3-decomposition tree for $\Gamma^i, \Delta^r \Rightarrow \phi^i$.*

(ii) *If there exists a successful out^+_3-decomposition tree $\mathcal{T}$ for $\Gamma^i, \Delta^r \Rightarrow \phi^o$, then there exists a successful out^+_3-decomposition tree $\mathcal{T}'$ for $\psi^o, \Gamma^i, \Delta^r \Rightarrow \psi^o$.*

(iii) *If there exists a successful out^+_3-decomposition tree for $\Gamma^i, \Delta^r \Rightarrow \phi^i$ and $\phi, \Gamma \vdash_{\mathsf{L}} \psi$, then there exists a successful out^+_3-decomposition tree for $\Gamma^i, \Delta^r \Rightarrow \psi^i$.*

Lemma 10. *Let $\Delta^r_1 \neq \varnothing$ and $\Delta^r_2 \neq \varnothing$.*

(i) *If there are successful out^+_3-decomposition trees for $\Gamma^i, \Delta^r_1 \Rightarrow \phi^i$ and $\phi^i, \Gamma^i, \Delta^r_2 \Rightarrow \psi^i$, there is a successful out^+_3-decomposition tree for $\Gamma^i, \Delta^r_1, \Delta^r_2 \Rightarrow \psi^i$.*

(ii) *If there are successful out^+_3-decomposition trees for $\Gamma^i, \Delta^r \Rightarrow \phi^i$ and $\phi^i, \Gamma^i, \Delta^o, \Delta^r_2 \Rightarrow \psi^o$, there is a successful out^+_3-decomposition tree for $\Gamma^i, \Delta^o, \Delta^r_1, \Delta^r_2 \Rightarrow \psi^o$.*

(iii) *If there are successful out^+_3-decomposition trees for $\Gamma^i, \Delta^r \Rightarrow \phi^i$ and $\phi^i, \Gamma^i, \Delta^o, \Delta^r_2 \Rightarrow \psi^o$, there is a successful out^+_3-decomposition tree for $\Gamma^i, \Delta^o, \Delta^r_1, \Delta^r_2 \Rightarrow \psi^o$.*

Theorem 2. *Let $\Gamma^r \neq \varnothing$ and $\mathcal{S} = \{\textbf{SDet}, \textbf{TP}\}$. Then $\vdash^{\mathsf{io}}_{\mathcal{S}} \Gamma^i, \Gamma^r \Rightarrow \phi^o$ iff there exists (at least) one successful out^+_3-decomposition tree for $\Gamma^i, \Gamma^r \Rightarrow \phi^o$.*

Proof. As for Thm. 1, leveraging Lem. 8, 9 and 10. ☐

Corollary 2. *Let $\Gamma^r \neq \varnothing$. Then $\vdash^{\mathsf{dl}}_{\mathcal{S}} \Gamma^i, \Gamma^r \Rightarrow \phi^o$ iff there exists (at least) one successful out^+_3-decomposition tree for $\Gamma^i, \Gamma^r \Rightarrow \phi^o$.*

Proof. Straightforward from Thm. 2, given that $\mathcal{S} = \{\mathbf{SDet}, \mathbf{TP}\}$ for normal default logic. ☐

4.3 out_4-Decomposition Trees

Logics $\mathsf{out}_2^{(+)}$ and $\mathsf{out}_4^{(+)}$ incorporate disjunctive reasoning by cases, with and without throughput (cf. **Cases**). Intuitively, this means that we check whether a formula follows from all "disjunctive reasoning paths," formalized as maximal consistent extensions of the input set. In particular, out_4 combines disjunctive reasoning with reusability, whereas out_2 incorporates reasoning by cases without reusability: we use $(+)$ to denote 'with or without throughtput.'

An out_4-decomposition tree of $\Gamma^i, \Delta^o, \Pi^r \Rightarrow \phi^x$ uses disjunctive normal form to represent those paths, that is, the disjunctive normal form whose disjuncts are maximal consistent sets of the literals obtained from the atoms in $\Gamma^i, \Delta^o, \Pi^r$. As a matter of fact, such disjuncts form finite bases for maximal consistent extensions.

Definition 6 (Bases for maximal consistent extensions)**.** *Let $x \in \{i, o\}$ and Γ be Γ^i stripped from its labels. For any sequent $\Gamma^i, \Delta^o, \Pi^r \Rightarrow \phi^x$, we take $\bigwedge D_1 \vee \dots \vee \bigwedge D_m$ to be the full disjunctive normal form of $\bigwedge \Gamma$ generated from the atoms in $\Gamma^i, \Delta^o, \Pi^r$.*

Definition 7 (Decomposition trees for out_4)**.** *Let $x \in \{i, o\}$ and k be any sequence of indices starting with 0. We generate a terminated out_4-decomposition tree for $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \theta^x$ by applying the following procedure, starting from $\Gamma^r_0 = \Gamma^r$ and $\Gamma^o_0 = \Gamma^o$:*

1. *First, we apply the rule*
$$\frac{\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \phi^x}{\{D^i_h, \Gamma^o, \Gamma^r \Rightarrow \phi^x\}_{1 \le h \le m}}$$
with $\bigwedge D_1 \vee \dots \vee \bigwedge D_m$ being as in Definition 6;
2. *if $\Gamma^r_{\mathsf{k}} = \Delta^r, (\phi_n, \psi_n)$ and the succedent is an o-labelled formula, then*
   - *(a) if there is no (ϕ, ψ) in Δ^r with $D_h \vdash_{\mathsf{L}} \phi$, apply the rule*
$$\frac{D^i_h, \Gamma^o_{\mathsf{k}}, \Delta^r, (\phi_n, \psi_n) \Rightarrow \theta^o}{D^i_h \Rightarrow \phi^i_n \qquad \Gamma^o_{\mathsf{k}}, \psi^o_n \Rightarrow \theta^o}$$
and take $\Gamma^r_{\mathsf{k}n} = \varnothing$, $\Gamma_{\mathsf{k}n} = \Gamma_{\mathsf{k}}, \psi_n$;
   - *(b) if there exists (at least) one (ϕ, ψ) in Δ^r with $D_k \vdash_{\mathsf{L}} \phi$, apply the rule*
$$\frac{D^i_h, \Gamma^o_{\mathsf{k}}, \Delta^r, (\phi_n, \psi_n) \Rightarrow \theta^o}{D^i_h, \Gamma^o_{\mathsf{k}}, \Delta^r \Rightarrow \phi^i_n \qquad D^i_h, \Gamma^o_{\mathsf{k}}, \psi^o_n, \Delta^r \Rightarrow \theta^o}$$
taking $\Gamma^r_{\mathsf{k}n} = \Delta^r$ and $\Gamma_{\mathsf{k}n} = \Gamma_{\mathsf{k}}, \psi_n$;
3. *if $\Gamma^r_{\mathsf{k}} = \Delta^r, (\phi_n, \psi_n)$ and the succedent is an i-labelled formula, then*
   - *(a) if there is no (ϕ, ψ) in Δ^r with $D_h, \Gamma_{\mathsf{k}} \vdash_{\mathsf{L}} \phi$, apply the rule*
$$\frac{D^i_h, \Gamma^o_{\mathsf{k}}, \Delta^r, (\phi_n, \psi_n) \Rightarrow \theta^i}{D^i_h, \Gamma^i_{\mathsf{k}} \Rightarrow \phi^i_n \qquad D^i_h, \Gamma^i_{\mathsf{k}}, \psi^i_n \Rightarrow \theta^i}$$
and take $\Gamma^r_{\mathsf{k}n} = \varnothing$, $\Gamma_{\mathsf{k}n} = \Gamma_{\mathsf{k}}, \psi_n$;
   - *(b) if there exists (at least) one (ϕ, ψ) in Δ^r such that $D_h, \Gamma_{\mathsf{k}} \vdash_{\mathsf{L}} \phi$, apply the rule*
$$\frac{D^i_h, \Gamma^o_{\mathsf{k}}, \Delta^r, (\phi_n, \psi_n) \Rightarrow \theta^i}{D^i_h, \Gamma^o_{\mathsf{k}}, \Delta^r \Rightarrow \phi^i_n \qquad D^i_h, \Gamma^o_{\mathsf{k}}, \psi^i_n, \Delta^r \Rightarrow \theta^i}$$
taking $\Gamma^r_{\mathsf{k}n} = \Delta^r$ and $\Gamma_{\mathsf{k}n} = \Gamma_{\mathsf{k}}, \psi_n$.

We say that a terminated out_4-decomposition tree for $\Gamma^i, \Gamma^r \Rightarrow \theta^x$ is successful *whenever any leaf $\Pi^x \Rightarrow \xi^x$ is such that $\Pi \vdash_{\mathsf{L}} \xi$ – and* unsuccessful *otherwise.*

Example 6. *A terminated out_4-decomposition tree for $p \vee q^i, (p, q), (q, p) \Rightarrow (p \wedge q)^o$ has immediate subtrees $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$, whose roots are labelled by $p^i, q^i, (p, q), (q, p) \Rightarrow (p \wedge q)^o$, $p^i, \neg q^i, (p, q), (q, p) \Rightarrow p \wedge q^o$ and $q^i, \neg p^i, (p, q), (q, p) \Rightarrow (p \wedge q)^o$, respectively. $\mathcal{T}_1$ has the following form:*

$$\dfrac{p^i, q^i, (p,q), (q,p) \Rightarrow (p \wedge q)^o}{\dfrac{p^i, q^i, (q,p) \Rightarrow p^i}{p^i, q^i \Rightarrow q^i \qquad p^i, q^i \Rightarrow p^i}\,3a \qquad \dfrac{q^o, p^i, q^i, (q,p) \Rightarrow (p \wedge q)^o}{p^i, q^i \Rightarrow q^i \qquad q^o, p^o \Rightarrow (p \wedge q)^o}\,2a}\,2b$$

$\mathcal{T}_2$ has the following form:

$$\dfrac{p^i, \neg q^i, (p,q), (q,p) \Rightarrow (p \wedge q)^o}{\dfrac{p^i, \neg q^i, (p,q) \Rightarrow q^i}{p^i, \neg q^i \Rightarrow p^i \qquad p^i, \neg q^i, q^i \Rightarrow q^i}\,3a \qquad \dfrac{p^o, p^i, \neg q^i, (p,q) \Rightarrow (p \wedge q)^o}{p^i, \neg q^i \Rightarrow p^i \qquad p^o, q^o \Rightarrow (p \wedge q)^o}\,2a}\,2b$$

Lastly, $\mathcal{T}_3$ has the following form:

$$\dfrac{q^i, \neg p^i, (p,q), (q,p) \Rightarrow (p \wedge q)^o}{\dfrac{q^i, \neg p^i, (q,p) \Rightarrow p^i}{q^i, \neg p^i \Rightarrow q^i \qquad q^i, \neg p^i, p^i \Rightarrow p^i}\,3a \qquad \dfrac{q^o, q^i, \neg p^i, (q,p) \Rightarrow (p \wedge q)^o}{q^i, \neg p^i \Rightarrow q^i \qquad q^o, p^o \Rightarrow (p \wedge q)^o}\,2a}\,2b$$

Example 7. *In a terminated out_4-decomposition tree for $p \vee q^i, (p, r), (q, r) \Rightarrow r^o$, the sequents $p^i, q^i, \phi^i, (p, r), (q, r) \Rightarrow r^o$, $p^i, \neg q^i, \phi^i, (p, r), (q, r) \Rightarrow r^o$ and $q^i, \neg p^i, \phi^i, (p, r), (q, r) \Rightarrow r^o$ label the immediate children of the root – with ϕ^i being r^i or $\neg r^i$. The trees $\mathcal{T}_{1,2}$ having the sequents $p^i, q^i, \phi^i, (p, r), (q, r) \Rightarrow r^o$ at the root have the following form:*

$$\dfrac{p^i, q^i, \phi^i, (p,r), (q,r) \Rightarrow r^o}{\dfrac{p^i, q^i, \phi^i, (p,r) \Rightarrow q^i}{p^i, q^i, \phi^i \Rightarrow p^i \qquad r^i, p^i, q^i, \phi^i \Rightarrow q^i}\,3a \qquad \dfrac{r^o, p^i, q^i, \phi^i, (p,r) \Rightarrow r^o}{p^i, q^i, \phi^i \Rightarrow p^i \qquad r^o \Rightarrow r^o}\,2a}\,2b$$

The trees $\mathcal{T}_{3,4}$ displaying the sequents $p^i, \neg q^i, \phi^i, (p, r), (q, r) \Rightarrow r^o$ at the root have the following form:

$$\dfrac{p^i, \neg q^i, \phi^i, (p,r), (q,r) \Rightarrow r^o}{p^i, \neg q^i, \phi^i \Rightarrow p^i \qquad r^o \Rightarrow r^o}\,2a$$

We say that (ϕ, ψ) is *triggerable* in a sequent of the form $\Delta^i, \Pi^o, \Sigma^r \Rightarrow \psi^o$ (resp. $\Delta^i, \Pi^o, \Sigma^r \Rightarrow \psi^i$) if $\Delta \vdash_{\mathsf{L}} \phi$ (resp. $\Delta, \Pi \vdash_{\mathsf{L}} \phi$). It is easy to verify that the out_4-decomposition trees displayed in Examples 6 and 7 are successful, and that each rule in the sequent labelling the root is triggerable in the sequent labelling some node of the tree. This is no coincidence, as the following result shows.

Proposition 5. *Let $x \in \{i, o\}$ and $\mathcal{T}$ be a terminated out_4-decomposition tree for a sequent $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \phi^x$. Each rule in Γ^r is triggerable in some sequent occurring in $\mathcal{T}$.*

We first introduce some terminology and then state a number of preliminary lemmas (Lem. 11-14), which underpin the main proof of this subsection (Thm. 3).

Definition 8 (Extended bases for mce's)**.** *Let $x \in \{i, o\}$. For any sequent $\Gamma^i, \Delta^o, \Pi^r \Rightarrow \phi^x$, we take $\bigwedge E_1 \vee \dots \vee \bigwedge E_m$ to be the full disjunctive normal form of $\bigwedge \Gamma$ generated from a set of atoms A extending the set of atoms in $\Gamma^i, \Delta^o, \Pi^r$.*

Definition 9 (Pseudo-decomposition trees for out_4)**.** *Let $x \in \{i, o\}$. For any terminated out_4-decomposition tree $\mathcal{T}$ for $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \theta^x$, the set of $\mathcal{T}$-based terminated*

out_4*-pseudo-decomposition trees for* $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \theta^x$ *is the smallest set containing* $\mathcal{T}$ *and any tree* $\mathcal{T}'$ *obtained from* $\mathcal{T}$ *by replacing* $D_1, \ldots, D_m$ *with* $E_1, \ldots, E_{m'}$ *extending* $D_1, \ldots, D_m$[1].

For any $\mathcal{T}$*-based* out_4*-pseudo-decomposition tree* $\mathcal{T}'$ *and* $\mathcal{T}''$*, if the sets of* i*-labelled literals occurring in the sequents labelling the immediate children of the root of* $\mathcal{T}$ *and* $\mathcal{T}'$ *are* $E_1, \ldots, E_{m'}$ *and* $F_1, \ldots, F_{m''}$*, respectively,* $\mathcal{T}''$ *extends* $\mathcal{T}'$ *whenever* $F_1, \ldots, F_{m''}$ *extend* $E_1, \ldots, E_{m'}$*.*

Lemma 11 below ensures that out_4-decomposition trees can be composed so as to internalize applications of **Cases**, whereas Lemmas 12–14 serve the same purpose for **Cut**.

Lemma 11. *Let* $\Gamma^r_1, \Gamma^r_2 \neq \varnothing$*. If there are successful* out_4*-decomposition trees for* $\phi^i, \Gamma^r_1 \Rightarrow \theta^o$ *and* $\psi^i, \Gamma^r_2 \Rightarrow \theta^o$*, there is a successful* out_4*-decomposition for* $\phi^i \vee \psi^i, \Gamma^r_1, \Gamma^r_2 \Rightarrow \theta^o$*.*

Lemma 12. *Let* $\Delta^r = (\phi_1, \psi_1), \ldots, (\phi_m, \psi_m)$*, with* $m \geq 1$*.*

- **(i)** *If there exists a successful* out_4*-decomposition tree for* $\Gamma^i, \Delta^r \Rightarrow \phi^x$*, there exists a successful* out_4*-decomposition tree for* $\Pi^i, \Gamma^i, \Delta^r \Rightarrow \phi^x$ *for any* Π^i*.*
- **(ii)** *If* $\Gamma \vdash_{\mathsf{L}} \phi$ *and there exists a successful* out_4*-decomposition tree for* $\phi^i, \Delta^i, \Delta^r \Rightarrow \theta^x$*, there exists a successful* out_4*-decomposition tree for* $\Gamma^i, \Delta^i, \Delta^r \Rightarrow \theta^x$*.*
- **(iii)** *If there exists a successful* out_4*-decomposition tree for* $\Gamma^i, \Delta^r \Rightarrow \phi^x$ *and* $\psi_1, \ldots, \psi_m \vdash_{\mathsf{L}} \psi$*, there exists a successful* out_4*-decomposition tree for* $\Gamma^i, \Delta^r \Rightarrow \psi^x$*.*

Lemma 13. *Let* $\Delta^r \neq \varnothing$*.*

- **(i)** *If there exists a successful* out_4*-decomposition tree for* $\Gamma^i, \Gamma^o, \Delta^r \Rightarrow \phi^o$*, then there exists a successful* out_4*-decomposition tree for* $\Gamma^i, \Gamma^o, \Delta^r \Rightarrow \phi^i$*.*
- **(ii)** *If there exists a successful* out_4*-decomposition tree* $\mathcal{T}$ *for* $\Gamma^i, \Gamma^r \Rightarrow \phi^o$*, then there exists a successful* out_4*-decomposition tree* $\mathcal{T}'$ *for* $\psi^o, \Gamma^i, \Gamma^r \Rightarrow \psi^o$*.*
- **(iii)** *If there exists a successful* out_4*-decomposition tree for* $\Gamma^i, \Delta^r \Rightarrow \phi^i$ *and* $\phi, \Gamma \vdash_{\mathsf{L}} \psi$*, then there exists a successful* out_4*-decomposition tree for* $\Gamma^i, \Delta^r \Rightarrow \psi^i$*.*

Lemma 14. *Let* $\Delta^r_1 \neq \varnothing$ *and* $\Delta^r_2 \neq \varnothing$*.*

- **(i)** *If there are successful* out_4*-decomposition trees for* $\Gamma^i, \Delta^r_1 \Rightarrow \phi^i$ *and* $\phi^i, \Gamma^i, \Delta^r_2 \Rightarrow \psi^i$*, then there is a successful* out_4*-decomposition tree for* $\Gamma^i, \Delta^r_1, \Delta^r_2 \Rightarrow \psi^i$*.*
- **(ii)** *If there are successful* out_4*-decomposition trees for* $\Gamma^i, \Delta^r \Rightarrow \phi^i$ *and* $\phi^i, \Gamma^i, \Delta^o, \Delta^r_2 \Rightarrow \psi^i$*, there is a successful* out_4*-decomposition tree for* $\Gamma^i, \Delta^o, \Delta^r_1, \Delta^r_2 \Rightarrow \psi^i$*.*
- **(iii)** *If there are successful* out_4*-decomposition trees for* $\Gamma^i, \Delta^r \Rightarrow \phi^i$ *and* $\phi^i, \Gamma^i, \Delta^o, \Delta^r_2 \Rightarrow \psi^o$*, there is a successful* out_4*-decomposition tree for* $\Gamma^i, \Delta^o, \Delta^r_1, \Delta^r_2 \Rightarrow \psi^o$*.*

Theorem 3. *Let* $\Gamma^r \neq \varnothing$ *and* $\mathcal{S} = \{$***SDet***, ***Cases***$\}$*. Then* $\vdash^{\mathsf{io}}_{\mathcal{S}} \Gamma^i, \Gamma^r \Rightarrow \phi^o$ *iff there exists (at least) one successful* out_4*-decomposition tree for* $\Gamma^i, \Gamma^r \Rightarrow \phi^o$*.*

Proof. ($\Rightarrow$) We reason by induction over the height of an $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$-derivation π of $\Gamma^i, \Gamma^r \Rightarrow \phi^o$. If $h(\pi) = 1$, the last rule applied is **BDet**: a successful out_4-decomposition tree is obtained by applying the rules (1) and (2a). If $h(\pi) > 1$ and the last rule applied is **Cases**, π has the following form:

$$\dfrac{\begin{matrix}\vdots \\ \psi^i, \Gamma^i_1, \Gamma^r_1 \Rightarrow \phi^o\end{matrix} \qquad \begin{matrix}\vdots \\ \theta^i, \Gamma^i_2, \Gamma^r_2 \Rightarrow \phi^o\end{matrix}}{\psi^i \vee \theta^i, \Gamma^i_1, \Gamma^i_2, \Gamma^r_1, \Gamma^r_2 \Rightarrow \phi^o}$$

with $\Gamma^i = \psi^i \vee \theta^i, \Gamma^i_1, \Gamma^i_2$ and $\Gamma^r = \Gamma^r_1, \Gamma^r_2$. By inductive hypothesis, there exist successful out_4-decomposition trees for $\psi^i, \Gamma^i_1, \Gamma^r_1 \Rightarrow \phi^o$ and $\theta^i, \Gamma^i_2, \Gamma^r_2 \Rightarrow \phi^o$: by Lem. 11, this entails the existence of successful out_4-decomposition trees for $\psi^i \vee \theta^i, \Gamma^i_1, \Gamma^i_2, \Gamma^r_1, \Gamma^r_2 \Rightarrow \phi^o$. If the last rule applied is **Cut**, we argue as for Thm. 1, leveraging Lem. 12-14.

($\Leftarrow$) Let $\mathcal{T}$ be a successful out_4-decomposition tree for $\Gamma^i, \Gamma^r \Rightarrow \phi^o$ and $\mathcal{T}_h$ be any of its immediate subtrees. For each $\mathcal{T}_h$, we proceed as in the proof of right-to-left direction of Thm. 1, limiting ourselves to consider the rules triggerable in $\mathcal{T}_h$ – i.e., the rules in some set $\Gamma^r_h \subseteq \Gamma^r$ (remark that $\Gamma^r_h \neq \varnothing$, since $\mathcal{T}$ is successful).
For any $\mathcal{T}_h$, if $D^i_h = \psi^i_{1,h}, \ldots, \psi^i_{n,h}$, we take the following $\mathsf{AC}^{\mathsf{io}}_{\mathcal{S}}$-derivation π_h:

$$\dfrac{\dfrac{}{\bigwedge D^i_h \Rightarrow \psi^i_{n,h}}\,\textbf{Ax} \qquad \begin{matrix} \dfrac{\dfrac{}{\bigwedge D^i_h \Rightarrow \psi^i_{1,h}}\,\textbf{Ax} \qquad \begin{matrix}\vdots \\ D^i_h, \Gamma^r_h \Rightarrow \phi^o\end{matrix}}{\bigwedge D^i_h, D^i_h \setminus \{\psi^i_{1,h}\}, \Gamma^r_h \Rightarrow \phi^o}\,\textbf{Cut} \\ \vdots \\ \bigwedge D^i_h, D^i_h \setminus \{\psi^i_{1,h}, \ldots, \psi^i_{n-1,h}\}, \Gamma^r_h \Rightarrow \phi^o \end{matrix}}{\bigwedge D^i_h, \Gamma^r_h \Rightarrow \phi^o}\,\textbf{Cut}$$

Next, we consider the following derivation ρ:

$$\dfrac{\begin{matrix}\dfrac{\dfrac{\begin{matrix}\vdots_{\pi_1} \\ \bigwedge D^i_1, \Gamma^r_1 \Rightarrow \phi^o\end{matrix} \qquad \begin{matrix}\vdots_{\pi_2} \\ \bigwedge D^i_2, \Gamma^r_2 \Rightarrow \phi^o\end{matrix}}{(\bigwedge D^i_1) \vee (\bigwedge D^i_2), \Gamma^r_1, \Gamma^r_2 \Rightarrow \phi^o}\,\textbf{Cases} \qquad \begin{matrix}\vdots_{\pi_3} \\ \bigwedge D^i_3, \Gamma^r_3 \Rightarrow \phi^o\end{matrix}}{(\bigwedge D^i_1) \vee (\bigwedge D^i_2) \vee (\bigwedge D^i_3), \Gamma^r_1, \Gamma^r_2, \Gamma^r_3 \Rightarrow \phi^o}\,\textbf{Cases} \\ \vdots \\ D^i_{\vec{m}}, \Gamma^r_{\vec{m}} \Rightarrow \phi^o\end{matrix} \qquad \begin{matrix}\vdots_{\pi_3} \\ \bigwedge D^i_m, \Gamma^r_m \Rightarrow \phi^o\end{matrix}}{D^i_{\vec{m}} \vee (\bigwedge D^i_m), \Gamma^r_{\vec{m}}, \Gamma^r_m \Rightarrow \phi^o}\,\textbf{Cases}$$

with $D^i_{\vec{m}} = (\bigwedge D^i_1) \vee (\bigwedge D^i_2) \vee (\bigwedge D^i_3) \vee \cdots \vee (\bigwedge D^i_{m-1})$ and $\Gamma^r_{\vec{m}} = \Gamma^r_1, \Gamma^r_2, \Gamma^r_3, \ldots, \Gamma^r_{m-1}$. By construction, $\Gamma^i \vdash_{\mathsf{L}} D^i_{\vec{m}} \vee (\bigwedge D^i_m)$ and Prop. 7 ensures $\Gamma^r_{\vec{m}}, \Gamma^r_m = \Gamma^r$. Hence, a **Cut** with $\Gamma^i \Rightarrow D^i_{\vec{m}} \vee (\bigwedge D^i_m)$ completes the proof. $\square$

4.4 out^+_4-Decomposition Trees

Here, we simply combine the ideas of out^+_3 and out_4.

Definition 10 (Decomposition trees for out^+_4)**.** *Let* $x \in \{i, o\}$*,* k *be any sequence of indices starting with* 0 *and* $D_1, \ldots, D_m$ *be sets of literals generated according to Definition 6. We generate a terminated* out_4*-decomposition tree for* $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \theta^x$ *by applying the following procedure, starting from* $\Gamma^r_0 = \Gamma^r$ *and* $\Gamma^o_0 = \Gamma^o$*:*

1. *First, we apply the rule*

$$\dfrac{\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \phi^x}{\{D^i_h, \Gamma^o, \Gamma^r \Rightarrow \phi^x\}_{1 \leq h \leq m}}$$

2. *if* $\Gamma^r_{\mathsf{k}} = \Delta^r, (\phi_n, \psi_n)$ *and the succedent is an* o*-labelled formula, then*
   - *(a) if there exists no* (ϕ, ψ) *in* Δ^r *such that* $D_h \vdash_{\mathsf{L}} \phi$*, apply the rule*

[1]Observe that every rule instance in $\mathcal{T}$ is preserved in $\mathcal{T}'$, irrespective of whether the replacement of $D_1, \ldots, D_m$ with $E_1, \ldots, E_{m'}$ violates the applicability conditions of (2a) or (3a).

$$\frac{D^i_h, \Gamma^o_{\mathsf{k}}, \Delta^r, (\phi_n, \psi_n) \Rightarrow \theta^o}{D^i_h \Rightarrow \phi^i_n \qquad D^o_h, \Gamma^o_{\mathsf{k}}, \psi^o_n \Rightarrow \theta^o}$$

and take $\Gamma^r_{\mathsf{k}n} = \varnothing$, $\Gamma_{\mathsf{k}n} = \Gamma_{\mathsf{k}}, \psi_n$;

(b) *if there exists (at least) one* (ϕ, ψ) *in* Δ^r *such that* $D_k \vdash_{\mathsf{L}} \phi$, *apply the rule*

$$\frac{D^i_h, \Gamma^o_{\mathsf{k}}, \Delta^r, (\phi_n, \psi_n) \Rightarrow \theta^o}{D^i_h, \Gamma^o_{\mathsf{k}}, \Delta^r \Rightarrow \phi^i_n \qquad D^i_h, \Gamma^o_{\mathsf{k}}, \psi^o_n, \Delta^r \Rightarrow \theta^o}$$

taking $\Gamma^r_{\mathsf{k}n} = \Delta^r$ *and* $\Gamma_{\mathsf{k}n} = \Gamma_{\mathsf{k}}, \psi_n$;

3. *if* $\Gamma^r_{\mathsf{k}} = \Delta^r, (\phi_n, \psi_n)$ *and the succedent is an* i*-labelled formula, then*
   (a) *if there exists no* (ϕ, ψ) *in* Δ^r *such that* $D_h, \Gamma_{\mathsf{k}} \vdash_{\mathsf{L}} \phi$, *apply the rule*

$$\frac{D^i_h, \Gamma^o_{\mathsf{k}}, \Delta^r, (\phi_n, \psi_n) \Rightarrow \theta^i}{D^i_h, \Gamma^i_{\mathsf{k}} \Rightarrow \phi^i_n \qquad D^i_h, \Gamma^i_{\mathsf{k}}, \psi^i_n \Rightarrow \theta^i}$$

   and take $\Gamma^r_{\mathsf{k}n} = \varnothing$, $\Gamma_{\mathsf{k}n} = \Gamma_{\mathsf{k}}, \psi_n$;

   (b) *if there exists (at least) one* (ϕ, ψ) *in* Δ^r *such that* $D_h, \Gamma_{\mathsf{k}} \vdash_{\mathsf{L}} \phi$, *apply the rule*

$$\frac{D^i_h, \Gamma^o_{\mathsf{k}}, \Delta^r, (\phi_n, \psi_n) \Rightarrow \theta^i}{D^i_h, \Gamma^o_{\mathsf{k}}, \Delta^r \Rightarrow \phi^i_n \qquad D^i_h, \Gamma^o_{\mathsf{k}}, \psi^i_n, \Delta^r \Rightarrow \theta^i}$$

   taking $\Gamma^r_{\mathsf{k}n} = \Delta^r$ *and* $\Gamma_{\mathsf{k}n} = \Gamma_{\mathsf{k}}, \psi_n$.

Theorem 4. *Let* $\Gamma^r \neq \varnothing$ *and* $\mathcal{S} = \{$***SDet***, ***Cases***, ***TP***$\}$. *Then* $\vdash^{\mathsf{io}}_{\mathcal{S}} \Gamma^i, \Gamma^r \Rightarrow \phi^o$ *iff there exists (at least) one successful* out^+_4*-decomposition tree for* $\Gamma^i, \Gamma^r \Rightarrow \phi^o$.

Proof. Proof analogous to those for Thms. 1, 2 and 3. □

4.5 $\mathsf{out}^{(+)}_1$- and $\mathsf{out}^{(+)}_2$-Decomposition Trees

The logics $\mathsf{out}^{(+)}_1$ and $\mathsf{out}^{(+)}_2$ do not permit successive detachment (cf. **SDet**). Hence, the decomposition trees for $\mathsf{out}^{(+)}_3$ and $\mathsf{out}^{(+)}_4$ can be straightforwardly adapted to $\mathsf{out}^{(+)}_1$ and $\mathsf{out}^{(+)}_2$, respectively, by requiring that all rules be decomposed simultaneously within a single inference step.

Example 8. *The following is a successful* out^+_1*-decomposition tree for the sequent* $p^i, q^i, (p, r), (q, s) \Rightarrow p \wedge r \wedge s^o$:

$$\frac{p^i, q^i, (p, r), (q, s) \Rightarrow p \wedge r \wedge s^o}{p^i, q^i \Rightarrow p^i \qquad p^i, q^i \Rightarrow q^i \qquad p^o, q^o, r^o, s^o \Rightarrow p \wedge r \wedge s^o}$$

Example 9. *A successful* out_2*-decomposition tree for* $p \vee q^i, (p, r), (q, s) \Rightarrow r \vee s^o$ *features the sequents* $\{p^i, \Gamma^i_j \Rightarrow p^i\}_{1 \leq j \leq 2}$, $\{q^i, \Delta^i_j \Rightarrow q^i\}_{1 \leq j \leq 2}$, $r^o \Rightarrow r \vee s^o$ *and* $s^o \Rightarrow r \vee s^o$ *as leaves, with* $\Gamma^i_1 = \{q\}$, $\Gamma^i_2 = \{\neg q\}$ *and* $\Delta^i_1 = \{p\}$, $\Delta^i_2 = \{\neg p\}$.

AC-derivations and decomposition trees play distinct yet complementary roles. The former implement forward reasoning over arguments, whereas the latter support backward reasoning by providing a local procedure for verifying derivability of individual arguments. The proof-search space for AC-derivations is readily seen to be infinite, due to the presence of the **Cut** rule. By contrast, decomposition trees can be composed so as to absorb instances of **Cut** (cf. Lemmas 6-7 for out_3-decomposition trees, Lemmas 9-10 for out^+_3-decomposition trees and Lemmas 13-14 for out_4-decomposition trees). As a result, decomposition trees satisfy a generalized form of the subformula property, which makes them suitable for automated reasoning procedures.

Fact 1 (Generalized subformula property)**.** *For any* $\mathsf{out}^{(+)}_{\mathsf{i}}$*-decomposition tree* $\mathcal{T}$ *for* $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \theta^x$, *with* $\mathsf{i} \in \{1, 2, 3, 4\}$ *and* $x \in \{i, o\}$, *each formula occurring in* $\mathcal{T}$ *is a subformula of some formula occurring in* $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \theta^x$ *or a literal obtained from the atoms in* $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \theta^x$. *For* $\mathsf{i} = \{1, 3\}$, *this statement can be strengthened: each formula occurring in a decomposition tree* $\mathcal{T}$ *is a subformula of some formula in* $\Gamma^i, \Gamma^o, \Gamma^r \Rightarrow \theta^x$.

5 Credulous Decision Procedures

We provide a finite decision procedure for credulous reasoning under stable semantics, using decomposition trees.

Definition 11 (Decision method for credulous reasoning)**.** *Let* $\mathbb{K} = \langle \mathcal{I}, \mathcal{R}, \mathcal{C} \rangle$ *be a knowledge base and* $\mathcal{AF}^l_{\mathcal{S}}(\mathbb{K}) = \langle \mathsf{arg}, \mathsf{att} \rangle$ *be an* $\mathsf{AC}^l_{\mathcal{S}}$*-induced argumentation framework based on* $\mathbb{K}$. *For any sequent* s *of the form* $\Gamma, (\phi_1, \psi_1), \ldots, (\phi_m, \psi_m) \Rightarrow \theta^o$, *we apply the following procedure to decide whether* $s \in \bigcup \mathtt{Stable}$ *or not.*

1. *If there exists no successful decomposition tree for* s, *then* $s \notin \bigcup \mathtt{Stable}$.
2. *If there exists (at least) one successful decomposition tree for* s, *two possible cases arise.*
   (a) *If* $\mathcal{C} \vdash_{\mathsf{L}} \neg \bigwedge_{i=1}^m \psi_i$, *then* $s \notin \bigcup \mathtt{Stable}$.
   (b) *If* $\mathcal{C} \nvdash_{\mathsf{L}} \neg \bigwedge_{i=1}^m \psi_i$, *then* $s \in \bigcup \mathtt{Stable}$.

Proposition 6. *The decision method in Def. 11 gives (i) credulous inference in the corresponding input/output and normal default logic and (ii) credulous inference in the corresponding* AC*-induced argumentation framework for* $\bigcup \mathsf{Stable}$.

Proof. Let $s = \Delta \Rightarrow \phi^o$ with $\Delta \subseteq \mathcal{I} \cup \mathcal{R} \cup \mathcal{C}$. For (i) we observe that if there exists one successful decomposition tree for s, by Thms. 1, 2, 3, 4, Cor. 2 and analogous results for $\mathsf{out}^{(+)}_{1,2}$ we know that s can be derived using the corresponding $\mathsf{AC}^l_{\mathcal{S}}$ proof-system, which subsequently by Proposition 1 captures detachment of ϕ^o from Δ in the corresponding input/output and normal default logic. Then, if (b) in Def. 11 obtains, we know there is a MCS (wrt $\mathcal{C}$) for $\Delta \cap \mathcal{L}^r$, and we are done. For (ii) we observe that by Prop. 1 there is a stable extension in the corresponding AC-induced argumentation framework for s. □

6 Related and Future Work

We provided proof-search methods for arguments generated by argument calculi for normative and doxastic reasoning, using decomposition trees that decompose norms, respectively, defaults step-by-step. As a consequence, we obtained proof-search for determining maximal consistent set membership for constrained input/output logic (Makinson and van der Torre 2001) and normal default logic (Reiter 1980). Our approach also showed strong links with nonmonotonic reasoning in logical argumentation (Dung 1995).

Alternative proof-theoretic methods exist for establishing nonmonotonic inference of input/output and default logics (Bonatti and Olivetti 2002; Lellmann 2021; Ciabattoni and Rozplokhas 2023; Arieli, van Berkel, and Straßer 2024; Piazza and Sabatini 2025; Piazza and Sabatini 2026). To the best of our knowledge, our approach is the first to characterize credulous reasoning for all constrained input/output logics beyond mere logical consistency (unlike (Lellmann 2021; Ciabattoni and Rozplokhas 2023)), while avoiding explicit underivability statements in the calculus (unlike (Lellmann 2021; Piazza and Sabatini 2026)). Moreover, our method readily extends to stable skeptical semantics under shared arguments due to its correspondence with grounded semantics in logical argumentation. In future work, we will prove such correspondence. We conjecture that the out_2- and out_4^+-decomposition procedures can be further refined to match the complexity bounds established in (Ciabattoni and Rozplokhas 2023), and that underivability statements are in fact necessary to capture full default logic with justifications (as in (Bonatti and Olivetti 2002; Piazza and Sabatini 2025)). Last, we note that the generalized subformula property of decomposition trees requires further investigation.

Due to the modularity of our approach, it is promising to study proof-search for other nonmonotonic logics, such as those for reasoning with preferences over norms (Horty 2012; Parent 2011; Straßer and Pardo 2021), disjunctive default reasoning (Gelfond et al. 1991; Zhou, Straßer, and van Berkel 2025) and causal reasoning (Bochman 2023).

The fact that decomposition trees also determine the derivability of arguments expressing attacks opens the door to formal argumentation methods. Argumentation is promising for explainable AI (Čyras et al. 2021). Notably, dialogue models and argumentative games can be modified for explanatory purposes (Fan and Toni 2015; Modgil and Caminada 2009; Bex and Walton 2016; van Berkel and Straßer 2024) using a variety of explanatory notions (Ho and Schlobach 2025; Bengel and Thimm 2025; Alfano et al. 2024; Saribatur, Wallner, and Woltran 2020). Existing work typically assumes that all derivable arguments are available to explainer and explainee. Leveraging proof-search, we may target the dynamic construction of explanatory dialogues where new arguments can be derived throughout the dialogue, justifications of the (un)derivability of (counter-)arguments can be queried, and arguments can be retracted that are shown underivable.

Acknowledgments

This work is partially supported by the Austrian Science Fund under the *Logical Methods for Deontic Explanations* project (FWF - 6372-N) and the cluster of excellence *Bilateral AI* (FWF - 10.55776/COE12), as well as by the European Union funded Rise-MSCA-2020, Project 101007627.

AI Declaration

No AI was used at any stage in writing this paper.

References

Alfano, G.; Greco, S.; Parisi, F.; and Trubitsyna, I. 2024. Counterfactual and semifactual explanations in abstract argumentation: Formal foundations, complexity and computation. In *Proceedings of the 22nd International Conference on Principles of Knowledge Representation and Reasoning, (KR 2024)*.

Arieli, O., and Straßer, C. 2015. Sequent-based logical argumentation. *Argument and Computation* 6(1):73–99.

Arieli, O., and Straßer, C. 2019. Logical argumentation by dynamic proof systems. *Theoretical Computer Science* 781:63–91.

Arieli, O.; Borg, A.; Heyninck, J.; and Straßer, C. 2021. Logic-based approaches to formal argumentation. In Gabbay, D.; Giacomin, M.; Simari, G. R.; and Thimm, M., eds., *Handbook of Formal Argumentation, Volume 2*. College Publications. 1793–1898.

Arieli, O.; van Berkel, K.; and Straßer, C. 2024. Defeasible normative reasoning: A proof-theoretic integration of logical argumentation. In *Proceedings of AAAI24, the 38th Annual AAAI Conference on Artificial Intelligence.* AAAI Press. 10450–10458.

Baroni, P.; Caminada, M.; and Giacomin, M. 2011. An introduction to argumentation semantics. *The Knowledge Engineering Review* 26(4):365–410.

Bengel, L., and Thimm, M. 2025. Sequence Explanations for Acceptance in Abstract Argumentation. In *Proceedings of the 22nd International Conference on Principles of Knowledge Representation and Reasoning, (KR 2025)*, 121–131.

van Berkel, K., and Straßer, C. 2022. Reasoning with and about norms in logical argumentation. In Toni, F.; Polberg, S.; Booth, R.; Caminada, M.; and Kido, H., eds., *Frontiers in Artificial Intelligence and Applications: Computational Models of Argument, proceedings (COMMA22)*, volume 353, 332 – 343. IOS press.

van Berkel, K., and Straßer, C. 2024. Towards deontic explanations through dialogue. In Čyras, K.; Kampik, T.; Cocarascu, O.; and Rago, A., eds., *2nd International Workshop on Argumentation for eXplainable AI (ArgXAI)*, 29–40. CEUR Workshop Proceedings.

van Berkel, K.; Straßer, C.; and Zhou, Z. 2024. Towards an argumentative unification of default reasoning. In *Frontiers in Artificial Intelligence and Applications: Computational Models of Argument, proceedings (COMMA24)*, TBA. IOS press.

van Berkel, K. 2023. *A Logical Analysis of Normative Reasoning: Agency, Action, and Argumentation*. Ph.D. Dissertation, TU Wien.

Bex, F., and Walton, D. 2016. Combining explanation and argumentation in dialogue. *Argument & Computation* 7(1):55–68.

Bochman, A. 2023. Default logic as a species of causal reasoning. In *Proceedings of the International Conference on Principles of Knowledge Representation and Reasoning*, volume 19, 117–126.

Bonatti, P. A., and Olivetti, N. 2002. Sequent calculi for propositional nonmonotonic logics. *ACM Transactions on Computational Logic (TOCL)* 3(2):226–278.

Ciabattoni, A., and Rozplokhas, D. 2023. Streamlining input/output logics with sequent calculi. In *Proceedings of the International Conference on Principles of Knowledge Representation and Reasoning (KR23)*, volume 19, 146–155.

Čyras, K.; Rago, A.; Albini, E.; Baroni, P.; and Toni, F. 2021. Argumentative XAI: a survey. *arXiv preprint arXiv:2105.11266*.

Dung, P. M. 1995. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and n-person games. *Artificial Intelligence* 77(2):321–357.

Fan, X., and Toni, F. 2015. On computing explanations in argumentation. In Bonet, B., and Koenig, S., eds., *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence (AAAI 2015)*, 1496–1502.

Gelfond, M.; Lifschitz, V.; Przymusinska, H.; and Truszczynski, M. 1991. Disjunctive defaults. In Allen, J.; Fikes, R.; and Sandewall, E., eds., *Proceedings of the Second International Conference on Principles of Knowledge Representation and Reasoning*, 230–237. Citeseer.

Ho, L., and Schlobach, S. 2025. Dialogue-based explanations for logical reasoning using structured argumentation. *arXiv preprint arXiv:2502.11291*.

Horty, J. F. 2012. *Reasons as defaults*. Oxford University Press.

Lellmann, B. 2021. From input/output logics to conditional logics via sequents – with provers. In Das, A., and Negri, S., eds., *Automated Reasoning with Analytic Tableaux and Related Methods*, 147–164. Springer International Publishing.

Liao, B.; Oren, N.; van der Torre, L.; and Villata, S. 2018. Prioritized norms in formal argumentation. *Journal of Logic and Computation* 29(2):215–240.

Makinson, D., and van der Torre, L. 2001. Constraints for Input/Output logics. *Journal of Philosophical Logic* 30(2):155–185.

Modgil, S., and Caminada, M. 2009. Proof theories and algorithms for abstract argumentation frameworks. In Rahwan, I., and Simari, G. R., eds., *Argumentation in Artificial Intelligence*. Springer. 105–129.

Parent, X. 2011. Moral particularism in the light of deontic logic. *Artificial Intelligence and Law* 19(2):75–98.

Piazza, M., and Sabatini, A. 2025. Hypersequent calculi for propositional default logics. *ACM Transactions on Computational Logic (TOCL)* 26(3):1–36.

Piazza, M., and Sabatini, A. 2026. A unified framework for input/output and default logics via hypersequents. *Journal of Logic and Computation* 36(2).

Pollock, J. L. 1987. Defeasible reasoning. *Cognitive science* 11(4):481–518.

Reiter, R. 1980. A logic for default reasoning. *Artificial intelligence* 13(1-2):81–132.

Saribatur, Z. G.; Wallner, J. P.; and Woltran, S. 2020. Explaining non-acceptability in abstract argumentation. In Giacomo, G. D.; Catala, A.; Dilkina, B.; Milano, M.; Barro, S.; Bugarín, A.; and Lang, J., eds., *Frontiers in Artificial Intelligence and Applications, 24th European Conference on Artificial Intelligence - ECAI 2020*, volume 325, 881–888. IOS Press.

Straßer, C., and Pardo, P. 2021. Prioritized defaults and formal argumentation. In Liu, F.; Marra, A.; Portner, P.; and Putte, F. V. D., eds., *Deontic Logic and Normative Systems - 15th International Conference, (DEON 2021/21)*. College Publications. 427–446.

Strasser, C. 2025. Nonmonotonic logic. *Elements in Philosophy and Logic*.

Toulmin, S. E. 1958. *The Uses of Argument*. Cambridge University Press.

Zhou, Z.; Straßer, C.; and van Berkel, K. 2025. Hypothesis-driven disjunctive reasoning in logical argumentation. In *International Workshop on Logic, Rationality and Interaction*, 179–194. Springer.