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Expressiveness of Epistemic Spaces for Iterated Belief Change Operators

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Abstract

Recently epistemic spaces have been introduced to formalize instantiations of iterated belief change operators and their translations from one concrete representation (epistemic space) to another. In this work, we build on these notions to deepen the understanding of iterated belief change and propose a general method for comparing the expressiveness of existing epistemic spaces. We introduce the notion of canonicity for epistemic spaces as a tool for identifying those that are sufficient, or necessary, to realize certain classes of iterated belief change operators. In particular, we give the canonical epistemic space (up to equivalence) that allows one to instantiate any iterated belief change operator.

1 Introduction

Belief revision concerns the correction of erroneous beliefs when an agent receives reliable information that contradicts them. This ability is fundamental for autonomous agents, such as robots, enabling them to react appropriately to situations that were previously considered impossible under their initial or learned beliefs.

The seminal AGM framework (Alchourrón, Gärdenfors, and Makinson 1985) provided the first formal characterization of belief revision, but it addressed only single-step changes. It imposes no constraints on successive revisions, which may lead to unintuitive or irrational change sequences, in particular regarding the preservation of conditional information. This difficulty is closely related to the classical frame problem (McCarthy and Hayes 1969; Reiter 1991).

Darwiche and Pearl (Darwiche and Pearl 1994; Darwiche and Pearl 1997) proposed four additional postulates to regulate successive revisions. They also argued that iterated change requires a richer representation of belief. In AGM, beliefs are represented by a propositional formula (or, equivalently in finite propositional logic, a deductively closed theory). Such a representation does not retain the epistemic information needed to guide further revisions. To overcome this limitation, Darwiche and Pearl introduced epistemic states: abstract objects associated with a formula encoding the agent's current beliefs, while also storing additional information that determines how future revisions are performed. Iterated revision operators satisfying the AGM and Darwiche and Pearl's postulates on epistemic states are referred to here as DP operators.

Concrete iterated revision operators must be defined within a specific representation, called epistemic space (Schwind, Konieczny, and Pino Pérez 2022). Examples include Boutilier's natural revision on sets of conditional formulas (Boutilier 1996), Nayak's lexicographic revision on epistemic entrenchments (Nayak 1994), Spohn's conditionalizations (Spohn 1988) and Williams' transmutations (Williams 1995) on ordinal conditional functions (OCFs), Lehmann's operators (Lehmann 1995) and Konieczny and Pino-Pérez's operators (Konieczny and Pino Pérez 2000) on sequences of formulas, and Booth–Meyer's restrained revision on total preorders over worlds (TPOs). Many operators can also be reformulated in other spaces; for instance, Boutilier and Nayak operators are often expressed on TPOs.

A substantial body of work has focused on TPOs, which provide a convenient epistemic space. However, they are limited: some natural operators cannot be defined within them (Aravanis, Peppas, and Williams 2019; Schwind and Konieczny 2020). Until recently, there was little systematic comparison of epistemic spaces. The study in (Schwind, Konieczny, and Pino Pérez 2022) introduced formal definitions of epistemic space and instantiation (translating an operator from one space to another), leading to two main results: (i) OCFs can be considered "universal" for DP operators satisfying an additional property requiring that every epistemic state can be reached from a single empty state via successive revisions, in the sense that every such operator can be instantiated on OCFs, and (ii) without this condition, OCFs are not universal.

Epistemic spaces are therefore central to understanding iterated belief change. This paper develops a systematic study of such spaces. We examine which operators are realizable (Sauerwald and Thimm 2024) in a given space, compare the expressiveness of different spaces, and identify spaces of particular interest for specific classes of operators. We adopt a broader perspective by considering improvement operators (Konieczny and Pino Pérez 2008; Konieczny, Medina Grespan, and Pino Pérez 2010; Borges et al. 2024), which generalize the class of DP operators by allowing new information to increase in plausibility without necessarily being fully accepted as a belief.

The central questions we address are the following. When are two operators, possibly defined on very different epistemic spaces, essentially the same? When can two epistemic spaces be considered equally expressive for a given class of operators? What is the smallest (least expressive) space on which some operator from a given class can be realized? Does there exist a universal, canonical representative space for all operators of a given class, and, more generally, for all iterated belief change operators? The latter question is particularly important, as it asks whether there is an epistemic space, unique up to expressiveness equivalence, that can realize every iterated belief change operator.

To answer these questions, we introduce the following formal tools:

- Canonization: instantiating an iterated belief change operator to an equivalent one on a minimal canonical space.
- Expressiveness comparison: determining when one space can equivalently represent all operators of another.
- Degree of an epistemic space: characterizing expressiveness relationships.

Using these tools, we identify, for DP and improvement operators (and several subclasses), lower and upper bounds on the expressiveness of epistemic spaces. We show that each class admits a unique (up to expressiveness equivalence) canonical representative space. In particular, we characterize a canonical space capable of realizing every possible IBC operator.

Proofs of propositions are available online.[1]

2 Preliminaries

Basic notions. We consider a propositional language $\mathcal{L}$ built up from a finite set of propositional variables P (of cardinality at least 2) and the usual connectives. A world is a mapping from P to $\{0, 1\}$. The set of all worlds on $\mathcal{L}$ is denoted by Ω. A world ω satisfies a formula φ if ω makes φ true; the set of such worlds is denoted by $[\varphi]$. A formula is consistent if $[\varphi] \neq \emptyset$. The set of consistent formulae is denoted by $\mathcal{L}_*$. The symbol $\models$ denotes logical entailment, and $\equiv$ logical equivalence: $\varphi \models \psi$ iff $[\varphi] \subseteq [\psi]$, and $\varphi \equiv \psi$ iff $[\varphi] = [\psi]$. The symbol $\mathcal{P}(\Omega)_*$ denotes the set of non-empty subsets of worlds, i.e., $\mathcal{P}(\Omega)_* = \{S \subseteq \Omega \mid S \neq \emptyset\}$. We assume throughout the paper the existence of an arbitrarily fixed mapping associating each set of worlds $W \in \mathcal{P}(\Omega)_*$ with a formula $\psi_W \in \mathcal{L}_*$ such that $[\psi_W] = W$.

We denote by Card the class of all cardinal numbers, by $\aleph_0$ the smallest infinite cardinal, and by $\mathfrak{c}$ the cardinal of the set of real numbers. Given a set S, $Card(S)$ denotes the cardinal of S, which is sometimes denoted by $|S|$.

Given a set E and a preorder (a reflexive and transitive relation on E) $\preceq$ over E, $\prec$ denotes the strict part of $\preceq$, i.e $x \prec y$ iff $x \preceq y$ and $y \not\preceq x$. Given $F \subseteq E$, $\min(F, \preceq)$ denotes the set of minimum elements of F; formally, $\min(F, \preceq) = \{x \in F \mid \nexists y \in F, y \prec x\}$.

Epistemic spaces. An *epistemic space* is a pair $\mathcal{E} = \langle U, B\rangle$, where U is a non-empty set of so-called *epistemic states* and B is a mapping $B : U \to \mathcal{L}_*$ (Schwind and Konieczny 2020; Schwind, Konieczny, and Pino Pérez 2022). Throughout this paper, we assume a form of "non-triviality" of epistemic spaces, i.e., that every epistemic space $\mathcal{E} = \langle U, B\rangle$ is such that not all epistemic states from U are associated with equivalent beliefs, i.e., there exist $\Psi, \Psi' \in U$ such that $B(\Psi) \not\equiv B(\Psi')$. Note that this requires in particular that $U \neq \emptyset$. We denote by $\mathcal{S}p$ the class of all possible epistemic spaces.

Let us now give some standard examples of epistemic spaces.

The TPO-*based epistemic space* is the space $\mathcal{E}^{\mathsf{tpo}} = \langle U^{\mathsf{tpo}}, B^{\mathsf{tpo}}\rangle$, where U^{tpo} is the set of all total preorders (TPOs) over Ω, and B^{tpo} is the mapping associating each TPO $\preceq$ from U^{tpo} with the formula $\psi_{\min(\Omega,\preceq)}$.

Another example of epistemic space is ordinal conditional functions (OCFs) (Spohn 1988; Williams 1995). An OCF κ is a mapping associating each world with a non-negative integer[2] such that there is a world $\omega \in \Omega$ with $\kappa(\omega) = 0$. Then, the OCF-*based epistemic space* is the epistemic space $\mathcal{E}^{\mathsf{ocf}} = \langle U^{\mathsf{ocf}}, B^{\mathsf{ocf}}\rangle$, where U^{ocf} is the set of all OCFs over Ω, and B^{ocf} is the mapping associating each OCF $\kappa \in U^{\mathsf{ocf}}$ with the formula $\psi_{\{\omega\in\Omega \mid \kappa(\omega)=0\}}$.

These are the most standard epistemic spaces found in the (iterated) belief change literature, but other spaces exist, which will be discussed and studied in a further section.

Iterated belief change. Given an epistemic space $\mathcal{E} = \langle U, B\rangle$, an *iterated belief change (IBC) operator on* $\mathcal{E}$ is a mapping $\circ : U_\circ \times \mathcal{L}_* \to U_\circ$, where $U_\circ$, called the *domain of* $\circ$, is such that $\emptyset \subsetneq U_\circ \subseteq U$. An IBC operator $\circ$ models how an agent changes his mind upon receiving new information: for any epistemic state $\Psi \in U_\circ$ and propositional formula $\mu \in \mathcal{L}_*$ representing evidence, it yields a new state $\Psi \circ \mu$ capturing how Ψ changes after incorporating μ. Note that typically (e.g., in (Schwind, Konieczny, and Pino Pérez 2022)), IBC operators $\circ$ are defined as mappings $U \times \mathcal{L}_* \to U$, so that their domain is the "full" set of epistemic states from the underlying space. The generalization we make here, for technical reasons, is harmless: when considering logical properties of IBC operators $\circ$, the epistemic states $U \setminus U_\circ$ are always ignored. We often write $(\mathcal{E}, \circ)$ to denote an IBC operator, so as to explicitly refer to the epistemic space $\mathcal{E}$ on which $\circ$ is defined. The symbol IBC denotes the class of all possible IBC operators.

The best-known class of IBC operators is Darwiche and Pearl's (DP) revision operators (Darwiche and Pearl 1994; Darwiche and Pearl 1997), and its extensions (Booth and Meyer 2006; Jin and Thielscher 2007). One of the most well-known postulates of DP revision operators is the success postulate, which requires the new incoming information μ to be accepted in the beliefs of every revised state, i.e., $B(\Psi \circ \mu) \models \mu$, for every $\Psi \in U_\circ$. We refer the reader to (Darwiche and Pearl 1997) for a list of postulates defining DP operators. Another important class of IBC operators is the class of General Improvement (GI)

[1] https://nicolas-schwind.github.io/SKP-KR26-proofs.pdf

[2] In the original definition, OCFs are defined on ordinals (Spohn 1988). But here, as in most cases, the integer restriction is enough.

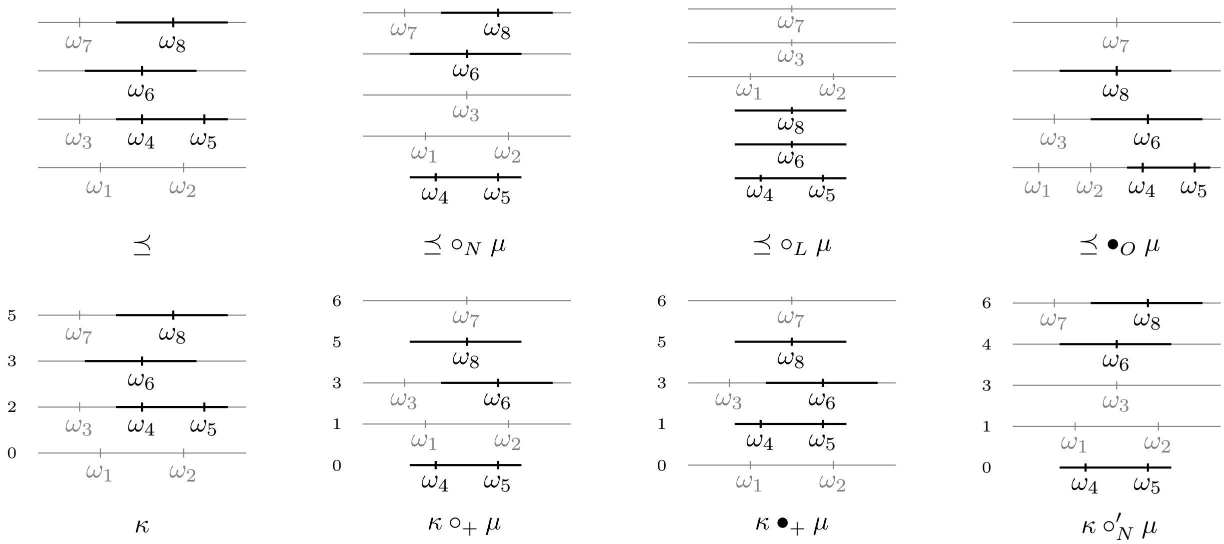

Figure 1: Illustration of the behavior of the GI operators $\circ_N$, $\circ_L$, $\bullet_O$ defined on the TPO-based epistemic space and $\circ_+$, $\bullet_+$, $\circ'_N$ defined on the OCF-based epistemic space, with $[\mu] = \{\omega_4, \omega_5, \omega_6, \omega_8\}$.

operators (Borges et al. 2024). This class is a slight generalization of the better-known class of improvement operators introduced in (Konieczny and Pino Pérez 2008; Konieczny, Medina Grespan, and Pino Pérez 2010) which impose an additional "admissibility" property that excludes certain DP operators, e.g., natural revision (Boutilier 1996). GI operators avoid such discards. GI operators generalize DP revision operators in the sense that the success postulate is weakened: the new piece of information is not forced to be believed after the change, but its plausibility is increased. Finally, DP revision postulates a form of syntax-independence comprised in the property (R*4) (if $\mu \equiv \mu'$, then $B(\Psi \circ \mu) \equiv B(\Psi \circ \mu')$). This property has been criticized as being too weak in a number of works (Konieczny and Pino Pérez 2008; Booth and Meyer 2011), since it requires syntax-independence only after one revision step. A stronger, more appropriate version of syntax independence was introduced together with the framework of improvement operators and is satisfied by GI operators as well:

(I4) for each $n \in \mathbb{N}$ and each $i \in \{1, \ldots, n\}$, if $\mu_i \equiv \mu'_i$, then $B(((\Psi \circ \mu_1) \circ \ldots) \circ \mu_n) \equiv B(((\Psi \circ \mu'_1) \circ \ldots) \circ \mu'_n)$

In fact, all existing examples of DP operators from the literature satisfy (I4). We denote by GI the class of GI operators, and by DP the class of DP operators satisfying (I4). Remark that $\mathsf{DP} \subseteq \mathsf{GI} \subseteq \mathsf{IBC}$.

Let us give some examples of IBC operators defined on the TPO-epistemic space $\mathcal{E}^{\mathsf{tpo}}$. The natural revision operator $(\mathcal{E}^{\mathsf{tpo}}, \circ_N)$ associates each TPO $\preceq$ and each formula μ with a TPO $\preceq'$ satisfying $\min(\Omega, \preceq') = \min([\mu], \preceq)$, and if $\omega, \omega' \notin \min([\mu], \preceq)$ then $\omega \preceq \omega'$ iff $\omega \preceq' \omega'$: it defines the first level of $\preceq'$ as the models of μ minimal according to $\preceq$ while leaving the remaining relationships unchanged (Boutilier 1996). The lexicographic revision $(\mathcal{E}^{\mathsf{tpo}}, \circ_L)$ ensures that the revised TPO $\preceq'$ satisfies $\omega \prec' \omega'$ whenever $\omega \in [\mu]$ and $\omega' \notin [\mu]$, and maintains the remaining relationships unchanged (Nayak 1994). The one-improvement operator $(\mathcal{E}^{\mathsf{tpo}}, \bullet_O)$ essentially shifts all models of μ one layer down, and creates a new bottom layer for $\min([\mu], \preceq)$ if $\min([\mu], \preceq) \subseteq \min(\Omega, \preceq)$ (see (Konieczny and Pino Pérez 2008) for a formal definition of $\bullet_O$.) It is well-known that $\circ_N, \circ_L \in \mathsf{DP}$ and $\bullet_O \in \mathsf{GI} \setminus \mathsf{DP}$.

Let us now introduce some examples of IBC operators defined on the OCF-epistemic space $\mathcal{E}^{\mathsf{ocf}}$. The first one, which we call the shifting operator $(\mathcal{E}^{\mathsf{ocf}}, \bullet_+)$, associates with any OCF κ and formula μ the OCF $\kappa' = \kappa \bullet_+ \mu$ defined by $\kappa'(\omega) = \kappa(\omega) - x$ if $\omega \in [\mu]$, otherwise $\kappa'(\omega) = \kappa(\omega) + 1 - x$, where $x = 0$ if $B^{\mathsf{ocf}}(\kappa) \wedge \mu \not\models \bot$, and $x = 1$ otherwise. This operator uniformly decreases the value of all μ-worlds in a given OCF κ by 1, and ensures by a normalization step that $\min\{\kappa'(\omega) \mid \omega \in \Omega\} = 0$, as required by the definition of an OCF. The second example of an operator $(\mathcal{E}^{\mathsf{ocf}}, \circ_+)$, introduced in (Schwind, Konieczny, and Pino Pérez 2022), essentially has the same behavior as $\bullet_+$ except that it adds a step to ensure that $\min\{\kappa'(\omega) \mid \omega \in \Omega\} = \min\{\kappa(\omega) \mid \omega \in [\mu]\}$ (see (Schwind, Konieczny, and Pino Pérez 2022) for a formal definition.) Lastly, let $(\mathcal{E}^{\mathsf{ocf}}, \circ'_N)$ be the IBC operator that defines κ' as $\kappa'(\omega) = 0$ if $\omega \in [\mu]$ and $\kappa(\omega) = \min\{\kappa(\omega') \mid \omega' \in [\mu]\}$, otherwise $\kappa'(\omega) = \kappa(\omega) + 1$. It is not difficult to verify that $\circ_+, \circ'_N \in \mathsf{DP}$ and $\bullet_+ \in \mathsf{GI} \setminus \mathsf{DP}$.

Figure 1 illustrates the behavior of these operators.

Strong equivalence between epistemic states. A *scenario* is a (possibly empty) sequence of formulae $\sigma = (\mu_1, \ldots, \mu_k)$ with $k \in \mathbb{N}$ (when $k = 0$, σ is the empty sequence). For a scenario σ and epistemic state Ψ, define $\Psi \circ \sigma$ inductively as $\Psi \circ \emptyset = \Psi$ and, for $k > 0$, $\Psi \circ \sigma = (\Psi \circ (\mu_1, \ldots, \mu_{k-1})) \circ \mu_k$. Thus, $\Psi \circ \sigma$ is the epistemic state obtained by successive revisions of Ψ by each formula in σ.

Let $\mathcal{E} = \langle U, B\rangle$, $\mathcal{E}' = \langle U', B'\rangle$ be two epistemic spaces, and $(\mathcal{E}, \circ)$, $(\mathcal{E}', \circ')$ be two IBC operators. Given $\Psi \in U$ and $\Psi' \in U'$, we say that Ψ is *strongly equivalent* to Ψ' wrt. $(\circ, \circ')$, denoted by $\Psi \equiv_{(\circ,\circ')} \Psi'$, if $\Psi \in U_\circ$, $\Psi' \in U'_{\circ'}$, and for each scenario σ, $B(\Psi \circ \sigma) \equiv B'(\Psi' \circ' \sigma)$. In the particular case when $\mathcal{E} = \mathcal{E}'$ and $\circ = \circ'$, we simply say that Ψ and Ψ' are strongly equivalent wrt. $\circ$ and simplify the notation $\Psi \equiv_{(\circ,\circ')} \Psi'$ into $\Psi \equiv_\circ \Psi'$ (Schwind, Konieczny, and Pino Pérez 2022). Then given an IBC operator $\circ$ on $\mathcal{E} = \langle U, B\rangle$, we denote by $U_{\equiv_\circ}$ the quotient of $U_\circ$ by the equivalence relation $\equiv_\circ$. That is, if $(\Psi)_{\equiv_\circ}$ denotes the equivalence class to which an epistemic state $\Psi \in U_\circ$ belongs[3], then $U_{\equiv_\circ} = \{(\Psi)_{\equiv_\circ} \mid \Psi \in U_\circ\}$. Lastly, in this paper, we sometimes take advantage of the axiom of choice and assume that when $\circ$ is an IBC operator on $\mathcal{E} = \langle U, B\rangle$, there exists a choice function on $U_{\equiv_\circ}$ associating every equivalence class $V \in U_{\equiv_\circ}$ with a specific epistemic state of V, which is denoted by Ψ_V.

The notion of strong equivalence between epistemic states plays a central role. Each epistemic state, regardless of the epistemic space it belongs to, can be viewed as a "black box" that maps any scenario to a formula representing the agent's resulting beliefs. While epistemic states may contain additional internal structure, only this observable behavior, i.e., the beliefs they yield after the change, is relevant when analyzing iterated belief change. Accordingly, two epistemic states are considered strongly equivalent if they behave identically under two given IBC operators; that is, they are indistinguishable with respect to the considered belief change operator $\circ$.

Equivalence between IBC operators. The notion of strong equivalence between epistemic states extends naturally to an equivalence between operators over their respective epistemic spaces. Introduced in (Schwind, Konieczny, and Pino Pérez 2022) for revision, we adapt it here to the general setting of IBC operators:

Definition 1 (Equivalence between IBC operators)**.** *Let $\mathcal{E} = \langle U, B\rangle$, $\mathcal{E}' = \langle U', B'\rangle$ be two epistemic spaces, and $(\mathcal{E}, \circ)$, $(\mathcal{E}', \circ')$ be two IBC operators. We say that $\circ$ and $\circ'$ are* equivalent*, written $\circ \equiv \circ'$ (or $(\mathcal{E}, \circ) \equiv (\mathcal{E}', \circ')$), if there is a one-to-one correspondence g from $U_{\equiv_\circ}$ to $U'_{\equiv_{\circ'}}$ such that for each $V \in U_{\equiv_\circ}$ and for all $\Psi \in V$, $\Psi' \in g(V)$, we have that $\Psi \equiv_{(\circ,\circ')} \Psi'$.*

This notion of equivalence between operators is crucial for identifying cases where two operators behave identically. In such situations, they can be regarded as instances of the same operator (possibly defined over different epistemic spaces) just as logically equivalent formulae express the same semantic content despite syntactic differences.

It is straightforward to verify that the equivalence relation between IBC operators, as defined in Definition 1, is well-formed: it is reflexive, symmetric, and transitive.

[3] $(\Psi)_{\equiv_\circ} = \{\Psi' \in U_\circ \mid \Psi' \equiv_\circ \Psi\}$

3 A Characterization of Equivalence Between IBC Operators

In (Schwind, Konieczny, and Pino Pérez 2022), the notion of *safe translation* was introduced to build bridges between two epistemic spaces, and was used to prove equivalence between revision operators defined on possibly different spaces. However, that notion was formulated specifically for DP operators and was incomplete, as it did not capture the full equivalence relation. We provide here a refined version that applies to all IBC operators:

Definition 2 (Safe weak translation)**.** *Let $\mathcal{E} = \langle U, B\rangle$, $\mathcal{E}' = \langle U', B'\rangle$ be two epistemic spaces, and $(\mathcal{E}, \circ)$, $(\mathcal{E}', \circ')$ be two IBC operators. A mapping $f : U_\circ \to U'$ is said to be a* weak translation from $\circ$ to $\circ'$ *if for each $\Psi \in U_\circ$, $B(\Psi) \equiv B'(f(\Psi))$ and for each $\mu \in \mathcal{L}_*$, $f(\Psi \circ \mu) \equiv_{\circ'} f(\Psi) \circ' \mu$. A weak translation is said to be* safe *if there exists a weak translation $f' : U'_{\circ'} \to U$ from $\circ'$ to $\circ$.*

When a safe weak translation exists, we explicitly indicate both mappings f and f', and write (f, f') to denote a safe weak translation for $(\circ, \circ')$.

This notion of safe weak translation is reminiscent of bisimulation in labeled transition systems (Baier and Katoen 2008), and more generally of standard notions used in modal logic (Blackburn, de Rijke, and Venema 2001; van Benthem 2010). It provides a concrete characterization of equivalence between IBC operators:

Proposition 1. *Let $(\mathcal{E}, \circ)$, $(\mathcal{E}', \circ')$ be two IBC operators. We have that $\circ \equiv \circ'$ if and only if there exists a safe weak translation (f, f') for $(\circ, \circ')$.*

Proposition 1 yields a constructive criterion for operator equivalence. Rather than relying solely on the abstract relation of Definition 1, one can exhibit a pair of weak translations (f, f') that explicitly relate epistemic states across the two spaces. This perspective will be instrumental in several subsequent results. As an illustration, it allows us to show that the natural revision operator $(\mathcal{E}^{\mathsf{tpo}}, \circ_N)$ and the operator $(\mathcal{E}^{\mathsf{ocf}}, \circ'_N)$ introduced in the preliminaries are equivalent:

Proposition 2. $(\mathcal{E}^{\mathsf{tpo}}, \circ_N) \equiv (\mathcal{E}^{\mathsf{ocf}}, \circ'_N)$.

In other words, $\circ_N$ and $\circ'_N$ implement the same revision policy; they differ only in the epistemic space used to represent states.

4 Canonicity

We now introduce a central notion: *canonicity*, which allows us to abstract away from representational redundancy and to consider epistemic spaces that exactly fit the IBC operator in the following sense:

Definition 3 (Canonicity)**.** *An IBC operator $(\mathcal{E}, \circ)$ is said to be* canonically defined *if $U = U_\circ$, and for all $\Psi, \Psi' \in U_\circ$, if $\Psi \neq \Psi'$ then $\Psi \not\equiv_\circ \Psi'$. We say that an epistemic space $\mathcal{E}' = \langle U', B'\rangle$ is* $\circ$-canonical *if there exists a canonically defined IBC operator $(\mathcal{E}', \circ')$ such that $\circ' \equiv \circ$.*

When an IBC operator is canonically defined, each epistemic state corresponds to a distinct behavioral profile w.r.t. this operator. This means that the epistemic space contains no two states that induce the same change dynamics.

Canonicity therefore captures a "minimality" requirement: the epistemic space contains exactly as many states as are needed to represent the operator up to behavioral equivalence.

We now reconsider the DP and GI operators introduced in the preliminaries:

Proposition 3.

- $(\mathcal{E}^{\mathsf{tpo}}, \circ_N)$, $(\mathcal{E}^{\mathsf{tpo}}, \circ_L)$, $(\mathcal{E}^{\mathsf{tpo}}, \bullet_O)$, $(\mathcal{E}^{\mathsf{ocf}}, \circ_+)$ *and* $(\mathcal{E}^{\mathsf{ocf}}, \bullet_+)$ *are canonically defined.*
- $(\mathcal{E}^{\mathsf{ocf}}, \circ'_N)$ *is not canonically defined.*

The second item follows from the fact that the OCF representation contains redundancies for $\circ'_N$. Indeed, any two OCFs κ and κ' that induce the same TPO (i.e., such that $\kappa(\omega) \le \kappa(\omega')$ iff $\kappa'(\omega) \le \kappa'(\omega')$ for all $\omega, \omega' \in \Omega$) are strongly equivalent with respect to $\circ'_N$. Thus, distinct epistemic states may be strongly equivalent, so $(\mathcal{E}^{\mathsf{ocf}}, \circ'_N)$ is not canonically defined.

The fact that $(\mathcal{E}^{\mathsf{tpo}}, \circ_N)$, $(\mathcal{E}^{\mathsf{tpo}}, \circ_L)$, and $(\mathcal{E}^{\mathsf{tpo}}, \bullet_O)$ are canonically defined is a direct consequence of the following lemma:

Lemma 1. *If* $(\mathcal{E}^{\mathsf{tpo}}, \circ) \in \mathsf{GI}$ *and* $U_\circ^{\mathsf{tpo}} = U^{\mathsf{tpo}}$*, then* $(\mathcal{E}^{\mathsf{tpo}}, \circ)$ *is canonically defined.*

This lemma shows that the standard TPO-based representation of GI operators is already canonical: as long as every total preorder lies in the domain, distinct TPOs necessarily induce distinct behavior. It follows immediately that the GI operators $\circ_N$, $\circ_L$, and $\bullet_O$ are canonically defined on $\mathcal{E}^{\mathsf{tpo}}$.

Finally, the fact that $\circ_+$ and $\bullet_+$ are canonically defined on $\mathcal{E}^{\mathsf{ocf}}$ implies that for any two distinct OCFs κ and κ', there exists a change scenario in which the operators behave differently: $\kappa \not\equiv_{\circ_+} \kappa'$ and $\kappa \not\equiv_{\bullet_+} \kappa'$.

Definition 4 (Canonization)**.** *Let* $\mathcal{E} = \langle U, B\rangle$ *be an epistemic space, and* $(\mathcal{E}, \circ)$ *be an IBC operator:*

- *The* canonization of $\mathcal{E}$ w.r.t. $\circ$ *is the epistemic space* $\mathcal{E}_{\circ_\mathcal{C}} = \langle U_{\circ_\mathcal{C}}, B_{\circ_\mathcal{C}}\rangle$*, where* $U_{\circ_\mathcal{C}} = U_{\equiv_\circ}$ *and for each* $V \in U_{\equiv_\circ}$*,* $[B_{\circ_\mathcal{C}}(V)] = [B(\Psi_V)]$ [4]
- *The* canonization of $\circ$ *is the IBC operator* $(\mathcal{E}_{\circ_\mathcal{C}}, \circ_\mathcal{C})$ *defined for each* $V \in U_{\equiv_\circ}$ *and each formula* $\mu \in \mathcal{L}_*$ *as* $V \circ_\mathcal{C} \mu = (\Psi_V \circ \mu)_{\equiv_\circ}$

Canonization removes all behavioral redundancy from an epistemic space by identifying strongly equivalent states. The resulting operator is defined on a smaller space but preserves exactly the same change behavior. This is captured formally by the following proposition:

Proposition 4. *Let* $(\mathcal{E}, \circ)$ *be any IBC operator. The canonization* $(\mathcal{E}_{\circ_\mathcal{C}}, \circ_\mathcal{C})$ *of* $\circ$ *is canonically defined, and* $\circ_\mathcal{C} \equiv \circ$*.*

By Proposition 4, the set $U_{\circ_\mathcal{C}}$ introduced in Definition 4 coincides with the domain of $\circ$ in the sense of Definition 3. The notation $U_{\circ_\mathcal{C}}$ is therefore unambiguous.

[4] Recall that Ψ_V is a representative epistemic state of the equivalence class V determined by an arbitrary choice function (see Preliminaries).

More importantly, this proposition guarantees that every IBC operator admits an equivalent representation on a canonical epistemic space. Thus, canonicity is not a restriction on operators, but a kind of normal form for their representation.

Example 1. *Figure 2 illustrates the canonization process on a concrete IBC operator* $(\mathcal{E}, \circ)$*. The operator (shown at the top of the figure) is defined on the epistemic space* $\mathcal{E} = \langle U, B\rangle$*, where* $U = \{\Psi^0, \ldots, \Psi^9\}$*, and:*

$$\begin{aligned}
&[B(\Psi^0)] = [B(\Psi^1)] = [B(\Psi^2)] = S\\
&[B(\Psi^4)] = [B(\Psi^5)] = [B(\Psi^6)] = [B(\Psi^7)] = S'\\
&[B(\Psi^3)] = [B(\Psi^8)] = [B(\Psi^9)] = S''
\end{aligned}$$

The graph depicts transitions $\Psi^i \circ \mu$ *for* $\Psi^i \in U$ *and* $\mu \in \{\alpha, \beta, \gamma\}$*. We assume that* $\Psi^3 \notin U_\circ$ *and that, for every* $i \in \{0, \ldots, 9\} \setminus \{3\}$ *and every* $\mu \in \mathcal{L}_* \setminus \{\alpha, \beta, \gamma\}$*,* $\Psi^i \circ \mu = \Psi^i$*. These identity transitions are omitted from the figure for readability. Vertices correspond to epistemic states, and labeled edges represent change steps.*

Dashed rectangles indicate the equivalence classes in $U_{\equiv_\circ}$*, which consists of five elements:*

$$\begin{aligned}
&(\Psi^0)_{\equiv_\circ} = (\Psi^1)_{\equiv_\circ} = \{\Psi^0, \Psi^1\}\\
&(\Psi^2)_{\equiv_\circ} = \{\Psi^2\}\\
&(\Psi^4)_{\equiv_\circ} = (\Psi^5)_{\equiv_\circ} = \{\Psi^4, \Psi^5\}\\
&(\Psi^6)_{\equiv_\circ} = (\Psi^7)_{\equiv_\circ} = \{\Psi^6, \Psi^7\}\\
&(\Psi^8)_{\equiv_\circ} = (\Psi^9)_{\equiv_\circ} = \{\Psi^8, \Psi^9\}
\end{aligned}$$

For example, $\Psi^0 \equiv_\circ \Psi^1$ *since for every scenario* σ*,* $[B(\Psi^0 \circ \sigma)] = [B(\Psi^1 \circ \sigma)] = S$*. Similarly,* $\Psi^4 \equiv_\circ \Psi^5$ *because for every* $\mu \in \mathcal{L}_*$*,* $(\Psi^4 \circ \mu)_{\equiv_\circ} = (\Psi^5 \circ \mu)_{\equiv_\circ}$*. Note that* Ψ^3 *does not belong to any equivalence class, since* $\Psi^3 \notin U_\circ$*.*

The bottom part of the figure shows the canonized operator $(\mathcal{E}_{\circ_\mathcal{C}}, \circ_\mathcal{C})$*. The canonized epistemic space is* $\mathcal{E}_{\circ_\mathcal{C}} = \langle U_{\circ_\mathcal{C}}, B_{\circ_\mathcal{C}}\rangle$*, with* $U_{\circ_\mathcal{C}} = \{\Psi^{01}, \Psi^2, \Psi^{45}, \Psi^{67}, \Psi^{89}\}$*, where each state denotes the corresponding equivalence class. For each* $\Psi^i \in U_\circ$ *and each scenario* σ*, we have* $\Psi^i \equiv_{(\circ, \circ_\mathcal{C})} (\Psi^i)_{\equiv_\circ}$*; for instance,* $(\Psi^5 \circ \beta) \circ \gamma = \Psi^9$ *and* $(\Psi^{45} \circ_\mathcal{C} \beta) \circ_\mathcal{C} \gamma = \Psi^{89}$*.*

We now introduce the notion of *regularity*, which characterizes exactly those epistemic spaces that can serve as canonical domains for some IBC operator:

Definition 5 (Regular Epistemic Space)**.** *An epistemic space* $\mathcal{E} = \langle U, B\rangle$ *is said to be* regular *if there exists a canonically defined IBC operator* $(\mathcal{E}, \circ)$*.*

Let $\mathcal{S}p_*$ denote the class of all regular epistemic spaces.

Proposition 5. *An epistemic space* $\mathcal{E} = \langle U, B\rangle$ *is regular if and only if* $Card(U) \le \mathfrak{c}$*.*

Proposition 5 shows that regularity is entirely determined by cardinality. An epistemic space admits a canonical IBC operator if and only if its set of states has cardinality at most that of the continuum.

5 Expressiveness of Epistemic Spaces

We now build on the notions of canonical and regular epistemic spaces introduced in the previous section to compare

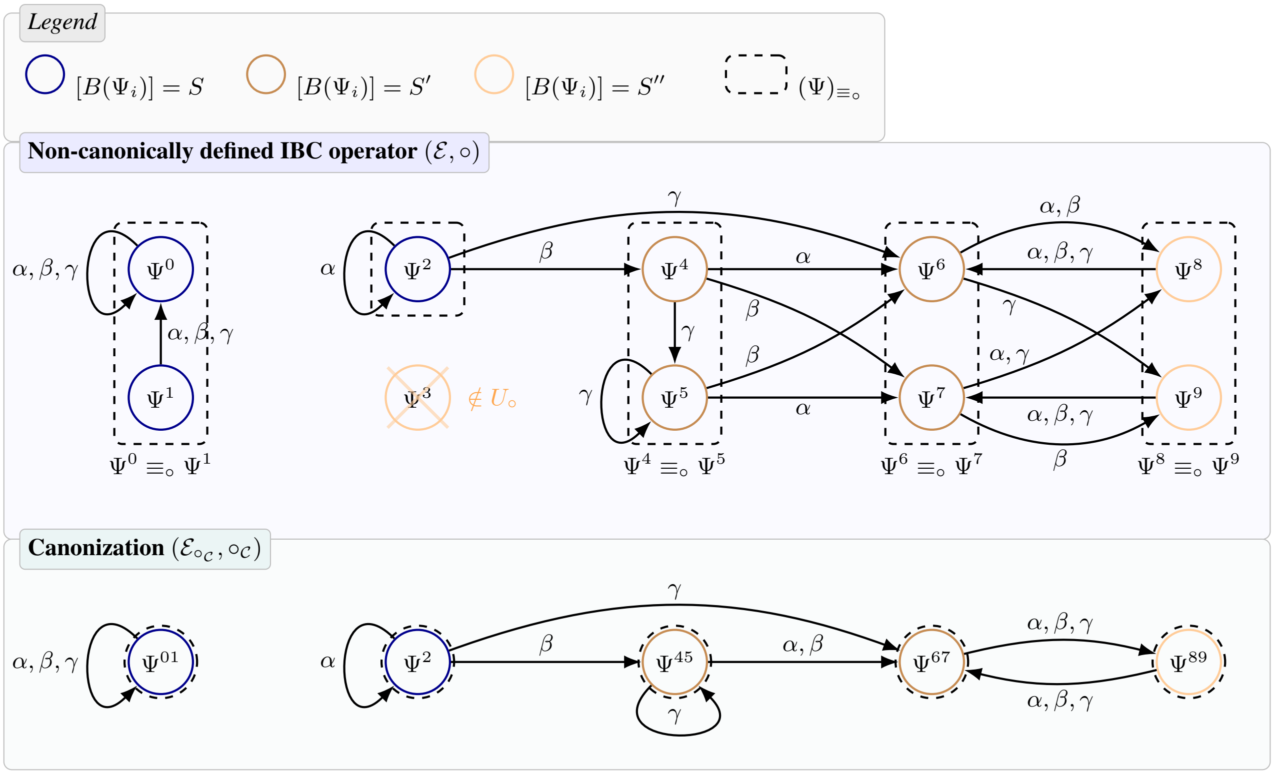

Figure 2: Graphical representations of (top) an IBC operator $(\mathcal{E}, \circ)$, projected to α, β, γ, and (bottom) its canonization $(\mathcal{E}_{\circ_\mathcal{C}}, \circ_\mathcal{C})$.

the expressiveness of different epistemic spaces. Intuitively, an epistemic space $\mathcal{E}$ is at least as expressive as another space $\mathcal{E}'$ if it has sufficient structure to realize every IBC operator that can be defined on $\mathcal{E}'$. This comparison is based on the notion of *realizability*, introduced in (Sauerwald and Thimm 2024) for (non-iterated) belief change operators[5], which asks whether a given belief change operator can be instantiated on a given epistemic space.

Definition 6 (Realizability)**.** *Let $(\mathcal{E}, \circ)$ be an IBC operator, and $\mathcal{E}' \in \mathcal{S}p$. We say that $\circ$ is* realizable *on $\mathcal{E}'$, equivalently, $\mathcal{E}'$* realizes *$\circ$, if there exists an IBC operator $(\mathcal{E}', \circ')$ such that $\circ' \equiv \circ$.*

We denote by $\mathcal{R}(\circ)$ the class of epistemic spaces that realize $\circ$.

Definition 7 (Expressiveness)**.** *Let $\mathcal{E}, \mathcal{E}' \in \mathcal{S}p$. We say that $\mathcal{E}'$ is more expressive than $\mathcal{E}$, denoted by $\mathcal{E} \preceq \mathcal{E}'$, if each IBC operator on $\mathcal{E}$ is realizable on $\mathcal{E}'$. And we say that $\mathcal{E}$ and $\mathcal{E}'$ are* equally expressive *(or simply, equivalent), denoted by $\mathcal{E} \equiv \mathcal{E}'$, if $\mathcal{E} \preceq \mathcal{E}'$ and $\mathcal{E}' \preceq \mathcal{E}$.*

By definition, $(\mathcal{S}p, \preceq)$ forms a partial preorder.

A natural question is then: *what makes one epistemic space more expressive than another?* To answer this, we introduce a simple structural measure capturing how many epistemic states have the same beliefs. Given an epistemic space $\mathcal{E} = \langle U, B\rangle$, its *degree* is the function $\theta_\mathcal{E}$ that assigns to each non-empty set of possible worlds $S \in \mathcal{P}(\Omega)_*$ a cardinal number

$$\theta_\mathcal{E}(S) = Card(\{\Psi \in U \mid [B(\Psi)] = S\})$$

Intuitively, $\theta_\mathcal{E}(S)$ counts how many epistemic states in $\mathcal{E}$ have beliefs whose models are exactly S.

Given two epistemic spaces $\mathcal{E}$ and $\mathcal{E}'$, their degrees $\theta_\mathcal{E}$ and $\theta_{\mathcal{E}'}$ can be compared pointwise: we write $\theta_\mathcal{E} \preceq_\theta \theta_{\mathcal{E}'}$ whenever for every $S \in \mathcal{P}(\Omega)_*$, $\theta_\mathcal{E}(S) \leq \theta_{\mathcal{E}'}(S)$. This relation is a partial order on the class of all degrees $\{\theta_\mathcal{E} \mid \mathcal{E} \in \mathcal{S}p\}$.

Example 1 (continued)**.** *Consider again the IBC operator $(\mathcal{E}, \circ)$ and its canonization $(\mathcal{E}_{\circ_\mathcal{C}}, \circ_\mathcal{C})$. The degree of $\mathcal{E}$ is given by:*

$$\begin{aligned} \theta_\mathcal{E}(S) &= |\{\Psi^0, \Psi^1, \Psi^2\}| = 3 \\ \theta_\mathcal{E}(S') &= |\{\Psi^4, \Psi^5, \Psi^6, \Psi^7\}| = 4 \\ \theta_\mathcal{E}(S'') &= |\{\Psi^3, \Psi^8, \Psi^9\}| = 3 \end{aligned}$$

Similarly, the degree of $\mathcal{E}_{\circ_\mathcal{C}}$ is:

$$\begin{aligned} \theta_{\mathcal{E}_{\circ_\mathcal{C}}}(S) &= |\{\Psi^{01}, \Psi^2\}| = 2 \\ \theta_{\mathcal{E}_{\circ_\mathcal{C}}}(S') &= |\{\Psi^{45}, \Psi^{67}\}| = 2 \\ \theta_{\mathcal{E}_{\circ_\mathcal{C}}}(S'') &= |\{\Psi^{89}\}| = 1 \end{aligned}$$

Thus, $\theta_{\mathcal{E}_{\circ_\mathcal{C}}} < \theta_\mathcal{E}$.

More generally, since epistemic states in $\mathcal{E}_{\circ_\mathcal{C}}$ correspond to strong equivalence classes, canonization always yields an epistemic space of smaller or equal degree. This suggests

[5]The term has also been used in other contexts, such as argumentation (Dunne et al. 2015), with a different meaning.

the following claim: if an epistemic space $\mathcal{E}'$ has a smaller degree than another space $\mathcal{E}$, then every IBC operator defined on $\mathcal{E}'$ should be realizable on $\mathcal{E}$, whereas the converse need not hold. This claim holds, as formalized by the next result:

Proposition 6. *Let $\mathcal{E}, \mathcal{E}' \in \mathcal{S}p$. If $\theta_{\mathcal{E}} \preceq_\theta \theta_{\mathcal{E}'}$, then $\mathcal{E} \preceq \mathcal{E}'$.*

For regular epistemic spaces, the converse implication also holds. This relies on the following proposition:

Proposition 7. *Let $\mathcal{E}, \mathcal{E}' \in \mathcal{S}p$. If $\mathcal{E}$ is canonical for $\circ$ and $\mathcal{E}' \in \mathcal{R}(\circ)$, then $\theta_{\mathcal{E}} \preceq_\theta \theta_{\mathcal{E}'}$.*

Proposition 7 shows that canonical spaces are minimal with respect to degree: the degree of a canonical space for $\circ$ is always less than or equal to that of any other epistemic space that can realize an equivalent operator. Using this result, we obtain the following converse to Proposition 6 for regular spaces.

Proposition 8. *Let $\mathcal{E} \in \mathcal{S}p_*$ and $\mathcal{E}' \in \mathcal{S}p$. If $\mathcal{E} \preceq \mathcal{E}'$, then $\theta_{\mathcal{E}} \preceq_\theta \theta_{\mathcal{E}'}$.*

Propositions 6 and 8 together yield the following characterization:

Theorem 1. *Let $\mathcal{E} \in \mathcal{S}p_*$ and $\mathcal{E}' \in \mathcal{S}p$. Then*

$$\mathcal{E} \preceq \mathcal{E}' \text{ iff } \theta_{\mathcal{E}} \preceq_\theta \theta_{\mathcal{E}'}$$

Theorem 1 shows that, for regular epistemic spaces, expressiveness is completely captured by degree. This leads to the following immediate consequences:

Corollary 1. *Let $\mathcal{E}, \mathcal{E}', \mathcal{E}'' \in \mathcal{S}p$, and $(\mathcal{E}'', \circ) \in \mathsf{IBC}$. Then:*

1. *if $\mathcal{E} \in \mathcal{S}p_*$, then $\mathcal{E} \equiv \mathcal{E}'$ iff $\theta_{\mathcal{E}} = \theta_{\mathcal{E}'}$*
2. *if $\mathcal{E}$ is canonical for $\circ$ and $\mathcal{E}' \in \mathcal{R}(\circ)$, then $\mathcal{E} \preceq \mathcal{E}'$*

Corollary 1.1 states that two regular epistemic spaces are equally expressive exactly when they have the same degree. Corollary 1.2 states that canonical spaces are the least expressive spaces that can still realize their corresponding operators.

We now illustrate how degrees can be computed for well-known epistemic spaces and used to compare their expressiveness. We begin with the TPO-epistemic space $\mathcal{E}^{\mathsf{tpo}}$ and the OCF-epistemic space $\mathcal{E}^{\mathsf{ocf}}$. Assume that the propositional language $\mathcal{L}$ is built from n propositional variables. The degree of $\mathcal{E}^{\mathsf{tpo}}$ is given, for each $S \in \mathcal{P}(\Omega)_*$, by:

$$\theta_{\mathcal{E}^{\mathsf{tpo}}}(S) = \begin{cases} 1 & \text{if } S = \Omega \\ \sum_{p=1}^{2^n-k} \left(\sum_{j=0}^{p} (-1)^{p-j} \binom{p}{j} j^{2^n-k}\right) & \text{otherwise} \end{cases}$$

Likewise, the degree of $\mathcal{E}^{\mathsf{ocf}}$ is:

$$\theta_{\mathcal{E}^{\mathsf{ocf}}}(S) = \begin{cases} 1 & \text{if } S = \Omega \\ \aleph_0 & \text{otherwise} \end{cases}$$

Besides $\mathcal{E}^{\mathsf{tpo}}$ and $\mathcal{E}^{\mathsf{ocf}}$, we also consider the following epistemic spaces:

• $\mathcal{E}^{\mathsf{lo}} = \langle U^{\mathsf{lo}}, B^{\mathsf{lo}} \rangle$, where $U^{\mathsf{lo}} \subseteq U^{\mathsf{tpo}}$ is the set of all linear orders over Ω, and for each linear order $\Psi \in U^{\mathsf{lo}}$, $B^{\mathsf{lo}}(\Psi) = B^{\mathsf{tpo}}(\Psi)$. Its degree is, for each $S \in \mathcal{P}(\Omega)_*$:

$$\theta_{\mathcal{E}^{\mathsf{lo}}}(S) = \begin{cases} (2^n - 1)! & \text{if } |S| = 1 \\ 0 & \text{otherwise} \end{cases}$$

This space can be used to define IBC operators in which an agent always strictly discriminates between any two worlds $\omega, \omega' \in \Omega$ in terms of relative plausibility. In such operators, ties between worlds never occur, so total preorders that are not linear orders do not take place.

• $\mathcal{E}^{\mathsf{rat}} = \langle U^{\mathsf{rat}}, B^{\mathsf{rat}} \rangle$, where U^{rat} consists of all mappings $r : \Omega \to \mathbb{Q}$, and $B^{\mathsf{rat}}(r) = \psi_{\mathsf{argmax}(\{r(\omega) \mid \omega \in \Omega\})}$. This space is known as the space of *rational rankings*. It was introduced in (Borges et al. 2024) to capture the dynamics of IBC operators in a flexible way. Compared to $\mathcal{E}^{\mathsf{ocf}}$, it allows one to insert intermediate "layers" of plausibility without modifying the relative ranking of other worlds. The degree of this space is, for each $S \in \mathcal{P}(\Omega)_*$:

$$\theta_{\mathcal{E}^{\mathsf{rat}}}(S) = \aleph_0$$

• $\mathcal{E}^{\mathsf{c}} = \langle U^{\mathsf{c}}, B^{\mathsf{c}} \rangle$, where $U^{\mathsf{c}} = U^{\mathsf{tpo}} \times \mathbb{B}^{\mathbb{N}}$ and $\mathbb{B}^{\mathbb{N}}$ is the set of all infinite binary sequences. For each $(\preceq, \delta) \in U^{\mathsf{c}}$, $B^{\mathsf{c}}((\preceq, \delta)) = B^{\mathsf{tpo}}(\preceq)$ (Schwind, Konieczny, and Pino Pérez 2022). The degree of this space is, for each $S \in \mathcal{P}(\Omega)_*$:

$$\theta_{\mathcal{E}^{\mathsf{c}}}(S) = \mathfrak{c}$$

This space is strictly larger than the previous ones, so distinct epistemic states can encode finer distinctions between change scenarios. Intuitively, in an epistemic state $(\preceq, \delta)$, the TPO $\preceq$ represents the agent's current plausibility ordering over worlds, while the binary sequence δ stores additional conditional information that guides how future revisions are performed.

By Proposition 5, all these epistemic spaces are regular. Comparing their degrees and applying Theorem 1, we obtain the following expressiveness hierarchy:

Theorem 2. $\mathcal{E}^{\mathsf{lo}} \prec \mathcal{E}^{\mathsf{tpo}} \prec \mathcal{E}^{\mathsf{ocf}} \prec \mathcal{E}^{\mathsf{rat}} \prec \mathcal{E}^{\mathsf{c}}$

Theorem 2 establishes a strict ordering of expressiveness among several standard epistemic spaces. Each strict inequality indicates the existence of IBC operators realizable in the space on the right but not in the space on the left. While some of these relationships were already known in specific cases, e.g., $\mathcal{E}^{\mathsf{tpo}} \prec \mathcal{E}^{\mathsf{ocf}}$ (Aravanis, Peppas, and Williams 2019; Schwind, Konieczny, and Pino Pérez 2022), by exhibiting operators that cannot be instantiated in the less expressive epistemic space. Our framework provides a direct and uniform method for deriving all of them.

As a final illustration, we show that different epistemic spaces may be used interchangeably when they have the same degree. Instead of working with the epistemic space of rational rankings $\mathcal{E}^{\mathsf{rat}}$, one can equivalently work with the space of free OCFs. Free OCFs are non-normalized OCFs, and the beliefs correspond to the worlds with minimal ranking (instead of 0 for standard OCFs): let $\mathcal{E}^{\mathsf{Focf}} = \langle U^{\mathsf{Focf}}, B^{\mathsf{Focf}} \rangle$, where U^{Focf} consists of all mappings from

Ω to the non-negative integers, and B^{Focf} maps each κ to $\psi_{\mathsf{argmin}(\{\kappa(\omega)|\omega\in\Omega\})}$ (Konieczny and Pino Pérez 2008).

It is straightforward to verify that $\theta_{\mathcal{E}^{\mathsf{rat}}} = \theta_{\mathcal{E}^{\mathsf{Focf}}}$. By Corollary 1.1, it follows that $\mathcal{E}^{\mathsf{rat}} \equiv \mathcal{E}^{\mathsf{Focf}}$.

Finally, Corollary 1 implies that all canonical spaces for a given operator have the same degree. As a consequence, degree is an invariant of the operator itself. Since every IBC operator can be canonized (Definition 4, Proposition 4), we can lift the notion of degree from epistemic spaces to IBC operators:

Definition 8 (Degree of an IBC Operator)**.** *Let $(\mathcal{E}, \circ)$ be an IBC operator. The* degree of $\circ$*, denoted by $\theta_\circ$, is the degree of the canonization of $\mathcal{E}$ w.r.t. $\circ$, i.e., $\theta_\circ = \theta_{\mathcal{E}_{\circ_C}}$ $(= \theta_{\circ_C})$.*

The degree of an operator thus reflects the expressive power required to realize it. For instance, from Propositions 2 and 3 we obtain:

Corollary 2.

- *$\circ_N$, $\circ_L$, $\bullet_O$ and $\circ'_N$ have degree $\theta_{\mathcal{E}^{\mathsf{tpo}}}$*
- *$\circ_+$ and $\bullet_+$ have degree $\theta_{\mathcal{E}^{\mathsf{ocf}}}$*

Since $\mathcal{E}^{\mathsf{tpo}} \prec \mathcal{E}^{\mathsf{ocf}}$ (Theorem 2), it follows that $\circ_+$ and $\bullet_+$ are strictly "more expressive" than $\circ_N$ (equivalently $\circ'_N$), $\circ_L$, and $\bullet_O$, in the following sense: there exist belief change scenarios that the more expressive operators can treat differently, while the less expressive operators necessarily treat them in the same way. Equivalently, an agent equipped with $\circ_+$ or $\bullet_+$ can represent and act on finer-grained distinctions between revision histories than an agent restricted to $\circ_N$, $\circ_L$, or $\bullet_O$.

6 Minimal and Universal Epistemic Spaces

Having developed the tools to compare the expressiveness of epistemic spaces and to characterize these relationships via their degrees, we now address four central questions for a given class C of IBC operators, which will then be instantiated for relevant classes (e.g., DP, GI):

1. What are the spaces capable of realizing *at least one* operator from C?
2. Among them, what are the least expressive spaces?
3. What are the spaces capable of realizing *all* operators from C?
4. Among them, is there a least expressive one?

Question 1 concerns realization of some operator in the class, while Question 2 asks for the minimal expressiveness required for such a realization. Question 3 asks for universality: spaces expressive enough to realize the *entire* class. Question 4 refines this further by asking whether there is a genuine lower bound among universal spaces, that is, a space whose degree exactly matches the expressive requirements of the class.

A positive answer to Question 4 is particularly informative: it identifies, up to $\equiv$, a minimally expressive universal space, i.e., a *canonical representative* of C. Such a space is both sufficient to realize every operator in C and necessary in the sense that no strictly less expressive space can be universal for C.

Definition 9 ($\exists(\mathsf{C})$ / $\forall(\mathsf{C})$)**.** *Let $\mathsf{C} \subseteq \mathsf{IBC}$. Then we define:*

- $\exists(\mathsf{C}) = \bigcup\{\mathcal{R}(\circ) \mid \circ \in \mathsf{C}\}$
- $\forall(\mathsf{C}) = \bigcap\{\mathcal{R}(\circ) \mid \circ \in \mathsf{C}\}$

We also denote by:

- $\exists_\star(\mathsf{C}) = \mathsf{Min}(\exists(\mathsf{C}), \preceq)$
- $\forall_\star(\mathsf{C}) = \mathsf{Min}(\forall(\mathsf{C}), \preceq)$

where $\mathsf{Min}(E, \preceq) = \{\mathcal{E} \in E \mid \nexists \mathcal{E}' \in E, \mathcal{E}' \prec \mathcal{E}\}$ denotes the set of $\preceq$-minimal elements of E.

The class $\exists(\mathsf{C})$ answers Question 1: it consists of the spaces that realize at least one operator in C. The class $\exists_\star(\mathsf{C})$ answers Question 2: it selects the $\preceq$-minimal spaces among them. The class $\forall(\mathsf{C})$ answers Question 3: it contains exactly the spaces that realize every operator in C. Finally, $\forall_\star(\mathsf{C})$ answers Question 4 by selecting the minimal elements of $\forall(\mathsf{C})$. If non-empty, $\forall_\star(\mathsf{C})$ contains exactly one epistemic space up to expressiveness equivalence.

We begin with the simplest case, which follows from very general considerations. Let $\mathbf{2}$ denote the class of epistemic spaces $\mathbf{2} = \{\mathcal{E} = \langle U, B\rangle \in \mathcal{S}p \mid Card(U) = 2\}$:

Proposition 9. $\exists(\mathsf{IBC}) = \{\mathcal{E} \in \mathcal{S}p \mid \exists \mathcal{E}' \in \mathbf{2},\ \mathcal{E}' \preceq \mathcal{E}\}$.

This case mainly serves as a baseline. More informative are the results for DP and GI, which already impose non-trivial expressiveness requirements.

Proposition 10. $\exists(\mathsf{GI}) = \exists(\mathsf{DP}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E}^{\mathsf{lo}} \preceq \mathcal{E}\}$.

This proposition shows that any space realizing at least one GI or DP operator must have degree at least $\theta_{\mathcal{E}^{\mathsf{lo}}}$, and that $\mathcal{E}^{\mathsf{lo}}$ already suffices. Thus, these classes require strictly more expressiveness than the full class IBC.

We now consider universal realization:

Proposition 11. $\forall(\mathsf{IBC}) = \forall(\mathsf{GI}) = \forall(\mathsf{DP}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E}^{\mathsf{c}} \preceq \mathcal{E}\}$.

This proposition has several important consequences. First, it shows that a universally expressive epistemic space for *all* IBC operators exists: the class $\forall(\mathsf{IBC})$ is non-empty. In particular, $\mathcal{E}^{\mathsf{c}} \in \forall(\mathsf{IBC})$, so $\mathcal{E}^{\mathsf{c}}$ can realize *every* IBC operator. Second, $\mathcal{E}^{\mathsf{c}}$ is not merely an example of a universal space: any space that realizes all operators in IBC, in GI, or in DP must be at least as expressive as $\mathcal{E}^{\mathsf{c}}$. In particular, since $\mathcal{E}^{\mathsf{c}} \in \forall(\mathsf{DP})$, there exist DP operators canonically defined on $\mathcal{E}^{\mathsf{c}}$. Together with Propositions 9 and 10, this yields the following minimality statements by taking the $\preceq$-minimal elements of the corresponding classes:

Theorem 3.

1. $\exists_\star(\mathsf{IBC}) = \mathbf{2}$
2. $\exists_\star(\mathsf{GI}) = \exists_\star(\mathsf{DP}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E} \equiv \mathcal{E}^{\mathsf{lo}}\}$
3. $\forall_\star(\mathsf{IBC}) = \forall_\star(\mathsf{GI}) = \forall_\star(\mathsf{DP}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E} \equiv \mathcal{E}^{\mathsf{c}}\}$

Thus, $\mathcal{E}^{\mathsf{lo}}$ is, up to $\equiv$, the least expressive space that realizes some operator in DP or GI, while $\mathcal{E}^{\mathsf{c}}$ is, up to $\equiv$, the least expressive space that realizes all operators in DP, GI, and IBC. The latter result is the more significant one: it identifies the exact expressiveness needed for a space to be universal for these classes.

We now refine the analysis by considering additional subclasses, combined with DP and GI.

	IBC	DP	DP ∩ TPO	DP ∩ $\mathsf{G}^{\equiv}$	DP ∩ $\mathsf{Gw}^{\equiv}$	GI	GI ∩ TPO	GI ∩ $\mathsf{G}^{\equiv}$	GI ∩ $\mathsf{Gw}^{\equiv}$
$\exists_\star(\mathsf{C})$	**2**	$\mathcal{E}^{\mathsf{lo}}$	$\mathcal{E}^{\mathsf{lo}}$	$\mathcal{E}^{\mathsf{tpo}}$	$\mathcal{E}^{\mathsf{tpo}}$	$\mathcal{E}^{\mathsf{lo}}$	$\mathcal{E}^{\mathsf{lo}}$	$\mathcal{E}^{\mathsf{tpo}}$	$\mathcal{E}^{\mathsf{tpo}}$
$\forall_\star(\mathsf{C})$	$\mathcal{E}^{\mathsf{c}}$	$\mathcal{E}^{\mathsf{c}}$	$\mathcal{E}^{\mathsf{tpo}}$	$\mathcal{E}^{\mathsf{ocf}}$	$\mathcal{E}^{\mathsf{rat}}$	$\mathcal{E}^{\mathsf{c}}$	$\mathcal{E}^{\mathsf{tpo}}$	$\mathcal{E}^{\mathsf{ocf}}$	$\mathcal{E}^{\mathsf{rat}}$

Table 1: Representative minimal spaces for realization and universal expressiveness.

- $\mathsf{TPO} = \{\circ \in \mathsf{IBC} \mid \mathcal{E}^{\mathsf{tpo}} \in \mathcal{R}(\circ)\}$: the class of IBC operators that are realizable in $\mathcal{E}^{\mathsf{tpo}}$
- $\mathsf{G}^{\equiv}$: the class of IBC operators for which: 1) there exists a unique equivalence class $(\Psi_\top)_{\equiv_\circ}$ such that $B(\Psi_\top) \equiv \top$, and 2) for each $\Psi \in U_\circ$, there exists a scenario σ such that $\Psi \equiv_\circ \Psi_\top \circ \sigma$
- $\mathsf{Gw}^{\equiv}$: the class of IBC operators for which there exists a state $\Psi_\top \in U_\circ$ such that ($B(\Psi_\top) \equiv \top$ and for each $\Psi \in U_\circ$, there exists a scenario σ such that $\Psi \equiv_\circ \Psi_\top \circ \sigma$)

The class TPO directly includes all operators defined on $\mathcal{E}^{\mathsf{tpo}}$, but it also includes operators that can be equivalently instantiated on $\mathcal{E}^{\mathsf{tpo}}$, such as $(\mathcal{E}^{\mathsf{ocf}}, \circ'_N)$ (cf. Proposition 2). Properties $\mathsf{G}^{\equiv}$ and $\mathsf{Gw}^{\equiv}$ both formalize the idea that all epistemic states are generated from an initial "empty" state $\Psi_\top$ by iterated change. The difference between $\mathsf{G}^{\equiv}$ and $\mathsf{Gw}^{\equiv}$ lies in the strength of the requirement. For $\mathsf{G}^{\equiv}$, the state $\Psi_\top$ must be unique up to strong equivalence. This property is a slight adaptation of the property G from (Schwind, Konieczny, and Pino Pérez 2022) which requires that the epistemic state $\Psi_\top$ must be unique. For $\mathsf{Gw}^{\equiv}$, uniqueness up to strong equivalence is not required: an agent may return to an empty state that is not strongly equivalent to the initial one, which allows, for instance, the design of agents whose change reluctance evolves over time (see (Perrotin and Schwind 2024) for examples).

Combining these subclasses with GI and DP yields the following expressiveness bounds:

Proposition 12.

1. $\exists(\mathsf{DP} \cap \mathsf{TPO}) = \exists(\mathsf{GI} \cap \mathsf{TPO}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E}^{\mathsf{lo}} \preceq \mathcal{E}\}$
2. $\exists(\mathsf{DP} \cap \mathsf{G}^{\equiv}) = \exists(\mathsf{GI} \cap \mathsf{G}^{\equiv}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E}^{\mathsf{tpo}} \preceq \mathcal{E}\}$
3. $\exists(\mathsf{DP} \cap \mathsf{Gw}^{\equiv}) = \exists(\mathsf{GI} \cap \mathsf{Gw}^{\equiv}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E}^{\mathsf{tpo}} \preceq \mathcal{E}\}$
4. $\forall(\mathsf{DP} \cap \mathsf{TPO}) = \forall(\mathsf{GI} \cap \mathsf{TPO}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E}^{\mathsf{tpo}} \preceq \mathcal{E}\}$
5. $\forall(\mathsf{DP} \cap \mathsf{G}^{\equiv}) = \forall(\mathsf{GI} \cap \mathsf{G}^{\equiv}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E}^{\mathsf{ocf}} \preceq \mathcal{E}\}$
6. $\forall(\mathsf{DP} \cap \mathsf{Gw}^{\equiv}) = \forall(\mathsf{GI} \cap \mathsf{Gw}^{\equiv}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E}^{\mathsf{rat}} \preceq \mathcal{E}\}$

From Proposition 12, we immediately obtain:

Theorem 4.

1. $\exists_\star(\mathsf{DP} \cap \mathsf{TPO}) = \exists_\star(\mathsf{GI} \cap \mathsf{TPO}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E} \equiv \mathcal{E}^{\mathsf{lo}}\}$
2. $\exists_\star(\mathsf{DP} \cap \mathsf{G}^{\equiv}) = \exists_\star(\mathsf{GI} \cap \mathsf{G}^{\equiv}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E} \equiv \mathcal{E}^{\mathsf{tpo}}\}$
3. $\exists_\star(\mathsf{DP} \cap \mathsf{Gw}^{\equiv}) = \exists_\star(\mathsf{GI} \cap \mathsf{Gw}^{\equiv}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E} \equiv \mathcal{E}^{\mathsf{tpo}}\}$
4. $\forall_\star(\mathsf{DP} \cap \mathsf{TPO}) = \forall_\star(\mathsf{GI} \cap \mathsf{TPO}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E} \equiv \mathcal{E}^{\mathsf{tpo}}\}$
5. $\forall_\star(\mathsf{DP} \cap \mathsf{G}^{\equiv}) = \forall_\star(\mathsf{GI} \cap \mathsf{G}^{\equiv}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E} \equiv \mathcal{E}^{\mathsf{ocf}}\}$
6. $\forall_\star(\mathsf{DP} \cap \mathsf{Gw}^{\equiv}) = \forall_\star(\mathsf{GI} \cap \mathsf{Gw}^{\equiv}) = \{\mathcal{E} \in \mathcal{S}p \mid \mathcal{E} \equiv \mathcal{E}^{\mathsf{rat}}\}$

It is worth noting that (Schwind, Konieczny, and Pino Pérez 2022) established that $\mathcal{E}^{\mathsf{ocf}}$ is universally expressive for DP ∩ G; because G is only a slight variant of $\mathsf{G}^{\equiv}$, this result coincides with point 5 of Proposition 12 for $\mathsf{DP} \cap \mathsf{G}^{\equiv}$. They also showed that $\mathcal{E}^{\mathsf{ocf}}$ is not universal for DP, which is here a consequence of Proposition 11. Our results extend this line of work by providing a comprehensive characterization of expressiveness for both DP and GI operators, including the subclasses based on TPO, $\mathsf{G}^{\equiv}$, and $\mathsf{Gw}^{\equiv}$, and by covering $\exists(\cdot)$ in addition to $\forall(\cdot)$. In particular, Theorem 3 establishes the existence of a universally expressive space for DP (namely $\mathcal{E}^{\mathsf{c}}$) and shows that it is canonical for the entire class IBC.

Table 1 summarizes Theorems 3 and 4. For each class C, it lists representative spaces, up to $\equiv$, for $\exists_\star(\mathsf{C})$ and $\forall_\star(\mathsf{C})$.

7 Conclusion

Iterated belief change (IBC) operators have been studied in several epistemic representations, most prominently the TPO-epistemic space $\mathcal{E}^{\mathsf{tpo}}$ and the OCF-epistemic space $\mathcal{E}^{\mathsf{ocf}}$. In $\mathcal{E}^{\mathsf{tpo}}$, epistemic states are total preorders over worlds and directly encode relative plausibility; in $\mathcal{E}^{\mathsf{ocf}}$, states are ranking functions that allow for a finer-grained representation of plausibility. Previous work has shown that $\mathcal{E}^{\mathsf{ocf}}$ is strictly more expressive than $\mathcal{E}^{\mathsf{tpo}}$ for DP operators (Aravanis, Peppas, and Williams 2019; Schwind, Konieczny, and Pino Pérez 2022), and also that $\mathcal{E}^{\mathsf{ocf}}$ is not expressive enough to instantiate all Darwiche-Pearl (DP) operators (Schwind, Konieczny, and Pino Pérez 2022). However, these results were obtained in a case-by-case manner and did not provide a general method for comparing the expressiveness of epistemic spaces or for identifying precise expressiveness bounds for operator classes.

This paper develops such a general framework. We introduced tools that allow the systematic comparison of expressiveness between epistemic spaces and studied the conditions under which an epistemic space is expressive enough to instantiate an IBC operator defined on another space. Three notions are central to this analysis: (i) canonization, which maps an IBC operator to an equivalent one defined on a minimal canonical space; (ii) a formal comparison of expressiveness, based on realizability of operators; and (iii) the degree of an epistemic space, which characterizes relative expressiveness for regular spaces.

Using this framework, we characterized, for IBC, DP, and General Improvement (GI) operators (as well as several subclasses), the minimally expressive spaces for each class, and the spaces that realize all its operators, which we call universal spaces. We also showed how to identify, among universal spaces, a canonical representative (up to expressiveness equivalence). These results determine the expressiveness thresholds associated with the different operator classes and make explicit how structural properties of operators translate into requirements on their epistemic domains.

A particularly important outcome is the characterization of the epistemic space $\mathcal{E}^{\mathsf{c}}$, whose states consist of a total preorder together with an infinite binary sequence encoding conditional information for future change scenarios. We proved that $\mathcal{E}^c$ is a canonical representative space for the full class of IBC operators: every such operator can be instantiated on it, and it is minimal among spaces with this universal property. Moreover, some DP operators (and therefore GI operators) are canonically defined on $\mathcal{E}^c$. Thus, $\mathcal{E}^c$ is not only a canonical representative space for all IBC operators, but also for both DP and GI operators. This closes a gap in the literature, where the existence of such a space had not previously been established.

Future directions include extending the analysis to other classes of iterated belief change, such as iterated contraction operators (Konieczny and Pino Pérez 2017), or incorporating additional structural properties, such as unbiasedness (Sauerwald and Thimm 2024), into the expressiveness map. More broadly, the framework developed here provides the tools needed to address long-standing open questions, including possible correspondences between iterated (DP) revision and iterated contraction in their canonical spaces, or between improvement and decrement operators (Sauerwald and Beierle 2019).

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AI Declaration

The authors used ChatGPT (OpenAI) for minor language editing, including English corrections and slight rephrasing to improve clarity and readability. No AI tools were used for generating scientific content, analysis, or conclusions.

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