Proceedings of the 23rd International Conference on Principles of Knowledge Representation and Reasoning
Main Track

Reasoning About Probabilities, Actions, and Knowledge in Fuzzy Modal Logic

Daniil Kozhemiachenko[1] , **Igor Sedlár**[2]
[1]Aix Marseille Univ, CNRS, LIS, Marseille, France
[2]Institute of Computer Science, The Czech Academy of Sciences, Prague, Czech Republic
daniil.kozhemiachenko@lis-lab.fr, sedlar@cs.cas.cz

Abstract

We explore a fuzzy modal logic that can formalise probabilistic reasoning about actions and knowledge. In particular, we deal with contexts involving statements about events expressed via modal formulas, e.g., 'after doing a, the probability of A knowing that p holds increases / decreases / is equal to 0.25', 'according to A, p is equally likely to happen after doing a or b', etc. We define the semantics of the logic on Kripke frames equipped with probability measures. We analyse the complexity of deciding the satisfiability of formulas of our logic over finitely branching models, for the full language and its fragments of varying expressivity. In particular, we identify several fragments of our logic where satisfiability is decidable in polynomial time.

1 Introduction

Probabilistic reasoning is a well-established field in artificial intelligence and knowledge representation and reasoning. It has found numerous applications in logic programming (Ng and Subrahmanian 1992; Lukasiewicz 1998; Riguzzi and Swift 2018), ontologies (Lukasiewicz 2008; Peñaloza 2020), and formal argumentation (Li, Oren, and Norman 2012; Fazzinga, Flesca, and Parisi 2015). An important approach to the formalisation of probabilistic reasoning is *probabilistic logics*. These are formal systems that formalise reasoning with probabilistic statements such as 'if the probability of p is twice as much as the probability of q, then the probability of q is less than 0.5'.

In this paper, we consider a fuzzy modal logic that can formalise probabilistic reasoning about actions and knowledge. We study the complexity of the full language and its fragments, including those for which satisfiability is decidable in polynomial time. This presents a perspective on the computational properties of formal verification and automated reasoning with probabilistic specifications that describe agents' knowledge and actions.

Probabilistic Logics Probabilistic logics were first proposed by Nilsson (1986). The propositional fragment of Nilsson's logic was further studied by Fagin et al. (1990). Later, Fagin and Halpern (1994) considered an expansion of probabilistic logic with epistemic modalities. Bacchus et al. (1999) propose a framework for a probabilistic logic of action and belief based on Reiter's situation calculus. Halpern and Pucella (2002) considered logics with operators expressing upper probabilities to model qualitative uncertainty about probabilities. Halpern and O'Neill (2005) apply probabilistic epistemic logic to reasoning about anonymity in multi-agent systems.

Since then, even more expressive systems of probabilistic logics have been extensively studied. In particular, Ognjanovic et al. (2012), Doder and Perovic (2020), and Doder and Ognjanovic (2024) consider temporal probabilistic logic. Tomovic et al. (2020) studied an epistemic expansion of the *first-order* probabilistic logic, and Doder et al. (2020) explored a multi-agent logic for reasoning about lower and upper probabilities.

Probabilistic Reasoning in Fuzzy Logics An important alternative approach to probabilistic logics is to use so-called *fuzzy logics* to reason about probabilities. Fuzzy logics were introduced by Zadeh (1965; 1975) to reason about imprecise statements such as 'it is cold outside', 'the symptoms are severe', etc. In fuzzy logics, formulas are evaluated over the interval $[0, 1]$, and their values are interpreted as *degrees of truth* from 0 (absolutely false) to 1 (absolutely true). Fuzzy logics have multiple applications in artificial intelligence. In particular, He et al. (2003) develop an autonomous bidding strategy based on fuzzy logic. Semantics of fuzzy logic is used to support learning and reasoning in real-world domains (Diligenti, Gori, and Saccà 2017; Badreddine et al. 2022) and statistical relational learning (Bach et al. 2017). Furthermore, van Krieken et al. (2022) analyse the interaction between different fuzzy logic semantics and learning. Chen et al. (2022) use fuzzy logic in reasoning problems with knowledge graphs.

Reasoning about uncertainty in fuzzy logics is usually formalised with *two-layered formulas*. Namely, the probability of an event α (*inner* or *event formula*) is the truth degree of the *probabilistic atom* — the formula $\mathsf{Pr}(\alpha)$, interpreted as 'α is probable'. Initially (Hájek and Harmancová 1995; Hájek, Godo, and Esteva 1995), the events were formalised using classical propositional formulas, and probabilistic atoms were combined with connectives of Łukasiewicz logic into *outer formulas*. Hájek et al. (2000) expand this framework into the language of Product fuzzy logic with rational constants (Esteva, Godo, and Montagna 2001) to express thresholds and conditional probabilities. It is shown by Baldi et al. (2020) that propositional fuzzy probabilistic logics have the same expressivity as logics of Fagin et al. (1990). It is argued that the same expressive power comes with a much simpler syntax and axiomatisation.

Modal expansions of fuzzy probabilistic logics have also been extensively studied. In particular, Godo et al. (2003) expressed Dempster–Shafer belief functions with formulas of the form $\mathsf{Pr}(\Box\alpha)$. Using the work of Marchioni (2008), Corsi et al. (2023) showed that lower probabilities can be expressed via formulas of the form $\Box\mathsf{Pr}(\alpha)$. They considered a logic where epistemic modalities are used to both express events and statements about their probabilities. Majer and Sedlár (2025) and Kozhemiachenko and Sedlár (2025) considered a fuzzy probabilistic modal logic with arbitrary $\Box$-like modalities and propositional event formulas, allowing for a broader model of lower probabilities where uncertainty is not necessarily epistemic in nature, but may arise as a result of non-determinism in outcomes of actions.

Contributions Although probabilistic fuzzy modal logics have received much attention, to the best of our knowledge, the framework proposed by Corsi et al. (2023) is the only one where modalities can appear on both layers. It also seems that the computational properties of such logics are considerably less studied. Thus, in this paper, we bring together the frameworks of (Corsi et al. 2023) and (Majer and Sedlár 2025) and explore a logic that we call $\mathsf{EA}^{\mathsf{Pr}}$, whose language is based upon a modal expansion of the combined Łukasiewicz and Product logics where epistemic and action modalities appear both in inner- and outer-layer formulas. As we will see in Section 3, this enables us to reason with a wide range of events and express various relations between their probabilities.

We show that the language can (i) express probabilistic statements about modal events involving knowledge or effects of actions (e.g. statements about the probability of an agent knowing a proposition or a proposition holding after a specific action is performed) and (ii) represent lower and upper probabilities of events arising from epistemic uncertainty or from uncertainty about outcomes of actions. As we discuss in more detail below, system specifications of these forms are important in areas such as multi-agent systems, cybersecurity, and robotics.

The main technical result in the paper is an extensive study of the complexity of $\mathsf{EA}^{\mathsf{Pr}}$. We establish that the validity (and satisfiability) problem for arbitrary formulas over finitely-branching models is PSPACE-complete. We also consider several fragments of our language where *validity is decidable in polynomial time*. Our contributions shed light on the computational features of formal verification and automated reasoning for probabilistic action-epistemic specifications within our expressive framework.

Plan of the Paper The text is structured as follows. In Section 2, we present the language and semantics of $\mathsf{EA}^{\mathsf{Pr}}$. We discuss its expressivity in Section 3. In Sections 4 and 5, we study the complexity of $\mathsf{EA}^{\mathsf{Pr}}$ and its fragments. In Section 6, we discuss related work in modal probabilistic logics. We summarise our results and outline the future work in Section 7. Omitted proofs are put in the appendix of the technical report (Kozhemiachenko and Sedlár 2026).

2 Language and Semantics

We begin with the definition of the *event language*.

Definition 1. *Let* $\mathtt{Act}$ *and* $\mathtt{Agt}$ *be two countable sets of labels, representing the set of basic actions and the set of agents, respectively. Let* $\mathtt{Var}$ *be a countable set of propositional variables, expressing basic events. The set* $\mathscr{L}_{\mathbf{evt}}$ *of* event formulas *is defined using the following grammar:*

$$\mathscr{L}_{\mathbf{evt}} \ni \alpha \coloneqq p \mid {\sim}\alpha \mid \alpha \wedge \alpha \mid \mathsf{K}_A\alpha \mid [a]\alpha$$

where $p \in \mathtt{Var}$, $A \in \mathtt{Agt}$ *and* $a \in \mathtt{Act}$.

As one can see, $\mathscr{L}_{\mathbf{evt}}$ is a variant of the classical multi-modal language containing knowledge modalities K_A for agents $A \in \mathtt{Agt}$ (where $\mathsf{K}_A\alpha$ means that 'the agent A knows that α') in combination with action modalities $[a]$ for actions $a \in \mathtt{Act}$ (where $[a]\alpha$ means that 'α holds in all states that can be obtained by doing a').

Definition 2 (Event frames and models)**.**

- *An* event frame *is a tuple* $\mathfrak{F} = \langle W, \mathsf{E}, \mathsf{R}\rangle$ *where* $W \neq \varnothing$, E *is a function from* $\mathtt{Agt}$ *to the set of equivalence relations on* W, *and* $\mathsf{R} : \mathtt{Act} \to 2^{W\times W}$.
- *An* event model *is a tuple* $\mathfrak{E} = \langle W, \mathsf{E}, \mathsf{R}, V\rangle$ *with* $\langle W, \mathsf{E}, \mathsf{R}\rangle$ *being an event frame and* $V : \mathtt{Var} \to 2^W$.

Given a model $\mathfrak{E}$, we will write $W_{\mathfrak{E}}$, $\mathsf{E}_{\mathfrak{E}}$, $\mathsf{R}_{\mathfrak{E}}$, and $V_{\mathfrak{E}}$ to denote, respectively, the set of states of $\mathfrak{E}$, its epistemic and action relations, and its valuation.

Informally, W is a set of possible states of the environment (or possible worlds). The equivalence relation E_A for $A \in \mathtt{Agt}$ represents the epistemic state of agent A: $\langle w, u\rangle \in \mathsf{E}_A$ (written also as E_Awu or $w\mathsf{E}_Au$) means that, in state w, the agent A lacks sufficient information to distinguish state w from state u. Similarly, the relation R_a represents possible outcomes of performing action $a \in \mathtt{Act}$: $\langle w, u\rangle \in \mathsf{R}_a$ (written also as R_awu or $w\mathsf{R}_au$) means that u is a possible outcome of performing a in state w. That is, we do not assume that actions are deterministic. We define $\mathsf{R}_a(w) \coloneqq \{w' \mid w\mathsf{R}_aw'\}$ and $\mathsf{E}_A(w) \coloneqq \{w' \mid w\mathsf{E}_Aw'\}$. Finally, the valuation function V assigns to each propositional variable $p \in \mathtt{Var}$ an *event*, that is, a subset of W. This event will also be denoted as $\|p\|_{\mathfrak{E}}$. Events $\|\alpha\|_{\mathfrak{E}}$ for complex event formulas are defined as usual in modal logic:

$$\begin{aligned}
\|{\sim}\alpha\|_{\mathfrak{E}} &= W \setminus \|\alpha\|_{\mathfrak{E}}\\
\|\alpha_1 \wedge \alpha_2\|_{\mathfrak{E}} &= \|\alpha_1\|_{\mathfrak{E}} \cap \|\alpha_2\|_{\mathfrak{E}}\\
\|\mathsf{K}_A\alpha\|_{\mathfrak{E}} &= \mathscr{K}_A\|\alpha\|_{\mathfrak{E}} \coloneqq \{w \mid \mathsf{E}_A(w) \subseteq \|\alpha\|_{\mathfrak{E}}\}\\
\|[a]\alpha\|_{\mathfrak{E}} &= \mathscr{R}_a\|\alpha\|_{\mathfrak{E}} \coloneqq \{w \mid \mathsf{R}_a(w) \subseteq \|\alpha\|_{\mathfrak{E}}\}
\end{aligned}$$

Definition 3 (Probability space)**.** *A* probability space *over an event model* $\mathfrak{E}=\langle W, \mathsf{E}, \mathsf{R}, V\rangle$ *is a pair* $\mathbb{S}=\langle\mathscr{F}, \mu\rangle$ *s.t.:*

- $\mathscr{F}$ *is a collection of subsets of* W *which contains* W *and* $V(p)$ *for all* $p \in \mathtt{Var}$*; and is closed under complements, finite unions and the operators* $\mathscr{K}_A$, $\mathscr{R}_a$ *for all* $A \in \mathtt{Agt}$ *and* $a \in \mathtt{Act}$*;*

- *μ is a finitely additive probability measure on $\mathscr{F}$.*

Definition 4 (Probabilistic models)**.** *A* probabilistic model *is a tuple $\mathfrak{F} = \langle W, \mathbf{F}, \mathsf{E}, \mathsf{R}, \mu, V \rangle$ s.t. $\langle W, \mathsf{E}, \mathsf{R}, V \rangle$ is an event model and $\langle \mathbf{F}, \mu \rangle = \langle \mathscr{F}_{w,A}, \mu_{w,A} \rangle_{w \in W, A \in \mathtt{Agt}}$ is a tuple of probability spaces over $\langle W, \mathsf{E}, \mathsf{R}, V \rangle$ for each $w \in W$ and $A \in \mathtt{Agt}$.*

Intuitively, $\mu_{w,A}(X)$ is the subjective probability of X for agent A in state w. To achieve generality, we do not assume that the same events are measurable for each agent in each state. This design choice gives rise to the plurality of probability spaces $\langle \mathbf{F}, \mu \rangle$. To express *reasoning* about events and their probabilities, we will use a language based on the fuzzy modal logic. We thus undertake the approach proposed by Hájek et al. (1995; 1998; 2000). The propositional fragment of our language is based on that of $\mathbf{Ł\Pi}\frac{1}{2}$ — the logic that combines connectives of Łukasiewicz and Product logics. We further expand it with epistemic and action modalities corresponding to those in the event language.

Definition 5. *The* full probabilistic language $\mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$ *is defined using the following grammar:*

$$\phi \coloneqq \mathsf{Pr}_A(\alpha) \mid \neg\phi \mid \phi{\to}\phi \mid \phi{\bullet}\phi \mid \phi{\to_\Pi}\phi \mid \overline{\tfrac{1}{2}} \mid \mathsf{K}_A\phi \mid [a]\phi$$

where $\alpha \in \mathscr{L}_{\mathbf{evt}}$, $A \in \mathtt{Agt}$ and $a \in \mathtt{Act}$.

Definition 6. *Given a probabilistic model $\mathfrak{M}$, we define* $\mathfrak{M}$-interpretation *as a function $\mathscr{I}_{\mathfrak{M}} : \mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}} \times W_{\mathfrak{M}} \to [0,1]$ s.t.:*

$$\begin{aligned}
\mathscr{I}_{\mathfrak{M}}(\mathsf{Pr}_A(\alpha), w) &= \mu_{w,A}(\|\alpha\|_{\mathfrak{M}})\\
\mathscr{I}_{\mathfrak{M}}(\neg\phi, w) &= 1 - \mathscr{I}_{\mathfrak{M}}(\phi, w)\\
\mathscr{I}_{\mathfrak{M}}(\phi{\to}\chi, w) &= \min(1, 1 - \mathscr{I}_{\mathfrak{M}}(\phi, w) + \mathscr{I}_{\mathfrak{M}}(\chi, w))\\
\mathscr{I}_{\mathfrak{M}}(\phi \bullet \chi, w) &= \mathscr{I}_{\mathfrak{M}}(\phi, w) \cdot \mathscr{I}_{\mathfrak{M}}(\chi, w)\\
\mathscr{I}_{\mathfrak{M}}(\phi{\to_\Pi}\chi, w) &= \begin{cases} 1 \textit{ if } \mathscr{I}_{\mathfrak{M}}(\phi, w) \leq \mathscr{I}_{\mathfrak{M}}(\chi, w)\\ \dfrac{\mathscr{I}_{\mathfrak{M}}(\chi, w)}{\mathscr{I}_{\mathfrak{M}}(\phi, w)} \textit{ otherwise;}\end{cases}\\
\mathscr{I}_{\mathfrak{M}}(\overline{\tfrac{1}{2}}, w) &= \tfrac{1}{2}\\
\mathscr{I}_{\mathfrak{M}}(\mathsf{K}_A\phi, w) &= \inf\{\mathscr{I}_{\mathfrak{M}}(\phi, w') \mid w\mathsf{E}_A w'\}\\
\mathscr{I}_{\mathfrak{M}}([a]\phi, w) &= \inf\{\mathscr{I}_{\mathfrak{M}}(\phi, w') \mid w\mathsf{R}_a w'\}
\end{aligned}$$

Definition 7. *Let $\Gamma \cup \{\chi\} \subseteq \mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$ be finite. We say that*

- *χ is* $\mathsf{EA}^{\mathsf{Pr}}$-satisfiable *(on a frame $\mathfrak{F}$) iff there is a probabilistic model $\mathfrak{M}$ (on $\mathfrak{F}$) and $w \in \mathfrak{M}$ s.t. $\mathscr{I}_{\mathfrak{M}}(\chi, w) = 1$;*
- *χ is* $\mathsf{EA}^{\mathsf{Pr}}$-valid *(on $\mathfrak{F}$) iff for every probabilistic model $\mathfrak{M}$ (on $\mathfrak{F}$) and every $w \in \mathfrak{M}$, it holds that $\mathscr{I}_{\mathfrak{M}}(\chi, w) = 1$;*
- *Γ* entails *χ iff $\mathscr{I}_{\mathfrak{M}}(\chi, w) = 1$ for every probabilistic model $\mathfrak{M}$ and $w \in \mathfrak{M}$ s.t. $\mathscr{I}_{\mathfrak{M}}(\phi, w) = 1$ for all $\phi \in \Gamma$.*

Convention 1. *We will further use $\langle a \rangle\phi$ and $\widehat{\mathsf{K}}_A\phi$ as shorthands for $\neg[a]\neg\phi$ and $\neg\mathsf{K}_A\neg\phi$, respectively. Modalities K_A and $[a]$ are called* box-like*, and modalities $\widehat{\mathsf{K}}_A$ and $\langle a \rangle$* diamond-like*. We will also use $\blacksquare$ to denote (sequences of) arbitrary box-like modalities: $[a]\mathsf{K}_B[c]$, $\mathsf{K}A\mathsf{K}B[c]$, etc. Similarly, $\blacklozenge$ stands for (sequences) of diamond-like modalities. In addition, we define*

$$\overline{1} \coloneqq \overline{\tfrac{1}{2}}{\to}\overline{\tfrac{1}{2}} \qquad \phi{\oplus}\chi \coloneqq \neg\phi{\to}\chi \qquad \triangle\phi \coloneqq \neg(\phi{\to_\Pi}\overline{0})$$
$$\overline{0} \coloneqq \neg\overline{1} \qquad \phi{\odot}\chi \coloneqq \neg(\phi{\to}\neg\chi)$$

The semantics of these connectives is derived from Definition 6:

$$\begin{aligned}
\mathscr{I}_{\mathfrak{M}}(\widehat{\mathsf{K}}_A\phi, w) &= \sup\{\mathscr{I}_{\mathfrak{M}}(\phi, w') \mid w\mathsf{E}_A w'\}\\
\mathscr{I}_{\mathfrak{M}}(\langle a \rangle\phi, w) &= \sup\{\mathscr{I}_{\mathfrak{M}}(\phi, w') \mid w\mathsf{R}_a w'\}\\
\mathscr{I}_{\mathfrak{M}}(\phi \oplus \chi, w) &= \min(1, \mathscr{I}_{\mathfrak{M}}(\phi, w) + \mathscr{I}_{\mathfrak{M}}(\chi, w))\\
\mathscr{I}_{\mathfrak{M}}(\phi \odot \chi, w) &= \max(0, \mathscr{I}_{\mathfrak{M}}(\phi, w) + \mathscr{I}_{\mathfrak{M}}(\chi, w) - 1)
\end{aligned}$$

$$\begin{aligned}\mathscr{I}_{\mathfrak{M}}(\overline{1}, w) &= 1\\ \mathscr{I}_{\mathfrak{M}}(\overline{0}, w) &= 0\end{aligned} \quad \mathscr{I}_{\mathfrak{M}}(\triangle\phi, w) = \begin{cases} 1 & \textit{if } \mathscr{I}_{\mathfrak{M}}(\phi, w) = 1\\ 0 & \textit{otherwise}\end{cases}$$

Let us briefly explain the semantics of connectives in Definition 6 and Convention 1. One can see that both implications, $\to$ and $\to_\Pi$ are order-preserving. $\oplus$ is the truncated sum on $[0,1]$, and $\triangle$ helps form crisp statements. Moreover, $\neg$, $\odot$, $\oplus$, and $\to$ interact in the classical way: $\neg\phi \oplus \chi$ is equivalent to $\phi \to \chi$, and $\neg(\phi{\oplus}\chi)$ is equivalent to $\neg\phi{\odot}\neg\chi$. Notice also that $\blacksquare\mathsf{Pr}_A(\alpha)$ and $\blacklozenge\mathsf{Pr}_A(\alpha)$ express *lower* and *upper probability* of α in accessible states.

Definition 8. *Let $W \neq \varnothing$ and $\mathscr{F} \subseteq 2^W$ be given as in Definition 3. Let further, $\mathscr{M}$ be a set of probability measures on $\mathscr{F}$ and $X \in \mathscr{F}$. We define the* lower *and* upper *probabilities of X ($\mu_*(X)$ and $\mu^*(X)$, respectively) as follows:*

$$\mu_*(X){=}\inf\{\mu(X)|\mu{\in}\mathscr{M}\} \quad \mu^*(X){=}\sup\{\mu(X)|\mu{\in}\mathscr{M}\}$$

We will conclude the section with a few brief observations. First, for every rational number $\frac{m}{n}$, one can define an $\mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$ formula $\overline{\frac{m}{n}}$ s.t. $\mathscr{I}_{\mathfrak{M}}(\overline{\frac{m}{n}}, w) = \frac{m}{n}$ in all probabilistic models, using $\{\neg, \oplus, \bullet, \to_\Pi, \overline{\frac{1}{2}}\}$ (cf. (Esteva, Godo, and Montagna 2001, Theorem 7)). Second, note that satisfiability and validity in $\mathsf{EA}^{\mathsf{Pr}}$ are reducible to one another in an expected way. Namely, ϕ is satisfiable iff $\neg\triangle\phi$ is *not valid*, and, conversely, ϕ is valid iff $\neg\triangle\phi$ is *unsatisfiable*. Third, finitary entailment is reducible to validity: ϕ entails χ iff $\triangle\phi \to \chi$ is valid.

3 Expressivity

The key feature of our language is its use of modal operators both in the event layer and in the probabilistic layer. Importantly, the modal operators are not limited to $\mathbf{S5}$-style epistemic operators expressing 'agent A knows that…', but they include general $\mathbf{K}$-style action operators expressing '…holds after action a is performed'. This combined language is rather expressive, as we show in this section.

We first point out that the event layer of our language can express various kinds of *modal events*.

Example 1 (Some modal events)**.**

- *'After action a is performed, p will hold':* $[a]p$.
- *'A knows that p':* K_Ap.
- *'After a is performed, A will know that p':* $[a]\mathsf{K}_Ap$.
- *'A knows that after a is performed, p will hold':* $\mathsf{K}_A[a]p$.

Our probabilistic operator Pr_A that expresses *subjective probability* of agent A can then be used to represent probabilities of these events. We will treat objective probability as a particular case of subjective probability. Formally, we will assume that $\mathtt{ob} \in \mathtt{Agt}$ and write $\mathsf{Pr}_{\mathtt{ob}}$ in our examples when dealing with objective probability.

Reasoning about probabilities of such modal events is crucial in various areas. In multi-agent systems, e.g., card games (Brown and Sandholm 2019), agents need to reason about what other agents know, and this reasoning is often based on probabilistic information. In cybersecurity, a security system needs to be able to reason probabilistically about what the attacker might know (Barthe et al. 2013). Probabilistic robotics (Thrun, Burgard, and Fox 2005) is well-studied. In particular, in human-robot interaction scenarios, human controllers often need to take into account what robots with possibly faulty sensors are likely to know in a given situation (Akkaladevi et al. 2021).

Using propositional connectives of the fuzzy logic $Ł\Pi\overline{\frac{1}{2}}$, we can express comparisons between probabilistic statements, including threshold comparisons.

- 'According to A, it's not much more likely that p is going to happen if action a is performed than if action b is performed':
$$\mathsf{Pr}_A([a]p) \to \mathsf{Pr}_A([b]p)$$
- 'Agent A likely knows that p':
$$\overline{\tfrac{1}{2}} \to \mathsf{Pr}_{\mathtt{ob}}(\mathsf{K}_A p)$$
- 'The probability that A knows that p is at least 0.7':
$$\triangle(\overline{0.7} \to \mathsf{Pr}_{\mathtt{ob}}(\mathsf{K}_A p))$$
- 'According to agent A, it's twice as likely for p to hold after a is performed that it is for p to fail':
$$\triangle(\overline{\tfrac{1}{2}} \bullet \mathsf{Pr}_A([a]p) \leftrightarrow \mathsf{Pr}_A([a]\neg p))$$

The following examples illustrate the expressivity of our logic in the setting of automated bidding agents.

Example 2. *Consider two competing automated bidding agents A and B. Agent A needs to decide whether to place a high bid. The decision depends on A's subjective probability of B's knowing that the item they bid for is rare.*

The subjective probability is represented by the atomic probabilistic formula $\mathsf{Pr}_A(\mathsf{K}_B\mathtt{rare})$. The truth degree of the implicational formula $\overline{0.7} \to \mathsf{Pr}_A(\mathsf{K}_B\mathtt{rare})$ is 1 if A's subjective probability of B knowing $\mathtt{rare}$ is no less that 0.7 and is the difference between A's subjective probability of B knowing $\mathtt{rare}$ and 0.7 otherwise (cf. Definition 6).

Using the delta operator, we can express the crisp statement that A's subjective probability of B knowing $\mathtt{rare}$ is at least 0.7. This is expressed by $\triangle(\overline{0.7} \to \mathsf{Pr}_A(\mathsf{K}_B\mathtt{rare}))$. Compare this to the imprecise statement that A's subjective probability of B knowing $\mathtt{rare}$ is not much *less than 0.7: $\overline{0.7} \to \mathsf{Pr}_A(\mathsf{K}_B\mathtt{rare})$.*

Example 3. *Assume that there is an auction with two participants: A and B. One of the things A can do is to place an* aggressive bid *(a). Now A's subjective probability that B fill fold after A places an aggressive bid is formalised by the formula $\mathsf{Pr}_A([a](B\,\mathtt{folds}))$. Using rational constants, we can express bounds on subjective probabilities. E.g., the truth degree of formula $\overline{0.7} \to \mathsf{Pr}_A([a](B\,\mathtt{folds}))$ is inversely proportional to the amount to which 0.7 exceeds A's subjective probability that B fill fold after A places an aggressive bid, and $\triangle(\overline{0.7} \to \mathsf{Pr}_A([a](B\,\mathtt{folds})))$ says that A's subjective probability that B fill fold after A places an aggressive bid is not less than 0.7.*

Moreover, we can express comparisons between subjective probabilities of the outcomes of different actions. Let m represent agent A placing a modest bid*; then $\mathsf{Pr}_A([m](B\,\mathtt{folds})) \to \mathsf{Pr}_A([a](B\,\mathtt{folds}))$ formalises the imprecise statement that A's subjective probability that B will fold after A places an aggressive bid is not much less than A's subjective probability that B will fold after A places a modest bid. Note that the delta operator $\triangle$ can turn an imprecise comparison into a precise one — the formula $\triangle(\mathsf{Pr}_A([m](B\,\mathtt{folds})) \to \mathsf{Pr}_A([a](B\,\mathtt{folds})))$ says that A's subjective probability that B fill fold after A places an aggressive bid is not less than A's subjective probability that B fill fold after A places a modest bid.*

Modal operators in the probabilistic layer can, in turn, formalise upper and lower probabilities (recall Definition 8) expressing qualitative uncertainty about probability. This uncertainty can arise from the lack of information on an agent's side, or from the fact that outcomes of actions are not determined, for example. Upper and lower probabilities are also useful when specifying the effects of actions on probabilities of events.

- 'Performing b cannot substantially decrease the probability that performing a will lead to p being true':
$$\mathsf{Pr}_{\mathtt{ob}}([a]p) \to [b]\mathsf{Pr}_{\mathtt{ob}}([a]p)$$
- 'Performing a cannot substantially increase the probability of agent A knowing that p':
$$[a]\mathsf{Pr}_{\mathtt{ob}}(\mathsf{K}_A p) \to \mathsf{Pr}_{\mathtt{ob}}(\mathsf{K}_A p)$$
- 'The amount of qualitative uncertainty of agent A concerning event p is not much higher than 0.2':
$$\overline{0.2} \to (\mathsf{K}_A\mathsf{Pr}_{\mathtt{ob}}(p) \leftrightarrow \widehat{\mathsf{K}}_A\mathsf{Pr}_{\mathtt{ob}}(p))$$

Example 4. *In the context of the previous bidding example, assume that A made a subtle, revealing bid. Agent A does not know what B's subjective probability of 'A has a strong hand' ($A\,\mathtt{strong}$) is after the revealing bid, A only has a range of possibilities. The formula $\mathsf{K}_A(\mathsf{Pr}_B(A\,\mathtt{strong}))$ expresses the infimum of the values of $\mathsf{Pr}_B(A\,\mathtt{strong})$ across these possibilities — B's lower subjective probability of $A\,\mathtt{strong}$ across the possibilities left open by agent A.*

Using Łukasiewicz implication, we can express comparisons between the lower probability and the 'actual' probability, for instance. This is done by the formula $\mathsf{Pr}_B(A\,\mathtt{strong}) \to \mathsf{K}_A(\mathsf{Pr}_B(A\,\mathtt{strong}))$ that expresses the truth degree of the imprecise statement that B's subjective probability of $A\,\mathtt{strong}$ is not much lower than lower B's probability of $A\,\mathtt{strong}$ (i.e. 'what A considers possible').

We can explicitly express the fact that we are looking at probabilities after the revealing bid *using the action modality $[r]$. In particular, the formula $[r]\mathsf{Pr}_B(A\,\mathtt{strong})$ expresses the infimum of B's subjective probabilities of $A\,\mathtt{strong}$ in all possible outcomes of the action r. Similarly, the formula $[r]\mathsf{K}_A(\mathsf{Pr}_B(A\,\mathtt{strong}))$ expresses the infimum of B's lower subjective probability of $A\,\mathtt{strong}$ across the possibilities left open by agent A after r. As* *before, we can use Łukasiewicz implication to express a comparison of these infima:* $[r]\mathsf{Pr}_B(A\,\mathtt{strong}) \rightarrow [r]\mathsf{K}_A(\mathsf{Pr}_B(A\,\mathtt{strong}))$.

We finish the section with the following observation. It is known from (Majer and Sedlár 2025) that the logic $\mathbf{K Ł \Pi}\frac{1}{2}^{\mathsf{Pr}}$ proposed there lacks the finite model property (FMP). In particular, $\phi = \triangle\langle a\rangle\mathsf{Pr}_A(p) \rightarrow \langle a\rangle\triangle\mathsf{Pr}_A(p)$ is valid in all finite frames but can be falsified in a model $\mathfrak{M}$ that contains a state w s.t. $\mathsf{R}_a(w) = \{w_i \mid i \in \mathbb{N}\}$; it suffices to set $\mathscr{I}_{\mathfrak{M}}(\mathsf{Pr}_A(p), w_i) = \frac{i}{i+1}$. Dually, $\neg\triangle\phi$ is unsatisfiable on every finite frame, but can be satisfied on the same infinitely branching model that falsifies ϕ. As $\mathbf{K Ł \Pi}\frac{1}{2}^{\mathsf{Pr}}$ is a fragment of $\mathsf{EA}^{\mathsf{Pr}}$, it follows that the latter also lacks FMP.

In most practical applications, however, it is sufficient to consider finite models. Still, the technical results presented in the next section apply to a slightly more general class of models: the *finitely branching* ones. These models represent situations in which an action can have only a finite number of possible outcomes, or in which an agent considers only a finite number of possibilities (i.e. their information in any state is consistent with only a finite number of states).

A similar restriction on models is often used in fuzzy description logics. Namely (cf. work by Hájek (2007), Bobillo et al. (2009; 2012) and Borgwardt et al. (2014a; 2014b; 2015; 2017)), one may consider *witnessed* interpretations, i.e., those with the following property. If a quantified assertion $[\forall \mathsf{R}.\mathsf{C}](a)$ has value x, then there must be an individual b s.t. $\mathsf{R}(a, b)$ and $\mathsf{C}(b)$ has value x.

In the remainder of the paper, we will mostly consider the satisfiability of $\mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$ formulas over *finitely branching* frames. In what follows, we will use $\mathsf{EA}^{\mathsf{Pr}}_{\mathsf{fb}}$ to stand for the logic $\mathsf{EA}^{\mathsf{Pr}}$ interpreted over finitely-branching frames. We will also say that $\phi \in \mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$ is $\mathsf{EA}^{\mathsf{Pr}}_{\mathsf{fb}}$-*valid* ($\mathsf{EA}^{\mathsf{Pr}}_{\mathsf{fb}}$-satisfiable) meaning that ϕ is $\mathsf{EA}^{\mathsf{Pr}}$-valid on all ($\mathsf{EA}^{\mathsf{Pr}}$-satisfiable on some) finitely branching frames.

4 Decidability of $\mathsf{EA}^{\mathsf{Pr}}_{\mathsf{fb}}$

In this section, we will establish the decidability of $\mathsf{EA}^{\mathsf{Pr}}_{\mathsf{fb}}$-satisfiability (and hence, validity). To do that, we will proceed as follows. We begin with constructing a terminating tableaux calculus for $\mathsf{EA}^{\mathbf{Ł\Pi}}_{\mathsf{fb}}$ — the fragment of $\mathsf{EA}^{\mathsf{Pr}}_{\mathsf{fb}}$ *without event formulas*. Then we will use these tableaux to devise a decision procedure for $\mathsf{EA}^{\mathsf{Pr}}_{\mathsf{fb}}$.

Let us first define $\mathscr{L}_{\mathsf{EA}}$ — the language of $\mathsf{EA}^{\mathbf{Ł\Pi}}_{\mathsf{fb}}$.

Definition 9. *Let* $p \in \mathtt{Var}$, $A \in \mathtt{Agt}$, *and* $a \in \mathtt{Act}$, *we define* $\phi \in \mathscr{L}_{\mathsf{EA}}$ *as follows:*

$$\phi \coloneqq p \mid \neg\phi \mid \phi \rightarrow \phi \mid \phi \bullet \phi \mid \phi \rightarrow_{\Pi} \phi \mid \overline{\tfrac{1}{2}} \mid \mathsf{K}_A\phi \mid [a]\phi$$

The semantics of $\mathsf{EA}^{\mathbf{Ł\Pi}}_{\mathsf{fb}}$ *can be obtained in the expected manner by restricting the interpretation functions from Definition 6 onto* finitely branching event frames.

Definition 10. *An* $\mathsf{EA}^{\mathbf{Ł\Pi}}_{\mathsf{fb}}$*-model is* $\mathfrak{M} = \langle W, \mathsf{E}, \mathsf{R}, v\rangle$ *with* $\langle W, \mathsf{E}, \mathsf{R}\rangle$ *being a finitely branching event frame and* $v\colon \mathtt{Var} \rightarrow [0, 1]$. *The* $\mathsf{EA}^{\mathbf{Ł\Pi}}_{\mathsf{fb}}$-interpretation induced by $\mathfrak{M}$ *is* $\mathscr{I}_{\mathfrak{M}}\colon W \times \mathscr{L}_{\mathsf{EA}} \rightarrow [0, 1]$ *s.t.* $\mathscr{I}_{\mathfrak{M}}(p, w) = v(p, w)$, *and the values of complex formulas are computed according to Definition 6. The notions of* $\mathsf{EA}^{\mathbf{Ł\Pi}}_{\mathsf{fb}}$*-satisfiability, validity, and entailment are also adapted from Definition 6.*

Remark 1. We note that $\mathsf{EA}^{\mathbf{Ł\Pi}}$ and $\mathsf{EA}^{\mathsf{Pr}}$ are connected in the following way. Given $\phi \in \mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$, one can think of it as an $\mathscr{L}_{\mathsf{EA}}$-formula $\phi^{\uparrow}$ where each probabilistic atom $\mathsf{Pr}_A(\alpha)$ is replaced by a fresh variable $p_{A,\alpha}$. In this interpretation, ϕ is satisfied in a probabilistic model $\mathfrak{M} = \langle W, \mathbf{F}, \mathsf{E}, \mathsf{R}, \mu, V\rangle$ iff $\phi^{\uparrow}$ is satisfied in the $\mathsf{EA}^{\mathbf{Ł\Pi}}$-*model* $\mathfrak{M}^{\uparrow} = \langle W, \mathsf{E}, \mathsf{R}, V^{\uparrow}\rangle$ on the same frame s.t. $V^{\uparrow}(p_{A,\alpha}, w) = \mathscr{I}_{\mathfrak{M}}(\mathsf{Pr}_A(\alpha))$.

Convention 2. *Given a formula* ϕ *and a set of formulas* Γ, *we use* $\ell(\phi)$ *and* $\ell[\Gamma]$ *to denote their* lengths, *i.e., the number of symbols in them.*

Let us now present a tableaux calculus for $\mathsf{EA}^{\mathbf{Ł\Pi}}_{\mathsf{fb}}$. We adapt the idea of 'constraint tableaux' by Hähnle (1999) for the modal logic framework. We first define *constraints* that encode the values of formulas in states and *relational terms* that represent relations between different states on the branch of the tableau.

Definition 11 (Constraints)**.** *We fix a countable set* $\mathsf{W} = \{w, w', w_0, w_1, \ldots\}$ *of* state labels *and define:*

- *a* labelled formula *as* $w{:}\phi$ *with* $w \in \mathsf{W}$ *and* $\phi \in \mathscr{L}_{\mathsf{EA}}$*;*
- *a* formulaic constraint *as* $w{:}\phi\triangledown P$ *s.t.* $w{:}\phi$ *is a labelled formula,* $\triangledown \in \{\leq, \geq\}$ *and* P *a polynomial over* $[0, 1]$ *with integer coefficients;*
- *a* numerical constraint *as* $w{:}P\triangleleft P'$ *with* $w \in \mathsf{W}$, $\triangleleft \in \{\leq, <\}$, *and* P *and* P' *polynomials.*

Definition 12 (Relational terms)**.** *For sets* $\mathtt{Act}$ *and* $\mathtt{Agt}$ *of* actions *and* agents, *we fix sets of* action *and* epistemic relational labels — $\mathfrak{Ac} = \{\mathsf{R}_a \mid a \in \mathtt{Act}\}$ *and* $\mathfrak{Cl} = \{\mathfrak{cl}^A_i \mid A \in \mathtt{Agt}, i \in \mathbb{N}\}$ *— and define:*

- *an* action relational term *to be* $w\mathsf{R}_a w'$ *with* $w, w' \in \mathsf{W}$ *and* $\mathsf{R}_a \in \mathfrak{Ac}$*;*
- *an* epistemic relational term *to be* $w \in \mathfrak{cl}^A_i$ *with* $w \in \mathsf{W}$, $i \in \mathbb{N}$, *and* $\mathfrak{cl}^A_i \in \mathfrak{Cl}$*.*

Note from the definition above that epistemic labels correspond to *subsets* of the model. Namely, given a model $\mathfrak{M}$ each $\mathfrak{cl}^A_i$ is an equivalence class on $\mathfrak{M}$ under E_A.

We are now ready to present the rules and formally define the notion of a *tableaux proof*. Our calculus $\mathscr{T}(\mathsf{EA}^{\mathbf{Ł\Pi}}_{\mathsf{fb}})$ expands $\mathscr{T}(\mathbf{K Ł \Pi}\overline{\tfrac{1}{2}}_{\mathsf{fb}})$ by Kozhemiachenko and Sedlár (2025) with rules for epistemic modalities.

Definition 13 ($\mathscr{T}(\mathsf{EA}^{\mathbf{Ł\Pi}}_{\mathsf{fb}})$ — constraint tableaux for $\mathsf{EA}^{\mathbf{Ł\Pi}}_{\mathsf{fb}}$)**.** *A* constraint tableau *is a downward branching tree of* formulaic constraints*,* numerical constraints*, and* relational terms*. A branch can be extended by an application of a rule from Fig. 1. Given a tableau branch* $\mathscr{B}$, *we let* $\mathsf{W}_{\mathscr{B}} = \{w \mid w \in \mathsf{W},\ w \text{ occurs on } \mathscr{B}\}$ *and define for every* $w \in \mathsf{W}_{\mathscr{B}}$ *the sets of* formulaic *(*$\mathscr{F}_w$*) and* numerical *(*$\mathscr{N}_w$*) constraints as well as their corresponding systems of inequalities* $\mathscr{F}^{\mathbb{I}}_w$ *and* $\mathscr{N}^{\mathbb{I}}_w$ *(below,* $x_{w:\phi}$ *is a variable over* $[0, 1]$*):*

$$\mathscr{F}_w = \{w{:}\phi\triangledown P \mid w{:}\phi\triangledown P \in \mathscr{B}\}$$

$$\mathscr{N}_w = \{w{:}P\triangledown P' \mid w{:}P\triangledown P' \in \mathscr{B}\} \cup \{w{:}\overline{\tfrac{1}{2}}\triangledown P \mid w{:}\overline{\tfrac{1}{2}}\triangledown P \in \mathscr{B}\}$$

$$\mathscr{F}^{\mathbb{I}}_w = \{x_{w:\phi}\triangledown P \mid w{:}\phi\triangledown P \in \mathscr{F}_w\}$$

$$\mathsf{K}_{\leq}\!:\dfrac{\begin{array}{c}w{:}\mathsf{K}_A\phi \leq P\\ w \in \mathfrak{cl}_i^A\end{array}}{\begin{array}{c}w'{:}\phi \leq P;\ w' \in \mathfrak{cl}_i^A\\ w' \in \mathfrak{cl}_j^{A'}(A' \in \mathtt{Agt})\end{array}} \qquad \mathsf{K}_{\geq}\!:\dfrac{\begin{array}{c}w{:}\mathsf{K}_A\phi \geq P\\ w \in \mathfrak{cl}_i^A\ u \in \mathfrak{cl}_i^A\end{array}}{u{:}\phi \geq P} \qquad \Box_{\leq}\!:\dfrac{w{:}[a]\phi \leq P}{\begin{array}{c}w\mathsf{R}_a w';\ w'{:}\phi \leq P\\ w' \in \mathfrak{cl}_j^{A'}(A' \in \mathtt{Agt})\end{array}} \qquad \Box_{\geq}\!:\dfrac{\begin{array}{c}w{:}[a]\phi \geq P\\ w\mathsf{R}_a u\end{array}}{u{:}\phi \geq P}$$

$$\neg_{\leq}\!:\dfrac{w{:}\neg\phi \leq P}{w{:}\phi \geq 1-P} \qquad \neg_{\geq}\!:\dfrac{w{:}\neg\phi \geq P}{w{:}\phi \leq 1-P} \qquad \rightarrow_{\leq}:\dfrac{w{:}\phi \rightarrow \chi \leq P}{P \geq 1 \left| \begin{array}{c}w{:}\phi \geq 1-P+y\\ w{:}\chi \leq y\\ w{:}y \leq P\end{array}\right.} \qquad \rightarrow_{\geq}:\dfrac{w{:}\phi \rightarrow \chi \geq P}{\begin{array}{c}w{:}\phi \leq 1-P+y\\ w{:}\chi \geq y\end{array}}$$

$$\bullet\!:\dfrac{w{:}\phi \bullet \chi \nabla P}{\begin{array}{c}w{:}\phi = y_1\\ w{:}\chi = y_2\\ w{:}y_1 \cdot y_2 \nabla P\end{array}} \qquad \rightarrow_{\Pi\leq}\!:\dfrac{w{:}\phi \rightarrow_{\Pi} \chi \leq P}{P \geq 1 \left| \begin{array}{c}w{:}\phi = y_1\\ w{:}\chi = y_2\\ w{:}y_2/y_1 \leq P\end{array}\right.} \qquad \rightarrow_{\Pi\geq}\!:\dfrac{w{:}\phi \rightarrow_{\Pi} \chi \geq P}{\begin{array}{c}w{:}\phi \geq y\\ w{:}\chi \leq y\end{array} \left| \begin{array}{c}w{:}\phi = y_1\\ w{:}\chi = y_2\\ w{:}y_2/y_1 \geq P\end{array}\right.}$$

Figure 1: Tableaux rules: $\nabla \in \{\leq, \geq\}$; y, y_1, and y_2 are new variables; w' is fresh on the branch, u is present on the branch; $w{:}\psi = y$ is a shorthand for $\{w{:}\psi \geq y, w{:}\psi \leq y\}$; for each state generated by $\mathsf{K}_{\leq}$ and $\Box_{\leq}$, we add a fresh relational term $w \in \mathfrak{cl}_j^{A'}$ for every $A' \in \mathtt{Agt}$ and $j \in \mathbb{N}$ being new on the branch.

$\mathscr{N}_w^{\mathbb{I}} = \{P \nabla P' \mid w{:}(P \nabla P') \in \mathscr{N}_w\}$

Furthermore, we set

$$\mathscr{F} = \bigcup_{w \in \mathsf{W}_{\mathscr{B}}} \mathscr{F}_w \quad \mathscr{N} = \bigcup_{w \in \mathsf{W}_{\mathscr{B}}} \mathscr{N}_w \quad \mathscr{B}_w^{\mathbb{I}} = \mathscr{F}_w^{\mathbb{I}} \cup \mathscr{N}_w^{\mathbb{I}}$$
$$\mathscr{F}^{\mathbb{I}} = \bigcup_{w \in \mathsf{W}_{\mathscr{B}}} \mathscr{F}_w^{\mathbb{I}} \quad \mathscr{N}^{\mathbb{I}} = \bigcup_{w \in \mathsf{W}_{\mathscr{B}}} \mathscr{N}_w^{\mathbb{I}} \quad \mathscr{B}^{\mathbb{I}} = \bigcup_{w \in \mathsf{W}_{\mathscr{B}}} \mathscr{B}_w^{\mathbb{I}}$$

A tableau branch $\mathscr{B}$ is called open *if its corresponding system of inequalities $\mathscr{B}^{\mathbb{I}}$ has a solution over $[0,1]$ and is* closed *otherwise. An open branch is* complete *if for every premise of a rule present on the branch, a conclusion of the rule also occurs on the branch. A tableau is closed when all its branches are closed. A formula ϕ* has a $\mathscr{T}(\mathsf{EA}_{\mathsf{fb}}^{\mathbf{Ł\Pi}})$-proof *if there is a closed tableau starting with $\{w{:}\phi \leq y, w{:}y<1\} \cup \{w \in \mathfrak{cl}_0^A \mid A \in \mathtt{Agt}\}$ with y being a variable.*

Observe briefly that for each generated state w' on the branch, we add terms $w' \in \mathfrak{cl}_i^A$ where $\mathfrak{cl}_i^A$ is a *fresh* epistemic relational label and $A \in \mathtt{Agt}$. This works as follows. E.g., we want to apply the $\Box_{\leq}$ rule to the constraint $w{:}[a]\phi \leq P$ on $\mathscr{B}$. Say that $\mathtt{Agt} = \{A, B\}$, and we already have labels $\mathfrak{cl}_1^A$ and $\mathfrak{cl}_1^B$ on the branch. Then, we extend $\mathscr{B}$ with $w\mathsf{R}_a w'$ and $w'{:}\phi \leq P$ with w' being a fresh state label. We also add relational terms $w' \in \mathfrak{cl}_2^A$ and $w' \in \mathfrak{cl}_2^B$. This ensures that the accessibility relations E_A and E_B that correspond to labels $\mathfrak{cl}_i^A$ and $\mathfrak{cl}_j^B$ are reflexive. The rule $\mathsf{K}_{\leq}$ works similarly. Note that $\Box_{\leq}$ and $\mathsf{K}_{\leq}$ are sound only w.r.t. finitely-branching frames because in general, $\inf\{\mathscr{I}_{\mathfrak{M}}(\phi, w') \mid w\mathsf{R}_a w'\} \leq x$ does not imply that there is some $w' \in \mathsf{R}_a(w)$ s.t. $\mathscr{I}_{\mathfrak{M}}(\phi, w') \leq x$.

To prove the completeness of the tableaux, we will show that every *complete open* branch $\mathscr{B}$ has a model that *realises* constraints on it. The following definition is an application of the standard notion to the semantics of $\mathsf{EA}_{\mathsf{fb}}^{\mathbf{Ł\Pi}}$.

Definition 14 (Realising model)**.** *Let $\mathscr{B}$ be a tableau branch, $\mathscr{B}^{\mathbb{I}}$ its corresponding system of inequalities, and $\mathcal{S}_{\mathscr{B}^{\mathbb{I}}}$ a solution of $\mathscr{B}^{\mathbb{I}}$. Given $x_{w:\phi}$ occurring in $\mathscr{F}^{\mathbb{I}}$, we use $\mathcal{S}_{\mathscr{B}^{\mathbb{I}}}(x_{w:\phi})$ to denote its value under $\mathcal{S}_{\mathscr{B}^{\mathbb{I}}}$. Let further, $\mathfrak{M} = \langle W, \mathsf{E}, \mathsf{R}, v\rangle$ be an $\mathsf{EA}^{\mathbf{Ł\Pi}}$-model. $\mathfrak{M}$* realises $\mathscr{B}$ under $\mathcal{S}_{\mathscr{B}^{\mathbb{I}}}$ *if there is a map $\mathsf{rl} : \mathsf{W}_{\mathscr{B}} \to W$ s.t. $\mathscr{I}_{\mathfrak{M}}(\phi, \mathsf{rl}(w)) = \mathcal{S}_{\mathscr{B}^{\mathbb{I}}}(x_{w:\phi})$. $\mathfrak{M}$ is* a realising model of $\mathscr{B}$ *if there is a solution $\mathcal{S}_{\mathscr{B}^{\mathbb{I}}}$ s.t. $\mathfrak{M}$ realises $\mathscr{B}$ under $\mathcal{S}_{\mathscr{B}^{\mathbb{I}}}$.*

The next statement can be proved in a standard manner. For soundness, we show that if $\mathfrak{M}$ realises a premise of a rule, then $\mathfrak{M}$ realises its conclusion. For completeness, we prove that *complete open* branches have realising models.

Theorem 1. *$\mathscr{T}(\mathsf{EA}_{\mathsf{fb}}^{\mathbf{Ł\Pi}})$ is sound and complete w.r.t. $\mathsf{EA}_{\mathsf{fb}}^{\mathbf{Ł\Pi}}$:*

1. *if ϕ has a $\mathscr{T}(\mathsf{EA}_{\mathsf{fb}}^{\mathbf{Ł\Pi}})$-proof, then it is $\mathsf{EA}_{\mathsf{fb}}^{\mathbf{Ł\Pi}}$-valid;*
2. *if ϕ is $\mathsf{EA}_{\mathsf{fb}}^{\mathbf{Ł\Pi}}$-valid, then it has a $\mathscr{T}(\mathsf{EA}_{\mathsf{fb}}^{\mathbf{Ł\Pi}})$-proof.*

Now, let us show that $\mathsf{EA}_{\mathsf{fb}}^{\mathsf{Pr}}$-satisfiability is decidable. For this, we will define an alternative semantics of $\mathsf{EA}_{\mathsf{fb}}^{\mathsf{Pr}}$ based on *sample-independent models* (SI models) proposed by Kozhemiachenko and Sedlar (2025).

Definition 15 (Sample-independent semantics)**.**

- *An* SI probabilistic frame over $\mathfrak{X}$ *(SI-frame) is a tuple $\mathfrak{F} = \langle W, \mathsf{E}, \mathsf{R}, \mathfrak{X}, \mu\rangle$ with $\langle W, \mathsf{E}, \mathsf{R}\rangle$ and $\mathfrak{X} = \langle W_{\mathfrak{X}}, \mathsf{E}_{\mathfrak{X}}, \mathsf{R}_{\mathfrak{X}}\rangle$ being event frames, and $\mu = \langle \mu_{w,A}\rangle_{w \in W, A \in \mathtt{Agt}}$ being a tuple of finitely additive probability measures on $2^{W_{\mathfrak{X}}}$.*
- *An* SI-$\mathsf{EA}^{\mathsf{Pr}}$-model *is a tuple $\mathfrak{M} = \langle W, \mathsf{E}, \mathsf{R}, \mathfrak{X}, \mu, V\rangle$ with $\langle W, \mathsf{E}, \mathsf{R}, \mathfrak{X}, \mu\rangle$ being an SI-frame and $V : \mathtt{Var} \to 2^{W_{\mathfrak{X}}}$.*

Given an SI-$\mathsf{EA}^{\mathsf{Pr}}$-model $\mathfrak{M} = \langle W, \mathsf{E}, \mathsf{R}, \mathfrak{X}, \mu, V\rangle$, we call $\mathfrak{X}$ the inner frame *and $\langle W, \mathsf{E}, \mathsf{R}\rangle$ the* outer frame. *States of inner and outer frames are called inner and outer states, respectively. The notions of probabilistic interpretations induced by $\mathfrak{M}$, validity, and satisfiability are adapted from Definition 6.*

From the definition above, it is clear that every 'standard' probabilistic model (as in Definition 4) can be treated as a sample-independent model. Thus, if $\phi \in \mathscr{L}_{\mathsf{EA}}^{\mathsf{Pr}}$ is valid on all SI-models, it is valid on all 'standard' probabilistic models. For the converse direction, we can take an SI-model $\mathfrak{M} = \langle W, \mathsf{E}, \mathsf{R}, \mathfrak{X}, \mu, V\rangle$ with $\mathfrak{X} = \langle W_{\mathfrak{X}}, \mathsf{E}_{\mathfrak{X}}, \mathsf{R}_{\mathfrak{X}}, V\rangle$ s.t. $\mathscr{I}_{\mathfrak{M}}(\phi, w){<}1$ for some $w{\in}W$. We then define the probabilistic model $\mathfrak{M}' = \langle W', \mathbf{F}', \mathsf{E}', \mathsf{R}', \mu', V'\rangle$:

- $W' = W \uplus W_{\mathfrak{X}}$;
- $\mathscr{F}'_{w,A} = 2^{W'}$ for all $w \in W'$ and $A \in \mathtt{Agt}$;
- $\mathsf{E}'_A = \mathsf{E}_A \uplus \mathsf{E}_{A_{\mathfrak{X}}}$ for every $A \in \mathtt{Agt}$;
- $\mathsf{R}'_a = \mathsf{R}_a \uplus \mathsf{R}_{a_{\mathfrak{X}}}$ for every $a \in \mathtt{Act}$;
- $\mu'_{w,A}(\{s\}){=}\mu_{w,A}(\{s\})$ if $s{\in}W_{\mathfrak{X}}$, $\mu'_{w,A}(\{s\}){=}0$ else;
- $V' = V$

Now, we can show by induction on $\phi \in \mathscr{L}_{\mathsf{EA}}^{\mathsf{Pr}}$ that $\mathscr{I}_{\mathfrak{M}'}(\phi, w) = \mathscr{I}_{\mathfrak{M}}(\phi, w)$ for every $w \in W$. This gives us the following statement.

Proposition 1. *A formula $\phi \in \mathscr{L}_{\mathsf{EA}}^{\mathsf{Pr}}$ is $\mathsf{EA}^{\mathsf{Pr}}$-valid iff it is valid on SI-frames.*

The idea of the sample-independent semantics is to decouple the event model from the states over which probabilistic formulas are evaluated. Relying on the equivalence between sample-independent semantics and standard semantics, we can prove the PSPACE-completeness of the satisfiability of $\mathscr{L}_{\mathsf{EA}}^{\mathsf{Pr}}$ formulas over finitely branching frames.

Theorem 2. *Given $\phi \in \mathscr{L}_{\mathsf{EA}}^{\mathsf{Pr}}$, it is* PSPACE*-complete to decide whether it is satisfiable over finitely branching frames.*

Proof sketch. We begin with PSPACE-hardness. For that, we consider a polynomial reduction from the satisfiability of $\mathscr{L}_{\mathbf{evt}}$-formulas to $\mathsf{EA}_{\mathsf{fb}}^{\mathsf{Pr}}$-satisfiability. Namely, $\alpha \in \mathscr{L}_{\mathbf{evt}}$ is satisfiable iff $\mathsf{Pr}_A(\alpha)$ is $\mathsf{EA}_{\mathsf{fb}}^{\mathsf{Pr}}$-satisfiable.

Let us now consider PSPACE-membership. Our proof combines the ideas of (Fagin and Halpern 1994) and (Hájek and Tulipani 2001). First of all, we note that all tableaux terminate: all rules have the branching factor of at most 2 and decompose formulas into their subformulas.

Now, we proceed as follows. Using a standard tableaux procedure akin to the one by Blackburn et al. (2010), we begin to construct the frame $\langle W, \mathsf{E}, \mathsf{R}\rangle$ by building a tableau for $\{w_0{:}\phi{\geq}1\}$. Note from Remark 1 that the tableaux rules are *sound for* $\mathsf{EA}_{\mathsf{fb}}^{\mathsf{Pr}}$. We apply the rules depth-first until we generate a state w s.t. no labelled formula $w{:}\psi$ contains an outer-layer modality. Then, we decompose formulas labelled with w and construct a system of polynomial inequalities $\mathscr{B}_w^{\mathbb{I}}$ (cf. Definition 13).

Now, for each $A \in \mathtt{Agt}$ on $\mathscr{B}$, we construct two sets

$$\begin{aligned}\mathsf{At}_A(w) =& \{\alpha \mid w{:}\mathsf{Pr}_A(\alpha)\triangledown P \in \mathscr{B}\}\\ \mathsf{SF}_A^{\sim}(w) =& \{\beta \mid \exists\alpha \in \mathsf{At}_A(w) : \beta \text{ is a subformula of } \alpha\}\cup\\ & \{\neg\beta \mid \exists\alpha \in \mathsf{At}_A(w) : \beta \text{ is a subformula of } \alpha\}\end{aligned}$$

We then consider the following event model:

$$\mathfrak{X}_{w,A} = \langle W^{\mathsf{SF}}, \mathsf{E}^{\mathsf{SF}}, \mathsf{R}^{\mathsf{SF}}, V^{\mathsf{SF}}\rangle$$

The states of $\mathfrak{X}_{w,A}$ correspond to the maximal consistent subsets of $\mathsf{SF}_A^{\sim}(w)$, the relations are defined as follows:

$$\mathsf{R}_a^{\mathsf{SF}}(\Xi, \Xi') \text{ iff } \forall\beta \in \mathsf{SF}_A^{\sim}(w) : [a]\beta \in \Xi \Rightarrow \beta \in \Xi'$$
$$\mathsf{E}_A^{\mathsf{SF}}(\Xi, \Xi') \text{ iff } \forall\beta \in \mathsf{SF}_A^{\sim}(w) : \mathsf{K}_A\beta \in \Xi \Leftrightarrow \mathsf{K}_A\beta \in \Xi'$$

and $V^{\mathsf{SF}}(p) = \{\Xi \mid p \in \Xi\}$ for all $p \in \mathtt{Var}(\phi)$. $\mathfrak{X}_{w,A}$ has exponential size, but for any $\Xi \subseteq \mathsf{SF}_A^{\sim}(w)$, one can check whether $\Xi \in W^{\mathsf{SF}}$ using polynomial space since event formulas contain only $\mathbf{S5}$ and $\mathbf{K}$ modalities. Moreover, one can show as in (Fagin and Halpern 1994, Theorem 4.1) that for every $\beta \in \mathsf{SF}_A^{\sim}(w)$ and $\Xi \in W^{\mathsf{SF}}$, we have $\mathfrak{X}_{w,A}, \Xi \vDash \beta$ iff $\beta \in \Xi$.

We now need to ensure that the values of $\mathsf{Pr}_A(\alpha)$'s at w are coherent with a measure on $\mathfrak{X}_{w,A}$, i.e., that there is some measure $\mu_{w,A}$ s.t. $\mu_{w,A}(\|\alpha\|_{\mathfrak{X}_{w,A}}) = \mathsf{Pr}_A(\alpha)$ for all $\alpha \in \mathsf{At}_A(w)$. We add fresh variables y_α^A for every $\alpha \in \mathsf{At}_A(w)$ and u_Ξ^A for every $\Xi \subseteq \mathsf{SF}_A^{\sim}(w)$. Here, y's are auxiliary variables that will stand for the values of $\mathsf{Pr}_A(\alpha)$'s at w, and u's are the values of $\mu_{w,A}(\{\Xi\})$'s (i.e., we encode the *probability distribution* induced by $\mu_{w,A}$). Furthermore, we set $a_{\Xi,\alpha} = 1$ if $\alpha \in \Xi$ and $a_{\Xi,\alpha} = 0$, otherwise. Now, we consider the following system of linear equations:

$$\left\{\sum_{\Xi\,\subseteq\,\mathsf{SF}_A^{\sim}(w)} u_\Xi^A = 1\right\}\cup \qquad (\mathsf{big}_w^A)$$
$$\left\{\sum_{\Xi\,\subseteq\,\mathsf{SF}_A^{\sim}(w)} (a_{\Xi,\alpha}\cdot u_\Xi^A) = y_\alpha^A \;\middle|\; \alpha \in \mathsf{At}_A(w)\right\}$$

In this system, the top equation ensures that the measures of all states in $\mathfrak{X}_{w,A}$ add up to 1, and the bottom equations guarantee that $\mathsf{Pr}_A(\alpha)$'s are obtained as the sums of measures of states where α's are true. Moreover, we note that it takes linear time w.r.t. $\ell(\phi)$ to determine the value of a given a and that (big_w^A) contains $|\mathsf{At}_A(w)|+1$ equations. Thus, applying (Chvátal 1983, Theorem 9.3), we can guess a list L containing $|\mathsf{At}_A(w)|+1$ variables u_Ξ^A s.t. Ξ's are maximal consistent sets and for every $\alpha \in \mathsf{At}_A(w)$, there is some Ξ s.t. $\alpha \in \Xi$. Then we set $u_\Xi^A = 0$ for every $u_\Xi^A \notin \mathsf{L}$. This gives us the following system of equations whose size is *polynomial* w.r.t. $\mathsf{At}_A(w)$:

$$\left\{\sum_{u_\Xi^A\,\in\,\mathsf{L}} u_\Xi^A = 1\right\}\cup \qquad (\mathsf{small}_w^A)$$
$$\left\{\sum_{u_\Xi^A\,\in\,\mathsf{L}} (a_{\Xi,\alpha}\cdot u_\Xi^A) = y_\alpha^A \;\middle|\; \alpha \in \mathsf{At}_A(w)\right\}$$

We repeat this process, obtaining (small_w^A)'s for every $A \in \mathtt{Agt}$. Then, we consider the following system of polynomial inequalities:

$$\mathsf{S}_w = \mathscr{B}_w^{\mathbb{I}} \cup \bigcup_{A\in\mathtt{Agt}} (\mathsf{small}_w^A) \cup \bigcup_{\substack{\alpha\in\mathsf{At}_A(w)\\ A\in\mathtt{Agt}}} \left\{y_\alpha^A = x_{w:\mathsf{Pr}_A(\alpha)}\right\}$$

Note that the size of S_w is polynomial in $\ell(\phi)$. Thus, it takes polynomial space to determine whether S_w has solutions. If it *does not*, then we close $\mathscr{B}$, and pick the next branch. If *it does*, we mark the constraint that generated w (say, $w' : [a]\psi \leq P$) as 'safe', delete $w'\mathsf{R}_a w$, S_w, and all formulas labelled with w from $\mathscr{B}$. We then pick another formulaic constraint of the form $w' : \blacksquare\psi \leq P'$ and repeat the process. Once all constraints of the form $w' : \blacksquare\psi \leq P'$ are 'safe', we mark w' as 'safe'. We proceed like this until w_0 is marked 'safe' (in which case, ϕ is $\mathsf{EA}^{\mathsf{Pr}}_{\mathsf{fb}}$-satisfiable) or all branches of the tableau are closed (i.e., ϕ is unsatisfiable). □

We finish the section with a brief remark.

Remark 2. Recall from (Hájek 2005, Lemma 3) and (Vidal 2022, Lemma 4.6) that modal Łukasiewicz logic has the *witnessed model property*. A witnessed model is a model $\mathfrak{M}$ where $\mathscr{I}_{\mathfrak{M}}([a]\phi, w) = x$ (or $\mathscr{I}_{\mathfrak{M}}(\mathsf{K}_A\phi', w') = x'$, respectively) implies the existence of $s \in \mathsf{R}_a(w)$ ($s' \in \mathsf{E}_A(w)$) s.t. $\mathscr{I}_{\mathfrak{M}}(\phi, s) = x$ ($\mathscr{I}_{\mathfrak{M}}(\phi', s') = x'$). In particular, this, together with Remark 1, means that the tableaux rules (Fig. 1) are sound w.r.t. $\{\bullet, \rightarrow_\Pi\}$-free $\mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$ formulas. Moreover, it follows that a $\{\bullet, \rightarrow_\Pi\}$-free $\phi \in \mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$ is $\mathsf{EA}^{\mathsf{Pr}}_{\mathsf{fb}}$-satisfiable iff it is $\mathsf{EA}^{\mathsf{Pr}}$-satisfiable. Thus, witnessed model property *implies finite model property*.

Thus, the next statement follows.

Theorem 3. *Given a $\{\bullet, \rightarrow_\Pi\}$-free $\phi \in \mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$, it is* PSPACE-*complete to determine whether it is* $\mathsf{EA}^{\mathsf{Pr}}$*-satisfiable.*

5 Polynomial Fragments of $\mathsf{EA}^{\mathsf{Pr}}$

As is well-known, classical logic has many polynomially decidable fragments. One of the most important such fragments is *Horn logic*. A *Horn theory* can be represented as a set of *rules* of two kinds: $\bigwedge_{i=1}^{n} p_i \rightarrow q$ and $\bigwedge_{i=1}^{n} p_i \rightarrow \bot$. In Łukasiewicz logic, there is a similar polynomially decidable fragment studied by Bofill et al. (2019). It is called *simple-clause fragment*: a simple clause theory is a theory that contains clauses of the form $\bigoplus_{i=1}^{m} p_i \oplus \bigoplus_{j=1}^{n} \neg q_j$. Using Definition 6 and Convention 1, simple clauses can also be represented in the rule form.

In this section, we will consider two modal probabilistic expansions of the simple clause fragment. The difference is that instead of propositional literals, our clauses (rules) will be built over *uniform probabilistic literals*.

Definition 16 (Uniform modal literals (UML's))**.** *Consider the following grammar that defines a fragment of $\mathscr{L}_{\mathbf{evt}}$:*

$$\pi \coloneqq p \mid \neg p \mid \langle a \rangle \pi \mid \widehat{\mathsf{K}}_A \pi$$
$$\nu \coloneqq p \mid \neg p \mid [a] \nu \mid \mathsf{K}_A \nu$$

Formulas of the type π and ν are called $\blacklozenge$-literals *and* $\blacksquare$-literals*, respectively. A literal is* proper *if it contains at least one modality and* propositional *otherwise.*

Definition 17 (Uniform probabilistic literals (UPL's))**.** *Consider the following grammar (λ is a $\neg$-free* UML*):*

$$\zeta \coloneqq \mathsf{Pr}_A(\lambda) \mid \neg \mathsf{Pr}_A(\lambda) \mid \langle a \rangle \zeta \mid \widehat{\mathsf{K}}_A \zeta$$
$$\eta \coloneqq \mathsf{Pr}_A(\lambda) \mid \neg \mathsf{Pr}_A(\lambda) \mid [a] \eta \mid \mathsf{K}_A \eta$$

Formulas of the type ζ and η are called $\blacklozenge^{\mathsf{Pr}}$-literals *and* $\blacksquare^{\mathsf{Pr}}$-literals*, respectively. A literal is* proper *if it contains at least one modality* on the outer layer *and* propositional *otherwise.*

Observe that all modal events from Example 1 are expressed via UML's and can occur in UPL's. Furthermore, it is important to note that $\blacksquare^{\mathsf{Pr}}$-literals can contain $\blacklozenge$, and $\blacklozenge^{\mathsf{Pr}}$-literals can contain $\blacksquare$ on the inner layer. Moreover, the fragments of $\mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$ that we are considering in this section will not use connectives $\bullet$ and $\rightarrow_\Pi$. Thus, again (cf. Remark 2), by (Hájek 2005, Lemma 3) and (Vidal 2022, Lemma 4.6), it follows that their $\mathsf{EA}^{\mathsf{Pr}}$-satisfiability can be w.l.o.g. established w.r.t. *witnessed models* and using rules from Fig. 1.

5.1 Universal Probabilistic Rules

Let us now define our first fragment — *universal probabilistic rules*. As the name suggests, we can only use box-like modalities on the outer layer. Note, however, that *inner formulas* can use both types of modalities.

Definition 18 (Universal probabilistic rules)**.** *A* universal probabilistic rule *(*UPR*) is*

- *either a formula of the form $\bigodot_{i=1}^{n} \eta_i \rightarrow \eta'$ s.t. η's are* proper $\blacksquare^{\mathsf{Pr}}$*-literals;*
- *or a formula of the form $\bigodot_{i=1}^{n} \eta_i \rightarrow \overline{0}$ s.t. η's are $\blacksquare^{\mathsf{Pr}}$-literals.*

A UPR theory *is a set of formulas containing* UPR*'s.*

For simplicity, we will write rules $\eta \rightarrow \overline{0}$ as $\blacklozenge^{\mathsf{Pr}}$-literals. E.g., $\langle a \rangle \neg \mathsf{Pr}_A(p)$ instead of $[a]\mathsf{Pr}_A(p) \rightarrow \overline{0}$.

Let us briefly illustrate the expressiveness of universal probabilistic rules. We recall several statements considered in Section 3. One can see we can *compare* two probabilities: $\mathsf{Pr}_A([a]p) \rightarrow \mathsf{Pr}_A([b]p)$ is a UPR. We can also express that performing a does not *increase* the probability with $[a]\mathsf{Pr}_{\mathsf{ob}}(\mathsf{K}_A p) \rightarrow \mathsf{Pr}_{\mathsf{ob}}(\mathsf{K}_A p)$. Moreover, $\overline{\tfrac{1}{2}} \rightarrow \mathsf{Pr}_{\mathsf{ob}}(\mathsf{K}_A p)$ can be rewritten as follows: $\{\mathsf{Pr}_{\mathsf{ob}}(q) \rightarrow \neg \mathsf{Pr}_{\mathsf{ob}}(q), \neg \mathsf{Pr}_{\mathsf{ob}}(q) \rightarrow \mathsf{Pr}_{\mathsf{ob}}(q), \mathsf{Pr}_{\mathsf{ob}}(q) \rightarrow \mathsf{Pr}_{\mathsf{ob}}(\mathsf{K}_A p)\}$.

On the other hand, we cannot express that an action does not *decrease* a given probability: $\mathsf{Pr}_{\mathsf{ob}}([a]p) \rightarrow [b]\mathsf{Pr}_{\mathsf{ob}}([a]p)$ is not a UPR since $\mathsf{Pr}_{\mathsf{ob}}([a]p)$ is not a proper $\blacksquare^{\mathsf{Pr}}$-literal. Likewise, qualitative uncertainty — $\mathsf{K}_A \mathsf{Pr}_{\mathsf{ob}}(p) \leftrightarrow \widehat{\mathsf{K}}_A \mathsf{Pr}_{\mathsf{ob}}(p)$ — is not expressible as a UPR.

Furthermore, crisp statements with $\triangle$ rational constants other than $\overline{\tfrac{1}{2}}$ (cf. Examples 2 and 3) are not expressible either. On the other hand, all formulas in Example 4 belong to the UPR fragment.

The next statement establishes the polynomial decidability of UPR theories. Observe from Remark 2 that since UPR theories *do not contain* $\bullet$ and $\rightarrow_\Pi$, our decidability result holds for *all* probabilistic models.

Theorem 4. *Given a* UPR *theory Γ, it takes polynomial time to verify whether it is* $\mathsf{EA}^{\mathsf{Pr}}$*-satisfiable.*

Let us sketch the proof. We first show that it takes polynomial time to decide whether a set of UPL's is satisfiable. Then, we consider *irreducible theories*, i.e., UPR theories $\Gamma = \Theta \cup \Xi$ where Θ is a set of UPL's and Ξ is a set of UPR's with at least two literals each s.t. Ξ does not contain any literal θ s.t. $\Theta \models_{\mathsf{EA}^{\mathsf{Pr}}} \neg\theta$. We show that it takes polynomial time to verify whether a given UPR theory is irreducible.

We then prove that every irreducible theory is $\mathsf{EA}^{\mathsf{Pr}}$-satisfiable. First, note that Θ is satisfiable. Second, we pick one *proper* $\blacksquare^{\mathsf{Pr}}$-literal from the left-hand side of each rule in Ξ (that contains proper $\blacksquare^{\mathsf{Pr}}$-literals) and show that there is a pointed probabilistic model $\langle \mathfrak{M}, w \rangle$ s.t. $\mathscr{I}_{\mathfrak{M}}(\theta, w) = 1$ for all $\theta \in \Theta$, and every literal we picked has value 0. Third, we show that the propositional literals in the rules that are not satisfied by that model can be evaluated at $\frac{1}{2}$ in w (whence all rules *without* proper $\blacksquare^{\mathsf{Pr}}$-literals have value 1 at w). This guarantees us that every formula in Γ has value 1 at w.

Let us now sketch a polynomial decision procedure for UPR theories. First, we transform the theory into the clause form, using that $\mathscr{I}_{\mathfrak{M}}(\phi \rightarrow \chi, w) = \mathscr{I}_{\mathfrak{M}}(\neg\phi \oplus \chi, w)$ in each probabilistic model $\mathfrak{M}$. Second, we apply the following rule:

$$\text{if } \Theta \models_{\mathsf{EA}^{\mathsf{Pr}}} \neg\theta, \text{ then } \Theta, \theta \oplus \kappa \models_{\mathsf{EA}^{\mathsf{Pr}}} \kappa$$

Using this rule, we need polynomial time to either transform Γ into an irreducible theory or infer a literal θ' s.t. $\Theta', \theta' \models_{\mathsf{EA}^{\mathsf{Pr}}} \overline{0}$ with Θ' being (possibly) augmented with other UPL's inferred via the above transformation. In the first case, Γ is $\mathsf{EA}^{\mathsf{Pr}}$-satisfiable, and in the second, it is not.

5.2 Existential Probabilistic Rules

Our second fragment is *existential* probabilistic rules. One can think of them as a probabilistic version of the Łukasiewicz description logic $\mathscr{EL}_\bot$ (cf., e.g., (Baader et al. 2017, Ch. 6) for a presentation of *classical* $\mathscr{EL}_\bot$). The main difference, of course, is that we will consider the complexity of *local* satisfiability (whether a formula is true in some state of a model) of theories containing existential probabilistic rules. This is because *global* satifiability (whether a formula is true in all states of a model) of Łukasiewicz $\mathscr{EL}_\bot$ is undecidable (Baader and Peñaloza 2011; Cerami and Straccia 2013; Borgwardt, Cerami, and Peñaloza 2015; Borgwardt, Cerami, and Peñaloza 2017; Baader, Borgwardt, and Peñaloza 2017; Bobillo and Straccia 2018).

Definition 19. *An* existential positive probabilistic rule *(EPR^+) is a formula of one of the following kinds:*
- *a* UPL*;*
- $\bigodot_{i=1}^{n} \zeta_i \rightarrow \overline{0}$ *with ζ's being propositional literals or $\neg$-free $\blacklozenge^{\mathsf{Pr}}$-literals;*
- $\bigodot_{i=1}^{n} \zeta_i \rightarrow \zeta$ *with ζ's being* proper *$\neg$-free $\blacklozenge^{\mathsf{Pr}}$-literals.*

There is also a syntactic difference between EPR^+ and $\mathscr{EL}_\bot$. Namely, we can freely use negation over propositional atoms. E.g., a rule $\neg\mathsf{Pr}_A([a]p)\odot\neg\mathsf{Pr}_B(\mathsf{K}_A\mathsf{K}_B q)\rightarrow\overline{0}$ is allowed, despite the negation in the left-hand side.

On the other hand, EPR^+'s and UPR's are not completely dual. In particular, in contrast to UPR theories, we cannot use negation in $\blacklozenge^{\mathsf{Pr}}$-literals when they occur in implications. Thus, existential rules such as $\langle a\rangle\neg\mathsf{Pr}_A(p) \rightarrow \widehat{\mathsf{K}}_A\mathsf{Pr}_B(q)$ are not allowed (while $[a]\neg\mathsf{Pr}_A(p) \rightarrow \mathsf{K}_A\mathsf{Pr}_B(q)$ is a UPR). Hence, no formula from Examples 2–4, except $\mathsf{Pr}_A([m]B\,\texttt{folds}) \rightarrow \mathsf{Pr}_A([a]B\,\texttt{folds})$, belongs to the EPR^+ fragment.

The next statement shows that $\mathsf{EA}^{\mathsf{Pr}}$-satisfiability of EPR^+ theories is also decidable in polynomial time. The proof is similar to that of Theorem 4.

Theorem 5. *Given an EPR^+ theory Γ, it takes polynomial time to verify whether it is $\mathsf{EA}^{\mathsf{Pr}}$-satisfiable.*

We conclude the section with a short sketch of the proof. Observe first from Theorem 4 that it takes polynomial time to check the $\mathsf{EA}^{\mathsf{Pr}}$-satisfiability of a given set of UPL's or whether an UPL is entailed from a set of UPL's.

Now, as was the case in Theorem 4, it remains to show that irreducible EPR^+ theories are $\mathsf{EA}^{\mathsf{Pr}}$-satisfiable. The difference is that instead of picking $\blacksquare^{\mathsf{Pr}}$-literals from each rule and assigning them value 0 at w, we show that there is a pointed probabilistic model $\langle\mathfrak{M}, w\rangle$ where one proper $\blacklozenge^{\mathsf{Pr}}$-*literal* from the left-hand side of each rule has value 0 at w. Thus, given an EPR^+ theory, we can use the same procedure as in Theorem 4 to either transform it into an irreducible (and hence, satisfiable) theory or infer a contradiction.

6 Related Work

As we mentioned in the introduction, the first family of logics that combined modal (epistemic) and probabilistic reasoning was proposed by Fagin and Halpern (1994). There are several important differences between their logic and ours. First, probabilistic formulas in (Fagin and Halpern 1994) are expressed via linear inequalities that are *two-valued*, and thus, are not well suited to express imprecise statements such as 'the probability of α *is not much* lower (higher) than the probability of β'. Second, the expressivity of the modal probabilistic language proposed by Fagin and Halpern is incomparable with that of $\mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$. On the one hand, there are no separate *inner* and *outer* formulas. Thus, e.g., the following formula (adapted to the notation in this paper) is permitted: $(\mathsf{Pr}_B(\mathsf{K}_A(\mathsf{Pr}_A(a)\geq\overline{\tfrac{2}{3}}))\leq\overline{\tfrac{1}{2}})\rightarrow\mathsf{K}_B q$. On the other hand, inequality formulas can contain only one agent label. Thus, statements such as 'A and B consider α equally likely' ($\triangle(\mathsf{Pr}_A(\alpha)\leftrightarrow\mathsf{Pr}_B(\alpha))$ in $\mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$) cannot be expressed, unless one specifies the degree of likelihood. Furthermore, the validity of some probabilistic logics is EXPTIME-complete (Fagin and Halpern 1994, Theorem 4.5). Observe, however, that all logics in (Fagin and Halpern 1994) have the finite model property, while in our case, this holds only for the $\{\bullet, \rightarrow_\Pi\}$-free fragment of $\mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$ (recall Remark 2).

Our work, however, continues the study of *fuzzy* modal probabilistic logics. In particular, we expand the fuzzy *epistemic* probabilistic logic proposed by Corsi et al. (2023) with *action* modalities as well as with additional connectives $\bullet$ and $\rightarrow_\Pi$. These allow us to express (subjective) *conditional probability* of α given β via $\mathsf{Pr}_A(\beta) \rightarrow_\Pi \mathsf{Pr}_A(\alpha \wedge \beta)$ as well as *degrees of independence* of α and β: $\mathsf{Pr}_A(\alpha \wedge \beta) \leftrightarrow (\mathsf{Pr}(\alpha) \bullet \mathsf{Pr}(\beta))$. Moreover, Corsi et al. study the axiomatisation and expressivity of their epistemic probabilistic logic, while we are concentrating on complexity. We also note that the logic in (Corsi et al. 2023) is the fragment of $\mathsf{EA}^{\mathsf{Pr}}$ without $\bullet$, $\rightarrow_\Pi$, and action modalities.

Another closely related work is by Majer and Sedlár (2025) and Kozhemiachenko and Sedlár (2025). There, a logic $\mathbf{K\text{Ł}\Pi\tfrac{1}{2}}^{\mathsf{Pr}}$ with action modalities and propositional event formulas is considered. Note, however, that even though $\mathsf{EA}^{\mathsf{Pr}}$ is more expressive than $\mathbf{K\text{Ł}\Pi\tfrac{1}{2}}^{\mathsf{Pr}}$, the satisfiability problem in both logics is PSPACE-complete. Thus, in this paper, we combined the frameworks of (Corsi et al. 2023) and (Kozhemiachenko and Sedlár 2025), by proposing a logic with both epistemic and action modalities on its outer and inner layers.

Finally, we note that to the best of our knowledge, there are no results on the complexity (and decidability) of Łukasiewicz modal logic extended with $\triangle$ over arbitrary frames (this logic fails the FMP as we observed in the end of Section 3), let alone of the modal logic over **ŁΠ** (combined Łukasiewicz and Product modal logic). Thus, there are no results on the decidability of their probabilistic expansions. Still, there is an interesting pattern: usually, the complexity of a two-layered logic built over logics $\mathbf{L}_1$ and $\mathbf{L}_2$ coincides with that of the harder logic. E.g., $\mathsf{FP}(\mathbf{Ł\Pi})$ by Hajek and Tulipani (2001) uses events expressed as Boolean formulas and is in PSPACE, just as the propositional **ŁΠ**. Likewise, the satisfiability of arbitrary probabilistic atoms $\mathsf{Pr}_A(\alpha)$ is PSPACE-hard because (classical) modal logic is PSPACE-hard. Thus, we conjecture that the complexity of $\mathsf{EA}^{\mathsf{Pr}}$ (over arbitrary frames) will coincide with that of $\mathsf{EA}^{\mathbf{Ł\Pi}}$.

7 Conclusion and Future Work

We introduced and studied $\mathsf{EA}^{\mathsf{Pr}}$, a modal probabilistic logic that unifies and expands the frameworks of Corsi et al. (2023) on one hand and Majer and Sedlár (2025) and Kozhemiachenko and Sedlár (2025), on the other hand. A distinguishing feature of our framework is the presence of epistemic and action modalities in both logical layers.

We showed that $\mathsf{EA}^{\mathsf{Pr}}$ can express probabilistic statements about modal events involving knowledge or the effects of actions and lower and upper probabilities of events arising from epistemic uncertainty or from uncertainty about outcomes of actions. Such specifications are important in domains such as robotics and cybersecurity. We established that the satisfiability problem for the full language over finitely-branching frames is PSPACE-complete. We also identified specific syntactic fragments for which validity is decidable in polynomial time.

There are several promising avenues for future research. A primary objective is to provide a sound and complete axiomatisation for $\mathsf{EA}^{\mathsf{Pr}}$. To enrich the framework and improve its applicability to reasoning about programs and multi-agent systems, we also intend to extend the current language to incorporate Propositional Dynamic Logic constructs and operators for group knowledge.

Finally, we plan to explore the semantic boundaries and other computational aspects of the logic. Our current complexity results in Section 4 for the full $\mathscr{L}^{\mathsf{Pr}}_{\mathsf{EA}}$ rely on a restriction to finitely branching frames. Investigating the complexity of the logic without this constraint is a necessary step towards generalising the framework. We also intend to expand the scope of uncertainty representation by incorporating fuzzy events and exploring other uncertainty measures alongside their qualitative counterparts. Identifying additional polynomially-decidable fragments of $\mathsf{EA}^{\mathsf{Pr}}$ remains a priority for practical implementation.

Acknowledgements

Igor Sedlár was supported by a grant from the Programme Johannes Amos Comenius under the Ministry of Education, Youth and Sports of the Czech Republic, CZ.02.01.01/00/23_025/0008711. The authors would also like to thank the reviewers for their constructive suggestions that improved the quality of the paper.

AI Declaration

The authors used Google Gemini to obtain feedback on the formulations of Examples in Section 3.

References

Akkaladevi, S. C.; Plasch, M.; Hofmann, M.; and Pichler, A. 2021. Semantic knowledge based reasoning framework for human robot collaboration. *Procedia CIRP* 97:373–378.

Baader, F., and Peñaloza, R. 2011. On the undecidability of fuzzy description logics with GCIs and product t-norm. In Tinelli, C., and Sofronie-Stokkermans, V., eds., *Proceedings of the 8th International Conference on Frontiers of Combining Systems (FroCoS-11)*, volume 6989 of *Lecture Notes in Artificial Intelligence*, 55–70. Berlin, Heidelberg: Springer-Verlag.

Baader, F.; Horrocks, I.; Lutz, C.; and Sattler, U. 2017. *An Introduction to Description Logic*. Cambridge University Press.

Baader, F.; Borgwardt, S.; and Peñaloza, R. 2017. Decidability and Complexity of Fuzzy Description Logics. *Künstliche Intelligenz* 31(1):85–90.

Bacchus, F.; Halpern, J. Y.; and Levesque, H. J. 1999. Reasoning about noisy sensors and effectors in the situation calculus. *Artificial Intelligence* 111(1–2):171–208.

Bach, S. H.; Broecheler, M.; Huang, B.; and Getoor, L. 2017. Hinge-loss markov random fields and probabilistic soft logic. *Journal of Machine Learning Research* 18(109):1–67.

Badreddine, S.; Avila Garcez, A. d.; Serafini, L.; and Spranger, M. 2022. Logic tensor networks. *Artificial Intelligence* 303(103649):103649.

Baldi, P.; Cintula, P.; and Noguera, C. 2020. Classical and Fuzzy Two-Layered Modal Logics for Uncertainty: Translations and Proof-Theory. *International Journal of Computational Intelligence Systems* 13:988–1001.

Barthe, G.; Köpf, B.; Olmedo, F.; and Zanella-Béguelin, S. 2013. Probabilistic relational reasoning for differential privacy. *ACM Transactions on Programming Languages and Systems* 35(3):1–49.

Blackburn, P.; Rijke, M. d.; and Venema, Y. 2010. *Modal logic*. Cambridge tracts in theoretical computer science 53. Cambridge University Press, 4. print. with corr. edition.

Bobillo, F., and Straccia, U. 2018. Reasoning within Fuzzy OWL 2 EL Revisited. *Fuzzy Sets and Systems* 351:1–40.

Bobillo, F.; Delgado, M.; Gómez-Romero, J.; and Straccia, U. 2009. Fuzzy description logics under Gödel semantics. *International Journal of Approximate Reasoning* 50(3):494–514.

Bobillo, F.; Delgado, M.; Gómez-Romero, J.; and Straccia, U. 2012. Joining Gödel and Zadeh fuzzy logics in fuzzy description logics. *International Journal of Uncertainty Fuzziness and Knowledge-Based Systems* 20(04):475–508.

Bofill, M.; Manyà, F.; Vidal, A.; and Villaret, M. 2019. New Complexity Results for Łukasiewicz Logic. *Soft Computing* 23(7):2187–2197.

Borgwardt, S.; Cerami, M.; and Peñaloza, R. 2015. The Complexity of Subsumption in Fuzzy EL. In Yang, Q., and Wooldridge, M. J., eds., *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, IJCAI 2015*, 2812–2818. AAAI Press.

Borgwardt, S.; Cerami, M.; and Peñaloza, R. 2017. The complexity of fuzzy EL under the łukasiewicz t-norm. *International Journal of Approximate Reasoning* 91:179–201.

Borgwardt, S.; Distel, F.; and Peñaloza, R. 2014a. Decidable Gödel description logics without the finitely-valued model property. In *Fourteenth International Conference on the Principles of Knowledge Representation and Reasoning*, 228–237.

Borgwardt, S.; Distel, F.; and Peñaloza, R. 2014b. Gödel description logics with general models. In *DL 2014: Informal Proceedings of the 27th International Workshop on Description Logics*, volume 1193, 391–403. CEUR.

Brown, N., and Sandholm, T. 2019. Superhuman AI for multiplayer poker. *Science* 365(6456):885–890.

Cerami, M., and Straccia, U. 2013. On the (Un)Decidability of Fuzzy Description Logics under Lukasiewicz t-Norm. *Information Sciences* 227:1–21.

Chen, X.; Hu, Z.; and Sun, Y. 2022. Fuzzy logic based logical query answering on knowledge graphs. In *Proceedings of the 36th AAAI Conference on Artificial Intelligence*, volume 36, 3939–3948. Association for the Advancement of Artificial Intelligence (AAAI).

Chvátal, V. 1983. *Linear programming*. W.H. Freeman and Company.

Corsi, E.; Flaminio, T.; Godo, L.; and Hosni, H. 2023. A modal logic for uncertainty: a completeness theorem. In Miranda, E.; Montes, I.; Quaeghebeur, E.; and Vantaggi, B., eds., *Proceedings of the Thirteenth International Symposium on Imprecise Probability: Theories and Applications*, volume 215 of *Proceedings of Machine Learning Research*, 119–129. PMLR.

Diligenti, M.; Gori, M.; and Saccà, C. 2017. Semantic-based regularization for learning and inference. *Artificial Intelligence* 244:143–165.

Doder, D., and Ognjanovic, Z. 2024. Probabilistic temporal logic with countably additive semantics. *Annals of Pure and Applied Logic* 175(9):103389.

Doder, D., and Perović, A. 2020. Probabilistic temporal logics. In Ognjanović, Z., ed., *Probabilistic Extensions of Various Logical Systems*. Springer International Publishing. 71–108.

Doder, D.; Savic, N.; and Ognjanovic, Z. 2020. Multi-agent Logics for Reasoning About Higher-Order Upper and Lower Probabilities. *Journal of Logic, Language and Information* 29(1):77–107.

Esteva, F.; Godo, L.; and Montagna, F. 2001. The $L\Pi$ and $L\Pi\frac{1}{2}$ logics: two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic* 40(1):39–67.

Fagin, R., and Halpern, J. 1994. Reasoning about knowledge and probability. *Journal of the ACM* 41(2):340–367.

Fagin, R.; Halpern, J.; and Megiddo, N. 1990. A logic for reasoning about probabilities. *Information and computation* 87(1–2):78–128.

Fazzinga, B.; Flesca, S.; and Parisi, F. 2015. On the complexity of probabilistic abstract argumentation frameworks. *ACM Transactions on Computational Logic* 16(3).

Godo, L.; Hájek, P.; and Esteva, F. 2003. A fuzzy modal logic for belief functions. *Fundamenta Informaticae* 57(2–4):127–146.

Hähnle, R. 1999. Tableaux for Many-Valued Logics. In Agostino, M. D.; Gabbay, D. M.; Hähnle, R.; and Posegga, J., eds., *Handbook of Tableaux Methods*, 529–580. Springer-Science+Business Media, B.V.

Hájek, P., and Tulipani, S. 2001. Complexity of fuzzy probability logics. *Fundamenta Informaticae* 45(3):207–213.

Hájek, P.; Godo, L.; and Esteva, F. 1995. Fuzzy logic and probability. In Besnard, P., and Hanks, S., eds., *UAI'95: Proceedings of the Eleventh Conference on Uncertainty in Artificial Intelligence*, 237–244. San Francisco: Morgan Kaufmann.

Hájek, P.; Godo, L.; and Esteva, F. 2000. Reasoning about probability using fuzzy logic. *Neural Network World* 10(5):811–824.

Hájek, P. 1998. *Metamathematics of Fuzzy Logic*. Trends in Logic 4. Dordrecht: Springer.

Hájek, P. 2005. Making fuzzy description logic more general. *Fuzzy Sets and Systems* 154(1):1–15.

Hájek, P. 2007. On witnessed models in fuzzy logic. *Mathematical Logic Quarterly* 53(1):66–77.

Halpern, J. Y., and O'Neill, K. R. 2005. Anonymity and information hiding in multiagent systems. *Journal of Computer Security* 13(3):483–514.

Halpern, J. Y., and Pucella, R. 2002. A logic for reasoning about upper probabilities. *Journal of Artificial Intelligence Research* 17:57–81.

He, M.; Leung, H.; and Jennings, N. R. 2003. A Fuzzy-Logic Based Bidding Strategy for Autonomous Agents in Continuous Double Auctions. *IEEE Transactions on Knowledge and Data Engineering* 15(6):1345–1363.

Hájek, P., and Harmancová, D. 1995. Medical fuzzy expert systems and reasoning about beliefs. In *Artificial Intelligence in Medicine. AIME 1995*, 403–404. Springer Berlin Heidelberg.

Kozhemiachenko, D., and Sedlár, I. 2026. Reasoning About Probabilities, Actions, and Knowledge in Fuzzy Modal Logic. Available at arXiv: 2604.22459.

Kozhemiachenko, D., and Sedlár, I. 2025. Complexity of Łukasiewicz modal probabilistic logics. In Bjorndahl, A., ed., *Proceedings Twentieth Conference on Theoretical Aspects of Rationality and Knowledge*, volume 437 of *Elec-*

tronic Proceedings in Theoretical Computer Science, 350–364.

Krieken, E. v.; Acar, E.; and Harmelen, F. v. 2022. Analyzing differentiable fuzzy logic operators. *Artificial Intelligence* 302(103602):103602.

Li, H.; Oren, N.; and Norman, T. J. 2012. Probabilistic argumentation frameworks. In *Theorie and Applications of Formal Argumentation*, Lecture Notes in Computer Science. Berlin, Heidelberg: Springer Berlin Heidelberg. 1–16.

Lukasiewicz, T. 1998. Probabilistic logic programming. In Prade, H., ed., *13th European Conference on Artificial Intelligence*, 388–392. John Wiley and Sons.

Lukasiewicz, T. 2008. Expressive probabilistic description logics. *Artificial Intelligence* 172(6–7):852–883.

Majer, O., and Sedlár, I. 2025. On Many-Valued Modal Probabilistic Logics. In *55th IEEE International Symposium on Multiple-Valued Logic, ISMVL 2025*, 26–31. IEEE.

Marchioni, E. 2008. Representing Upper Probability Measures over Rational Łukasiewicz Logic. *Mathware and Soft Computing* 15(2):159–173.

Ng, R., and Subrahmanian, V. S. 1992. Probabilistic logic programming. *Information and Computation* 101(2):150–201.

Nilsson, N. J. 1986. Probabilistic logic. *Artificial Intelligence* 28(1):71–87.

Ognjanovic, Z.; Markovic, Z.; Raskovic, M.; Doder, D.; and Perovic, A. 2012. A propositional probabilistic logic with discrete linear time for reasoning about evidence. *Annals of Mathematics and Artificial Intelligence* 65(2-3):217–243.

Peñaloza, R. 2020. Introduction to probabilistic ontologies. In *Reasoning Web. Declarative Artificial Intelligence*, Lecture Notes in Computer Science. Cham: Springer International Publishing. 1–35.

Riguzzi, F., and Swift, T. 2018. A survey of probabilistic logic programming. In *Declarative Logic Programming: Theory, Systems, and Applications*. ACM. 185–228.

Thrun, S.; Burgard, W.; and Fox, D. 2005. *Probabilistic Robotics*. MIT Press.

Tomovic, S.; Ognjanovic, Z.; and Doder, D. 2020. A First-order Logic for Reasoning about Knowledge and Probability. *ACM Trans. Comput. Log.* 21(2):16:1–16:30.

Vidal, A. 2022. Undecidability and non-axiomatizability of modal many-valued logics. *The Journal of Symbolic Logic* 87(4):1576–1605.

Zadeh, L. A. 1965. Fuzzy sets. *Information and Control* 8(3):338–353.

Zadeh, L. A. 1975. Fuzzy logic and approximate reasoning. *Synthese* 30(3-4):407–428.