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Representation Theorems for Cumulative Propositional Dependence Logics

Juha Kontinen[1*], **Arne Meier**[2*], **Kai Sauerwald**[3*]
[1]Department of Mathematics and Statistics, University of Helsinki, Helsinki, Finland
[2]Theoretical Computer Science, Leibniz University Hannover, Hannover, Germany
[3]Faculty of Mathematics and Computer Science, University of Hagen, Hagen, Germany

Abstract

This paper establishes and proves representation theorems for cumulative propositional dependence logic and for cumulative propositional logic with team semantics. Cumulative logics are famously given by System C. For propositional dependence logic, we show that System C entailments are exactly captured by cumulative models from Kraus, Lehmann and Magidor. On the other hand, we show that entailment in cumulative propositional logics with team semantics is exactly captured by cumulative and asymmetric models. For the latter, we also obtain equivalence with cumulative logics based on propositional logic with classical semantics. The proofs will be useful for proving representation theorems for other cumulative logics without negation and material implication.

1 Introduction

The ability to reason is one of the central features of intelligent agents and thus a major concern of artificial intelligence. In this paper, we study the fusion of non-monotonic reasoning with team-based reasoning. Such a combination is interesting, as it allows reasoning in which one can express settings involving a plurality of objects (the team-based component) while also taking into account extra-logical information, such as plausibility, exceptionality, reliability, preference, or typicality (the non-monotonic component). Notably, the combination of these approaches provides a setting that cannot be formalized by neither of the approaches alone.

Less is known about how to construct non-monotonic entailment relations for team-based logics. Known approaches for non-monotonic propositional logics or non-monotonic first-order logics rely heavily on the properties of the underlying logics, such as, e.g., the availability of all Boolean connectives, the law of excluded middle, and presence of material implication. However, these properties are not (always) available in team-based logics. There are many approaches to non-monotonic logics that take inspiration from (Brewka, Dix, and Konolige 1997), from which some are also generic, e.g., MAK models (Makinson 1989), KLM-style reasoning (Kraus, Lehmann, and Magidor 1990), characterization logics (Baumann and Strass 2025), or approximation fixpoint theory (Denecker, Marek, and Truszczyński 2000).

[*]The authors are ordered alphabetically.

So far, there are only two pioneering works that consider a combination of team semantics and non-monotonic reasoning. First, the work by Yan (2023), which considers a non-monotonic team-based modal logic in the context of formal analysis of natural language. The second is by Sauerwald, Meier, and Kontinen (2025) (SMK), which also contains a broader introduction and motivation, and focuses on non-monotonic reasoning in the style of Kraus, Lehmann, and Magidor (1990) (KLM).

KLM (1990) showed for classical propositional logic that cumulative entailment relations provide a stable theory of reasoning with multiple representations. Cumulative entailment relations are obtained by reasoning via the axiomatic system known as *System C*. Furthermore, a cumulative entailment relation $\mathrel{|\!\sim}$ can be represented by a cumulative model $\mathbb{C}$, by which $\varphi \mathrel{|\!\sim} \psi$ amounts (intuitively) to checking whether all minimal models of φ in $\mathbb{C}$ are models of ψ. SMK (2025) show that entailment via preferential models (which are specific cumulative models) satisfies *System C* in the context of propositional dependence logic. However, SMK provides no representation theorems for any kind of general cumulative reasoning in the context of team-based logics.

Contributions. This work establishes cumulative reasoning as a stable approach in the context of team-based logics. Specifically, we will encounter the cumulative counterparts of the following team-based logics:

- Propositional logic with team-based semantics (TPL)
- Propositional dependence logic (PDL)

For each of these logics, we consider, both, reasoning via *System C* and via cumulative models as in the approach by KLM. This leads to *System C*-based entailment relations for PDL, denoted by $\mathsf{PDL}^{\mathsf{c}}$. Also, we will consider PDL entailment relations via cumulative models, denoted by $\mathsf{PDL}^{\mathsf{cuml}}$. With $\mathsf{TPL}^{\mathsf{cuml}}$ and $\mathsf{TPL}^{\mathsf{c}}$, we denote the respective approaches for TPL. We will show the following representation results:

Representation Theorem for Cumulative PDL**:** $\mathsf{PDL}^{\mathsf{c}}$ and $\mathsf{PDL}^{\mathsf{cuml}}$ define the same entailment relations.

Representation Theorem for Cumulative TPL**:** $\mathsf{TPL}^{\mathsf{c}}$ and $\mathsf{TPL}^{\mathsf{cuml}}$ define the same entailment relations. Furthermore, we show that the novel class of asymmetric models are a sufficient subclass of cumulative models for representation.

The proofs for showing these results demonstrate, in a principled way, how similar relationships between these classes of entailment relations can be obtained for other logics without negation and material implication.

2 Preliminaries

In this paper, we consider propositional logics from a model-theoretic perspective. We denote by $\mathsf{Prop} = \{ p_i \mid i \in \mathbb{N} \}$ the countably infinite set of propositional variables. We consider propositional formulas in negation normal form, i.e., PL-formulas are formed by the grammar, where $p \in \mathsf{Prop}$:

$$\varphi ::= p \mid \neg p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi.$$

We write $\mathsf{Prop}(\varphi)$ for the set of variables occurring in φ.

Classical Propositional Logic (CPL). For a non-empty finite subset $N \subseteq \mathsf{Prop}$ of propositional variables, one defines for valuations $v\colon N \to \{0,1\}$ over N and PL-formulas φ:

$$[\![\varphi]\!]^c = \{ v\colon N \to \{0,1\} \mid v \models \varphi \}.$$

The valuation function v is extended to the *set of all* PL-*formulas satisfying* $\mathsf{Prop}(\varphi) \subseteq N$, $\mathrm{PL}(N)$, in the usual way. We denote by $\mathbb{A}_N$ the set of all assignments over N. We write $\varphi \models^c \psi$ for $[\![\varphi]\!]^c \subseteq [\![\psi]\!]^c$ and $\varphi \equiv^c \psi$ if both $\varphi \models^c \psi$ and $\psi \models^c \varphi$ are true.

Prop. Logic with Team Semantics (TPL). Next, we define *team semantics* for PL-formulas (cf. (Hannula et al. 2018; Yang and Väänänen 2016)). A *team* X is a set of valuations for some finite $N \subseteq \mathsf{Prop}$. The *domain* N *of* X is denoted by $\mathsf{dom}(X)$, and $\mathbb{T}_N$ denotes the set of all such teams.

Definition 1 (Team semantics of PL)**.** *Let X be a team. For any* PL-*formula φ with* $\mathsf{dom}(X) \supseteq \mathsf{Prop}(\varphi)$, *the* satisfaction relation, *$X \models \varphi$, is defined inductively as:*

$X \models p$	*if for all* $v \in X : v \models p$,
$X \models \neg p$	*if for all* $v \in X : v \not\models p$,
$X \models \top$	*is always the case*,
$X \models \bot$	*if* $X = \emptyset$,
$X \models \varphi \wedge \psi$	*if* $X \models \varphi$ *and* $X \models \psi$,
$X \models \varphi \vee \psi$	*if there exist* $Y, Z \subseteq X$ *s.t.* $X = Y \cup Z, Y \models \varphi,$ *and* $Z \models \psi$.

The set of all teams X with $X \models \varphi$ is denoted by $[\![\varphi]\!]^t$. For any two PL-formulas φ, ψ, we write $\varphi \models^t \psi$ if $[\![\varphi]\!]^t \subseteq [\![\psi]\!]^t$. Write $\varphi \equiv^t \psi$ if both $\varphi \models^t \psi$ and $\psi \models^t \varphi$ are true. We define the following properties for a formula φ:

- $X \models \varphi \iff$ for all $v \in X,\ \{v\} \models \varphi$. **(Flatness)**
- $\emptyset \models \varphi$. **(Empty team)**
- If $X \models \varphi$ and $Y \subseteq X$, then $Y \models \varphi$. **(Downward closure)**

Proposition 2. TPL *has the properties flatness, empty team, and downward closure.*

Due to the flatness property, logical entailment of propositional logic with team-based semantics $\models^t$ and classical semantics $\models^c$ coincide.

Propositional Dependence Logic (PDL). A *(propositional) dependence atom* is a string $=\!(\vec{a}, b)$, in which $\vec{a} = a_1, \ldots, a_k$ and b are propositional variables from Prop. A team X *satisfies a dependence atom*, $X \models =\!(\vec{a}, b)$, if for all $v, v' \in X$, $v(\vec{a}) = v'(\vec{a})$ implies $v(b) = v'(b)$. A dependence atom where the first component is empty will be abbreviated as $=\!(p)$ and called a *constancy atom*. The language of *propositional dependence logic* ($\mathrm{PL}(\mathsf{dep})$) is defined as PL-formulas extended by dependence atoms.

Example 3. *Consider the team X over $\{p, q, r\}$ defined by:*

	p	q	r
v_1	1	0	0
v_2	0	1	0
v_3	0	1	0

Here, we have that $X \models =\!(p, q)$ and $X \models =\!(r)$. Moreover, we have that $X \models =\!(p) \vee =\!(p)$, however it is true that $X \not\models =\!(p)$ as the value of p is not overall constant.

Proposition 4. PDL *has the empty team and the downward closure property, but not the flatness property.*

Generic View on Logics. Some parts of this paper require a generic perspective on logics, which we discuss next. A *satisfaction system* is a triple $\mathbb{S} = \langle \mathcal{L}, \Omega, \models \rangle$, where $\mathcal{L}$ is the set of *formulas*, Ω is the set of *interpretations*, and $\models\, \subseteq \Omega \times \mathcal{L}$ is the *satisfaction relation*. We write $[\![\varphi]\!]^{\mathbb{S}} = \{\omega \in \Omega \mid \omega \models \varphi\}$ for the set of all models of the formula $\alpha \in \mathcal{L}$. An *entailment relation* for a satisfaction system is a relation $\Vdash\, \subseteq \mathcal{L} \times \mathcal{L}$ such that $\varphi \Vdash \gamma$ if and only if $\psi \Vdash \gamma$, whenever $[\![\varphi]\!]^{\mathbb{S}} = [\![\psi]\!]^{\mathbb{S}}$. A satisfaction system $\mathbb{S}$ together with an entailment relation $\Vdash$ is called a *logic* and denoted by $\mathscr{L} = \langle \mathcal{L}, \Omega, \models, \Vdash \rangle$. The propositional logics discussed in this section fit into this general model-theoretic view for each $N \subseteq \mathsf{Prop}$ as follows:

$$\mathsf{CPL}_N = \langle \mathrm{PL}(N), \mathbb{A}_N, \models, \models^c \rangle$$
$$\mathsf{TPL}_N = \langle \mathrm{PL}(N), \mathbb{T}_N, \models, \models^t \rangle$$
$$\mathsf{PDL}_N = \langle \mathrm{PL}(\mathsf{dep}, N), \mathbb{T}_N, \models, \models^t \rangle$$

When there is no ambiguity, we will write $\models$ instead of $\models^t$. Moreover, we will use CPL to denote the class consisting of all logics CPL_N for any non-empty finite subset $N \subseteq \mathsf{Prop}$. The classes TPL and PDL are defined analogous.

3 *System C* and Cumulative Models

We consider the construction of entailment relations.

System C. We make use of the following rules for calculi:

$$\frac{\varphi \equiv \psi \quad \varphi \mathrel{|\!\sim} \gamma}{\psi \mathrel{|\!\sim} \gamma}\ \text{(LLE)} \qquad \frac{\varphi \models \psi \quad \gamma \mathrel{|\!\sim} \varphi}{\gamma \mathrel{|\!\sim} \psi}\ \text{(RW)}$$

$$\frac{\varphi \wedge \psi \mathrel{|\!\sim} \gamma \quad \varphi \mathrel{|\!\sim} \psi}{\varphi \mathrel{|\!\sim} \gamma}\ \text{(Cut)} \qquad \frac{\varphi \mathrel{|\!\sim} \psi \quad \varphi \mathrel{|\!\sim} \gamma}{\varphi \wedge \psi \mathrel{|\!\sim} \gamma}\ \text{(CM)}$$

Note that $\models$ is a placeholder for the entailment relation $\Vdash$ of an underlying logic $\mathscr{L} = \langle \mathcal{L}, \Omega, \models, \Vdash \rangle$, and $\equiv$ is the respective semantic equivalence from $\mathscr{L}$. *System C* consists of the rules (RW), (LLE), (CM), (Cut) and reflexivity (Ref) $\varphi \mathrel{|\!\sim} \varphi$ for all formulas (Kraus, Lehmann, and Magidor 1990; Gabbay 1984). We say that an entailment relation $\mathrel{|\!\sim}$ *satisfies System C* if $\mathrel{|\!\sim}$ is closed under all rules of *System C*.

Relational Models. For a relation $R \subseteq \mathcal{S} \times \mathcal{S}$ on a set $\mathcal{S}$ and a subset $S \subseteq \mathcal{S}$, an element $s \in S$ is called *minimal in S with respect to R* if for each $s' \in S$ does *not* hold $s' \mathrel{R} s$. Then, $\min(S, R)$ is the set of all $s \in S$ that are minimal in S

with respect to R. Moreover, for a set of interpretations M and a formula φ of $\mathscr{L} = \langle \mathcal{L}, \Omega, \models, \Vdash \rangle$, we write $M \models \varphi$ if for all $\omega \in M$ we have that $\omega \models \varphi$.

Definition 5 (Shoham 1988, Dix and Makinson 1989; 1992)**.** *Let $\mathscr{L} = \langle \mathcal{L}, \Omega, \models, \Vdash \rangle$ be a logic. A* relational model *for $\mathscr{L}$ is a triple $\mathbb{M} = \langle \mathcal{S}, \ell, R \rangle$ where $\mathcal{S}$ is a set, $\ell \colon \mathcal{S} \to \mathcal{P}(\Omega)$, and R is a binary relation on $\mathcal{S}$.*

Intuitively, a relational models 'orders' interpretations of the underlying logic $\mathscr{L}$. However, to permit multiple appearance of interpretations in the relation, states are introduced. This construction was shown to be necessary (KLM, 1990; Freund, 1993).

We will deal, in this paper, with specific types of relational models, which we define next. We say $S \subseteq \mathcal{S}$ is *smooth* if for each $s \in S$, we either have that $s \in \min(S, R)$, or there exists a state $s' \in \min(S, R)$ with $s' \; R \; s$. For $\varphi \in \mathcal{L}$, we denote by $\mathcal{S}(\varphi) = \{ s \in \mathcal{S} \mid \ell(s) \models \varphi \}$ the set of states that satisfy φ. With $\min(\llbracket \varphi \rrbracket, R) = \bigcup \{ \ell(s) \mid s \in \min(\mathcal{S}(\varphi), R) \}$, we denote the set of all interpretations that appear in $\ell(s)$ for any minimal state s that satisfy φ.

Definition 6 (KLM, 1990)**.** *A relational model $\mathbb{C} = \langle \mathcal{S}, \ell, R \rangle$ for a logic $\mathscr{L} = \langle \mathcal{L}, \Omega, \models, \Vdash \rangle$ is called* cumulative *if for all $\varphi \in \mathcal{L}$ the set $\mathcal{S}(\varphi)$ is smooth. A cumulative model $\mathbb{C}$ is* strong *if R is asymmetric (xRy implies $\neg(yRx)$) and $\min(\mathcal{S}(\varphi), R)$ is a singleton for each formula $\varphi \in \mathcal{L}$.*

Because cumulative models satisfy smoothness, they avoid the problem of reasoning via arbitrary relational models, in which the set $\min(\llbracket \varphi \rrbracket, R)$ might be empty, even when $\mathcal{S}(\varphi)$ is non-empty. Entailment relations, which are induced by a relational model, are defined as follows.

Definition 7 (KLM, 1990)**.** *Let $\mathscr{L} = \langle \mathcal{L}, \Omega, \models, \Vdash \rangle$ be a logic. The* entailment relation *$\mathrel{|\!\sim}_{\mathbb{M}} \subseteq \mathcal{L} \times \mathcal{L}$ for a relational model $\mathbb{M} = \langle \mathcal{S}, \ell, R \rangle$ for $\mathscr{L}$ is given by*

$$\varphi \mathrel{|\!\sim}_{\mathbb{M}} \psi \text{ if } \min(\llbracket \varphi \rrbracket, R) \subseteq \llbracket \psi \rrbracket \, .$$

An entailment relation $\mathrel{|\!\sim} \subseteq \mathcal{L} \times \mathcal{L}$ is called *(strongly) cumulative* if there is a (strong) cumulative model $\mathbb{C}$ for $\mathscr{L}$ such that $\mathrel{|\!\sim} = \mathrel{|\!\sim}_{\mathbb{C}}$.

Classes of Entailment Relations. If L is a class of logics (such as PDL, TPL, or CPL), we denote with L^{c} the class of all entailment relations $\mathrel{|\!\sim}$ for $\langle \mathcal{L}, \Omega, \models, \Vdash \rangle \in \mathsf{L}$ that satisfy *System C*. Similarly, we use $\mathsf{L}^{\mathsf{cuml}}$ ($\mathsf{L}^{\mathsf{cuml[str]}}$) for the class of all (strong) cumulative entailment relations.

4 Representation Theorems

In this section, we will establish representation theorems for cumulative logics, i.e., logics that satisfy *System C*. A classic result is that cumulative classical propositional logics coincide with classical propositional logics that satisfy *System C*.

Proposition 8 (KLM, 1990)**.** $\mathsf{CPL}^{\mathsf{cuml}} = \mathsf{CPL}^{\mathsf{cuml[str]}} = \mathsf{CPL}^{\mathsf{c}}$.

Here we will prove that when basing *System C* on propositional dependence logic, there is also the same connection between cumulative logics and cumulative models. We will also show that logics based on preferential and cumulative models for TPL correspond to *System C* logics for CPL.

4.1 Propositional Dependence Logic

We will show the following representation result that establishes an equivalence between the class of cumulative propositional dependence logics and propositional dependence logics that satisfy *System C*.

Theorem 9. $\mathsf{PDL}^{\mathsf{cuml}} = \mathsf{PDL}^{\mathsf{cuml[str]}} = \mathsf{PDL}^{\mathsf{c}}$.

First, we like to remark that the original proof of Proposition 8 makes heavy use of classical negation and material implication of the underlying propositional logic. As neither classical negation nor material implication are available (and also not definable) in PDL, the proof of Theorem 9 requires a new approach that circumvents the usage of implication and negation. We present the proof in the following, and mark those steps that are borrowed from KLM (1990).

Our first observation is that entailment relations in $\mathsf{PDL}^{\mathsf{cuml}}$ satisfy *System C*. This is a novel result as the proof of *System C* satisfaction for $\mathsf{CPL}^{\mathsf{pref}}$ given by KLM does not carry over to preferential propositional dependence logic.

Proposition 10. $\mathsf{PDL}^{\mathsf{cuml}} \subseteq \mathsf{PDL}^{\mathsf{c}}$.

Proof (idea). One checks case by case that every postulate of *System C* is satisfied. A full proof is provided in the arXiv version of this paper[1]. ☐

In the remainder of this section, we will show that every entailment relation $\mathrel{|\!\sim} \in \mathsf{PDL}^{\mathsf{c}}$ is strongly cumulative. For that, we will first show that one can characterize all the consequences of a formula φ via $\mathrel{|\!\sim}$ semantically.

For a given formula φ, we denote the set of all teams that satisfy all consequences of φ with respect to $\mathrel{|\!\sim}$, by

$$\mathrm{Norm}(\varphi, \mathrel{|\!\sim}) = \{ X \mid X \models \psi \text{ for all } \psi \text{ with } \varphi \mathrel{|\!\sim} \psi \} \, .$$

In the following, we will show that for every φ the set $\mathrm{Norm}(\varphi, \mathrel{|\!\sim})$ characterize semantically the consequences of $\mathrel{|\!\sim}$ drawn from φ. For all formulas φ, all sets of formulas $F \subseteq \mathcal{L}$, and all sets of teams M we define:

$$\mathrm{C}_{\mathrel{|\!\sim}}(\varphi) = \{ \psi \mid \varphi \mathrel{|\!\sim} \psi \}, \quad \mathrm{Cn}(\varphi) = \{ \psi \mid \varphi \models \psi \},$$
$$\mathrm{Th}(M) = \{ \varphi \in \mathcal{L} \mid X \models \varphi \text{ for all } X \in M \},$$
$$\mathrm{Cn}(F) = \{ \psi \mid \varphi \models \psi \text{ for all } \varphi \in F \}.$$

The following lemma provides basic insights into the closures $\mathrm{C}_{\mathrel{|\!\sim}}$ and Cn, and the theory operator Th.

Lemma 11. *The following statements are true:*

(a) For $\mathrel{|\!\sim} \in \mathsf{PDL}^{\mathsf{c}}$ we have that $\mathrm{C}_{\mathrel{|\!\sim}}(\varphi) = \mathrm{Cn}(\mathrm{C}_{\mathrel{|\!\sim}}(\varphi))$.

(b) For each formula φ there exists an $M \subseteq \mathcal{P}(\mathbb{T}_N)$ such that $\mathrm{Cn}(\varphi) = \mathrm{Th}(M)$ and $M = \llbracket \varphi \rrbracket$.

(c) For each set of formulas F with $F = \mathrm{Cn}(F)$ there exists a formula φ_F such that $F = \mathrm{Cn}(\varphi_F)$ is true.

Proof. Because (b) is a direct consequence of the semantics, we omit a proof here.

Next we consider the proof for (a). We obtain $\mathrm{C}_{\mathrel{|\!\sim}}(\varphi) \subseteq \mathrm{Cn}(\mathrm{C}_{\mathrel{|\!\sim}}(\varphi))$ by definition and reflexivity of $\models$. Now assume that $\mathrm{C}_{\mathrel{|\!\sim}}(\varphi) \subsetneq \mathrm{Cn}(\mathrm{C}_{\mathrel{|\!\sim}}(\varphi))$ is true, i.e., there is $\psi \in \mathrm{C}_{\mathrel{|\!\sim}}(\varphi)$ and $\gamma \in \mathrm{Cn}(\mathrm{C}_{\mathrel{|\!\sim}}(\varphi))$ such that $\psi \models \gamma$ and $\gamma \notin \mathrm{C}_{\mathrel{|\!\sim}}(\varphi)$.

[1]Consult the reference Kontinen, Meier, and Sauerwald (2026).

Now recall that $\mid\!\sim$ satisfies (RW), and hence, we also have $\varphi \mid\!\sim \gamma$, because $\psi \models \gamma$ and $\varphi \mid\!\sim \psi$ is true. From $\varphi \mid\!\sim \gamma$, we obtain the contradiction $\gamma \in \mathrm{C}_{\mid\sim}(\varphi)$.

Finally, we prove (c). We will employ the underlying logic PDL_N, where we have a finite signature $N \subseteq \mathsf{Prop}$. It has been shown that classical disjunction is expressible in PDL_N, i.e., for two formulas φ, ψ there is a formula $\varphi \varovee \psi$ with $[\![\varphi \varovee \psi]\!]^t = [\![\varphi]\!]^t \cup [\![\psi]\!]^t$ (Yang 2014). Moreover, every downward-closed set of teams is M definable, i.e., there is a φ_M with $[\![\varphi_M]\!]^t = M$. Because N is finite, for every set of formulas F the set $[\![F]\!]^t$ is a finite disjunction of downward-closed sets of teams, i.e., $[\![F]\!]^t = M_1 \cup \ldots \cup M_n$. Hence, for $\varphi_F = \varovee_{i=1}^n \varphi_{M_i}$ we have $[\![F]\!]^t = [\![\varphi_F]\!]^t$. ☐

Note that the proof of Lemma 11 relies on the fact that $\mathrm{PL}(\mathsf{dep})$ is expressively complete for team properties (Yang 2014), i.e., every set of teams that is closed under subteams is the models set of some formula. We show the central insight that $\mathrm{Norm}(\varphi, \mid\!\sim)$ characterizes all consequences of φ by $\mid\!\sim$.

Theorem 12 (Definability of Consequences)**.** *For all entailment relations $\mid\!\sim \in \mathsf{PDL}^{\mathsf{c}}$ and all formulas φ we have that:*

(a) For all formulas ψ we have that:
$\varphi \mid\!\sim \psi$ if and only if $\mathrm{Norm}(\varphi, \mid\!\sim) \models \psi$.

(b) There is a θ_φ such that for all formulas ψ we have that:
$\varphi \mid\!\sim \psi$ if and only if $\theta_\varphi \models \psi$.

Proof. Let $\mid\!\sim$ and φ be as above. We let θ_φ be a formula such that $[\![\theta_\varphi]\!] = \mathrm{Norm}(\varphi, \mid\!\sim)$ which exists due to Lemma 11.

[Case "$\varphi \mid\!\sim \psi \Rightarrow \mathrm{Norm}(\varphi, \mid\!\sim) \models \psi$"] If $\varphi \mid\!\sim \psi$, then we obtain $\theta_\varphi \models \psi$ by definition of $\mathrm{Norm}(\varphi, \mid\!\sim)$.

[Case "$\theta_\varphi \models \psi \Rightarrow \varphi \mid\!\sim \psi$"] By Lemma 11, there exist $M \subseteq \mathcal{P}(\mathbb{T}_N)$ such that $C_{\mid\sim}(\varphi) = \mathrm{Th}(M)$ is true.

We show that $M \subseteq \mathrm{Norm}(\varphi, \mid\!\sim)$ is true. Suppose not, i.e., there is an $X \in M$ such that $X \notin \mathrm{Norm}(\varphi, \mid\!\sim)$. From the latter, we obtain that there is a formula γ with $X \not\models \gamma$ and $\varphi \mid\!\sim \gamma$. Now, from $C_{\mid\sim}(\varphi) = \mathrm{Th}(M)$, we can deduce $X \models \delta$ for all $\delta \in C_{\mid\sim}(\varphi)$. Consequently, $X \models \gamma$, as we have that $\gamma \in C_{\mid\sim}(\varphi)$. This shows $M \subseteq \mathrm{Norm}(\varphi, \mid\!\sim)$.

We then obtain that $\mathrm{Th}(\mathrm{Norm}(\varphi, \mid\!\sim)) \subseteq \mathrm{Th}(M)$ by employing $M \subseteq \mathrm{Norm}(\varphi, \mid\!\sim)$. Consequently, from $\mathrm{Norm}(\varphi, \mid\!\sim) \models \psi$, we deduce that $M \models \psi$. Thus, by employing $[\![\theta_\varphi]\!] = \mathrm{Norm}(\varphi, \mid\!\sim)$ and $C_{\mid\sim}(\varphi) = \mathrm{Th}(M)$, from which we have that $\theta_\varphi \models \psi$ implies $\varphi \mid\!\sim \psi$. ☐

Next, we construct a cumulative model that captures $\mid\!\sim$. The construction is inspired by the construction of Kraus, Lehmann, and Magidor (1990). Let $\sim$ be the equivalence relation, respectively $\preceq_{\mid\sim}$ be the relation, defined by

$$\varphi \sim \psi \quad \text{if } \varphi \mid\!\sim \psi \text{ and } \psi \mid\!\sim \varphi,$$
$$[\varphi]_\sim \preceq_{\mid\sim} [\psi]_\sim \quad \text{if there exists } \gamma \in [\varphi]_\sim \text{ with } \psi \mid\!\sim \gamma\,.$$

Define $\mathbb{C}_{\mid\sim} = \langle \mathcal{S}, \ell, R\rangle$, with $\mathcal{S} := \mathrm{PL}(\mathsf{dep})/\!\sim$ as follows:

$$\ell := \{\, ([\varphi]_\sim, \mathrm{Norm}(\varphi, \mid\!\sim)) \mid \varphi \in \mathrm{PL}(\mathsf{dep}) \,\}\,,$$
$$R := \{\, ([\varphi]_\sim, [\psi]_\sim) \mid [\varphi]_\sim \neq [\psi]_\sim \text{ and } [\varphi]_\sim \preceq_{\mid\sim} [\psi]_\sim \,\}\,.$$

The states of $\mathbb{C}_{\mid\sim}$ are the equivalence classes of $\sim$. The function ℓ assign to every state $[\varphi]_\sim$ the normal worlds of φ, and we have that $[\varphi]_\sim\ R\ [\psi]_\sim$ if $[\varphi]_\sim$ and $[\varphi]_\sim$ are different and $[\varphi]_\sim \preceq_{\mid\sim} [\psi]_\sim$ is true.

Proposition 13. $\mathsf{PDL}^{\mathsf{c}} \subseteq \mathsf{PDL}^{\mathsf{cuml[str]}}$.

Proof. First, we show that $\mathbb{C}_{\mid\sim}$ is a strong cumulative model. One sees easily that $\mathbb{C}_{\mid\sim}$ is a relational model and R is asymmetric. It suffices to show that for every formula φ the state $[\varphi]_\sim$ is the unique element of $\min(\mathcal{S}(\varphi), R)$. Suppose not, i.e., there is a state $[\psi]_\sim \in \min(\mathcal{S}(\varphi), R)$ different from $[\varphi]_\sim$. By definition, we have $\ell([\psi]_\sim) = \mathrm{Norm}(\psi, \mid\!\sim)$ and hence, that $\mathrm{Norm}(\psi, \mid\!\sim) \subseteq [\![\varphi]\!]^t$ is true. We obtain from the latter that $\psi \mid\!\sim \varphi$ is true by employing Theorem 12. From $\varphi \in [\varphi]_\sim$ and $\psi \mid\!\sim \varphi$, we obtain $[\varphi]_\sim \preceq_{\mid\sim} [\psi]_\sim$. Consequently, we have that $[\varphi]_\sim\ R\ [\psi]_\sim$, because of $[\varphi]_\sim \preceq_{\mid\sim} [\psi]_\sim$ and $[\varphi]_\sim \neq [\psi]_\sim$. However, $[\varphi]_\sim\ R\ [\psi]_\sim$ is a contradiction to $[\psi]_\sim \in \min(\mathcal{S}(\varphi), R)$.

To complete the proof, we show that $\mid\!\sim$ and $\mid\!\sim_{\mathbb{C}_{\mid\sim}}$ coincide. Due to Theorem 12, we have that $\varphi \mid\!\sim \psi$ if and only if $\mathrm{Norm}(\varphi, \mid\!\sim) \models \psi$. By construction of $\mathbb{C}_{\mid\sim}$, it is true that $\min([\![\varphi]\!]^t, R) = \mathrm{Norm}(\varphi, \mid\!\sim)$. By definition, we get $\varphi \mid\!\sim_{\mathbb{C}_{\mid\sim}} \psi$ exactly when $\mathrm{Norm}(\varphi, \mid\!\sim) \models \psi$. Consequently, we have that $\varphi \mid\!\sim \psi$ if and only if $\varphi \mid\!\sim_{\mathbb{C}_{\mid\sim}} \psi$. ☐

Because $\mathsf{PDL}^{\mathsf{cuml[str]}} \subseteq \mathsf{PDL}^{\mathsf{cuml}}$ holds, we obtain Theorem 9 from Proposition 13 and Proposition 10.

4.2 Propositional Logic with Team-Semantics

In this section, we relate reasoning via System C and via cumulative models over propositional formulas when team-semantics and classical semantics is used. Intuitively, because, cumulative models for CPL order set of valuations and cumulative models for TPL order sets of teams, one would expect that the inclusion $\mathsf{CPL}^{\mathsf{cuml}} \subseteq \mathsf{TPL}^{\mathsf{cuml}}$ is strict. However, as we will show, $\mathsf{CPL}^{\mathsf{cuml}}$ and $\mathsf{TPL}^{\mathsf{cuml}}$ contain the same entailment relations.

First, we define cumulative models with singleton teams in labels and an asymmetric relation.

Definition 14. *A relational model $\mathbb{C} = \langle \mathcal{S}, \ell, R\rangle$ for a logic $\mathscr{L}$ is called* asymmetric *if $\mathbb{C}$ is cumulative, $\prec$ is asymmetric and for each state $s \in S$, the set $\ell(s)$ is a singleton.*

With $\mathsf{TPL}^{\mathsf{cuml[as]}}$ we denote the class of entailment relations based on asymmetric models over TPL. Note that in strong cumulative models $\min([\![\varphi]\!], \mathcal{S})$ is a singleton and each $\ell(s)$ may contain arbitrary many elements, where in asymmetric models, $\ell(s)$ is a singleton and the cardinality of $\min([\![\varphi]\!], \mathcal{S})$ is not restricted.

The next theorem provides equivalence of various reasoning approaches for CPL and TPL with cumulative models and System C. This includes the equivalence $\mathsf{TPL}^{\mathsf{c}} = \mathsf{TPL}^{\mathsf{cuml[as]}}$, which shows that asymmetric models for TPL can be seen as a standard representation for $\mathsf{TPL}^{\mathsf{c}}$.

Theorem 15. $\mathsf{TPL}^{\mathsf{cuml[as]}} = \mathsf{CPL}^{\mathsf{x}} = \mathsf{CPL}^{\mathsf{y}} = \mathsf{TPL}^{\mathsf{x}} = \mathsf{TPL}^{\mathsf{y}}$ *is true for all* $\mathsf{x}, \mathsf{y} \in \{\, \mathsf{cuml},\ \mathsf{cuml[str]},\ \mathsf{c} \,\}$.

Proof. Note first that Proposition 8 implies that $\mathsf{CPL}^{\mathsf{c}} = \mathsf{CPL}^{\mathsf{cuml}} = \mathsf{CPL}^{\mathsf{cuml[str]}}$. Furthermore, we have that $\mathsf{TPL}^{\mathsf{cuml[str]}} \subseteq \mathsf{TPL}^{\mathsf{cuml}}$ and $\mathsf{TPL}^{\mathsf{c}} = \mathsf{CPL}^{\mathsf{c}}$ are immediate consequences of the definitions. Finally, the analogue of Proposition 10 can be used to show that $\mathsf{TPL}^{\mathsf{cuml}} \subseteq \mathsf{TPL}^{\mathsf{c}}$ is true.

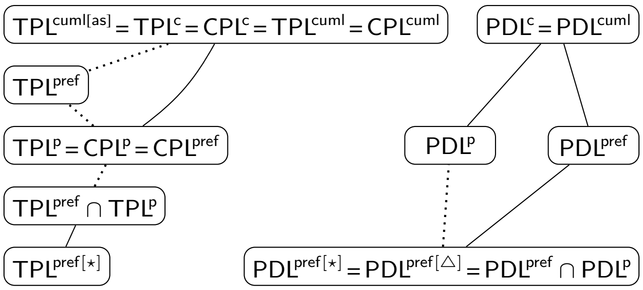

Figure 1: Landscape of classes of entailment relations. For not defined classes, consult SMK (2025).

Let us then show that $\mathsf{CPL}^{\mathsf{cuml[str]}} \subseteq \mathsf{TPL}^{\mathsf{cuml[str]}}$. Let $\mathrel{|\!\sim}_{\mathbb{C}} \in \mathsf{CPL}^{\mathsf{cuml[str]}}$ and $\mathbb{C} = \langle \mathcal{S}, \ell, R \rangle$ be the respective strong cumulative model for CPL. We reinterpret $\mathbb{C}$ as a strong cumulative model $\mathbb{C}' = \langle \mathcal{S}', \ell', R' \rangle$ for TPL with $\mathcal{S}' = \mathcal{S}$,, $\ell' = \{ (s, \{\ell(s)\}) \mid s \in \mathcal{S} \}$, $R' = R$. Because in TPL is holds that $(\star)$ $\{\nu\} \models \varphi$ if and only if $\nu \models^c \varphi$ for all singleton teams $\{\nu\}$, we obtain then $\mathrel{|\!\sim}_{\mathbb{C}} = \mathrel{|\!\sim}_{\mathbb{C}'}$. Next, we show that $\mathsf{CPL}^{\mathsf{cuml[str]}} \subseteq \mathsf{TPL}^{\mathsf{cuml[as]}}$ is true. Note that any strong cumulative model $\mathbb{C} = \langle \mathcal{S}, \ell, R \rangle$ for CPL can be reinterpreted ($\ell(s) = \{ \nu_1, \ldots, \nu_m \}$ becomes $\ell(s) = \{ \{\nu_1, \ldots, \nu_m\} \}$) as an asymmetric model for TPL, hence by the flatness property of TPL-formulas and $(\star)$ above the claim follows. Finally, we show that $\mathsf{TPL}^{\mathsf{cuml[as]}} \subseteq \mathsf{CPL}^{\mathsf{cuml}}$ holds. Note that any asymmetric model $\mathbb{C} = \langle \mathcal{S}, \ell, R \rangle$ for TPL can be interpreted ($\ell(s) = \{ \{\nu_1, \ldots, \nu_m\} \}$ becomes $\ell(s) = \{ \nu_1, \ldots, \nu_m \}$) as a cumulative model $\mathbb{C}$ for CPL. By the flatness property of TPL-formulas and $(\star)$ above it follows that $\mathrel{|\!\sim}_{\mathbb{C}} = \mathrel{|\!\sim}_{\mathbb{C}'}$. $\square$

5 Conclusion

In this paper, we proved representation theorems between entailment relations based on *System C* and based on cumulative reasoning for propositional dependence logic and propositional logic with team semantics. Hence, they are forming, in the context of these logics, a stable class which is justified to be denoted as *cumulative logics*. The results from SMK (2025) and KLM (1990), and the novel results from this paper, yield a diverse landscape as shown in Figure 1.

On our future agenda, we will extend our work to other team-based logics. Furthermore, we will study other subclasses of the rich landscape of cumulative reasoning, such as c-inference (Kern-Isberner 2001), lexicographic inference (Lehmann 1995) or System W (Komo and Beierle 2022). Of special interest is the open relationship of the classes considered here to $\mathsf{TPL}^{\mathsf{pref}}$ and $\mathsf{PDL}^{\mathsf{pref}}$, i.e., entailment relations via preferential models (Kraus, Lehmann, and Magidor 1990), i.e., cumulative models in which for every state s, the set $\ell(s)$ is a singleton, and R is a strict partial order. Related to this problem is the open axiomatics for $\mathsf{TPL}^{\mathsf{pref}}$ and $\mathsf{PDL}^{\mathsf{pref}}$. Note that reasoning via *System C* over PDL plus the following rule

$$\frac{\varphi \mathrel{|\!\sim} \gamma \qquad \psi \mathrel{|\!\sim} \gamma}{\varphi \vee \psi \mathrel{|\!\sim} \gamma}\,, \tag{Or}$$

denoted $\mathsf{PDL}^{\mathsf{p}}$, was shown to be different from $\mathsf{PDL}^{\mathsf{pref}}$ (SMK, 2025). Note that $\mathsf{TPL}^{\mathsf{p}} \subseteq \mathsf{TPL}^{\mathsf{pref}}$ is a consequence of the results shown here, but it is unknown if the inclusion is strict.

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AI Declaration

The authors have not employed any Generative AI tools.

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