Proceedings of the 23rd International Conference on Principles of Knowledge Representation and Reasoning
Main Track

Safely Decomposing Conditional Belief Bases Into c-LEG Networks

Gabriele Kern-Isberner[1], **Alexander Hahn**[1], **Lars-Phillip Spiegel**[2], **Marco Wilhelm**[3], **Christoph Beierle**[2]

[1]TU Dortmund, Dortmund, Germany
[2]FernUniversität in Hagen, Hagen, Germany
[3]Federal Institute for Occupational Safety and Health, Dortmund, Germany
gabriele.kern-isberner@cs.tu-dortmund.de, alexander.hahn@tu-dortmund.de,
lars-phillip.spiegel@fernuni-hagen.de, wilhelm.marco@baua.bund.de,
christoph.beierle@fernuni-hagen.de

Abstract

Like Pearl's System Z, c-representations provide a constructive approach to compute a ranking function from a conditional belief base from which further (conditional) beliefs can be derived, meeting major quality standards of nonmonotonic reasoning. This paper proposes a network-based structure for c-representations that allows for cutting down the complexity of reasoning significantly by decomposing the conditional belief base over a hypertree. We introduce c-LEG networks capturing the interactions among conditionals on a syntactical basis in full compatibility with the semantics of c-representations. This allows for reasoning in much smaller local contexts while still complying with the global information provided by the full conditional belief base. Moreover, we generalize the so-called safety property, which was recently presented in the context of conditional syntax splitting, to ensure that local c-representations of subbases over the hyperedges can be merged to yield global c-representations of the full conditional belief base. This allows for computing global c-representations step by step in local contexts, following the structure of the hypertree.

1 Introduction

Conditionals expressing plausible beliefs provide a suitable base for inductive (nonmonotonic) reasoning. In their seminal paper (Lehmann and Magidor 1992), Lehmann and Magidor elaborated in detail what a conditional belief base is supposed to entail, proposing the rational closure approach which is basically the same as Pearl's System Z (Pearl 1990). Like Pearl's System Z, c-representations (Kern-Isberner 2004) generate an ordinal conditional function (*OCF*, also called *ranking function*) (Spohn 1988) from such a conditional belief base from which then further (conditional) beliefs can be derived. By employing a sophisticated procedure to elaborate on the interactions among conditionals, c-representations yield high-quality inferences (see, e.g., (Kern-Isberner, Beierle, and Brewka 2020)). However, since basically all possible worlds need to be investigated, c-representations also suffer from a high complexity, like many other semantic-based nonmonotonic logics.

To reduce the complexity of nonmonotonic resp. conditional reasoning, various techniques for decomposing belief bases have been proposed. In the KR community, such techniques are often presented as splitting techniques (Kern-Isberner, Beierle, and Brewka 2020) where the inspiration comes from seminal work in belief revision (Parikh 1999). A bit more generally, decomposition of belief bases according to network-based structures also offers a high potential for developing more efficient algorithms. A most well-known example here are Bayesian networks (Pearl 1989) which had a huge impact on making probabilistic reasoning efficient.

In this paper, we combine ideas from syntax splitting and network-based techniques to propose an approach for representing and constructing c-representations of conditional belief bases Δ with the help of hypertrees, the so-called *c-LEG networks*. LEG networks are hypergraphs consisting of *local event groups, LEGs,* which are sets of variables occurring in the conditionals of a subset of Δ, together with a suitable semantics. In probabilistics, they have been used for the efficient computation of maximum entropy probabilities (Lemmer 1982; Meyer 1998). Here we equip LEGs with ranking functions which are supposed to be ranking models of the respective subset of Δ. The crucial idea of such networks is that local components like LEGs contain all relevant information to compute inferences in a global context, i.e., regarding all variables occurring in the network. This allows for reducing the complexity of reasoning in two ways: First, inferences concerning only a subset of the variables can often be computed in much smaller components; second, regarding inductive reasoning, local ranking functions on LEGs can be combined to define a global ranking function on all variables.

Now, to ensure that all local and global ranking functions are c-representations of the respective (sub)bases we make use of a generalized safety condition from a recent paper on generalized safe conditional syntax splitting (Spiegel et al. 2025). We define what it means that a conditional belief base is safely decomposed over a hypertree. Safety here ensures that ranking models of subbases over the hyperedges can be set up as local c-representations which can be merged to yield global c-representations of the full conditional belief base. This allows for computing global c-representations step by step in local contexts, following the structure of the hypertree. In turn, those local c-representations can be recovered from the global ones by marginalization. Hence the safety condition fills a gap left over in (Eichhorn and Kern-Isberner 2014) where it could not be guaranteed that all ranking functions involved in an OCF-LEG network are c-representations of the respective (sub)bases. We present results elaborating on both directions between global (over the whole conditional belief base Δ) and local (over subbases of Δ) views in detail. Moreover, we show that safe decompositions of conditional belief bases continue on previously published works in that they generalize the concept of generalized safe conditional splittings (Spiegel et al. 2025) and yield safe covers of the conditional belief base (Wilhelm and Kern-Isberner 2024).

This paper is organized as follows: In the next section, we summarize basic formal details and notations that we use in this paper. Section 3 recalls main techniques and results from (Eichhorn and Kern-Isberner 2014) and (Spiegel et al. 2025). Section 4 elaborates on generalized safe splittings for c-representations, in particular regarding the effects on their constraint systems. Section 5 then presents the c-LEG networks. Finally, we show connections to safe covers in Section 6, and conclude the paper by summarizing its main contributions and pointing out possible future work in Section 7. We provide proof sketches for all major technical results of this paper to point out the main arguments of the proof.

2 Formal Preliminaries

Let $\mathcal{L}$ be a finitely generated propositional language over a signature Σ with atoms $a, b, c, \ldots$ and with formulas $A, B, C, \ldots$. We may write AB instead of $A \wedge B$, and overline formulas to indicate negation, i.e., $\overline{A}$ means $\neg A$. Let Ω denote the set of *possible worlds* over $\mathcal{L}$, taken here simply as the set of all propositional interpretations over $\mathcal{L}$. $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in $\omega \in \Omega$; in this case ω is called a *model* of A, and the set of all models of A is denoted by $\mathit{Mod}\,(A)$. For propositions $A, B \in \mathcal{L}$, $A \models B$ holds iff $\mathit{Mod}\,(A) \subseteq \mathit{Mod}\,(B)$, as usual. We will use ω both for the model and the corresponding complete conjunction of all positive or negated atoms, allowing us to use ω both as an interpretation and a proposition.

For $\Theta \subseteq \Sigma$, let $\mathcal{L}(\Theta)$ denote the propositional language defined by Θ, with associated set of interpretations $\Omega(\Theta)$. Note that while each formula of $\mathcal{L}(\Theta)$ can also be considered as a formula of $\mathcal{L}$, the interpretations $\omega^\Theta \in \Omega(\Theta)$ are not elements of $\Omega(\Sigma)$ if $\Theta \neq \Sigma$. But each interpretation $\omega \in \Omega$ can be written uniquely in the form $\omega = \omega^\Theta \omega^{\overline{\Theta}}$ with concatenated $\omega^\Theta \in \Omega(\Theta)$ and $\omega^{\overline{\Theta}} \in \Omega(\overline{\Theta})$, where $\overline{\Theta} = \Sigma \backslash \Theta$. In the following, we will often denote subsignatures of Σ by $\Sigma_1, \Sigma_2, \ldots$ and write ω^i instead of ω^{Σ_i} to ease notation. If Σ_i, Σ_j are subsignatures of Σ with $\Sigma_i \cap \Sigma_j \neq \emptyset$, we often write $\omega^{i,j}$ instead of $\omega^{\Sigma_i \cap \Sigma_j}$.

By making use of a conditional operator $|$, we introduce the language $(\mathcal{L}|\mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$ of *conditionals* over $\mathcal{L}$. Conditionals $(B|A)$ are meant to express plausible, defeasible rules "If A then plausibly (usually, possibly, probably, typically etc.) B". For a world ω a conditional $(B|A)$ is either *verified* by ω if $\omega \models AB$, *falsified* by ω if $\omega \models A\overline{B}$, or *not applicable* to ω if $\omega \models \overline{A}$. Conditional belief bases Δ (over Σ) consist of finitely many conditionals from $(\mathcal{L} \mid \mathcal{L})$.

A popular semantic framework for interpreting conditionals are *ordinal conditional functions (OCFs)* $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$, also called *ranking functions*, introduced in a more general form by (Spohn 1988). Intuitively, less plausible worlds are assigned higher numbers. Formulas are assigned the rank of their most plausible models, i.e., $\kappa(A) := \min\{\kappa(\omega) \mid \omega \models A\}$. A conditional $(B|A)$ is *accepted* by κ, written as $\kappa \models (B|A)$, iff $\kappa(A) = \infty$, or $\kappa(AB) < \kappa(A\overline{B})$, i.e., iff AB is more plausible than $A\overline{B}$. This is lifted to belief bases via $\kappa \models \Delta$ if $\kappa \models (B|A)$ for all $(B|A) \in \Delta$; κ is then called a *ranking model of* Δ. Consistency of such a belief base Δ can be defined in terms of OCFs (Pearl 1990; Goldszmidt and Pearl 1996): Δ is consistent iff there is an OCF κ such that $\kappa \models \Delta$ and $\kappa(\omega) < \infty$ for all $\omega \in \Omega$. In this paper, we focus on consistent belief bases Δ, and we assume that Δ does not contain a self-fulfilling conditional $(B|A)$ with $A \models B$; hence, every conditional in Δ is both verifiable and falsifiable. This enables us to present the technical details of our approach most clearly, without having to take limiting cases like conditionals $(B|A)$ with $\kappa(A) = \infty$ into account. An adaptation of the approach to such cases is possible, but is left for future work.

Like probabilities, also ranking functions can be marginalized to subsignatures: The *marginal of* κ *on* $\Theta \subseteq \Sigma$, denoted by $\kappa_{|\Theta}$, is defined by $\kappa_{|\Theta}(\omega^\Theta) = \kappa(\omega^\Theta)$ for any $\omega^\Theta \in \Omega(\Theta)$. Here ω^Θ is treated as a world in $\kappa_{|\Theta}(\omega^\Theta)$ but as a formula in $\kappa(\omega^\Theta)$.

3 LEG-Networks and Splittings

In this section, we recall the basics of two decomposition techniques which are crucial for developing the technical contributions of this paper. We start with a network-based approach and then summarize briefly the idea of generalized safe conditional syntax splitting.

3.1 OCF-LEG Networks

OCF-LEG networks are hypergraphs that can be used to decompose ranking functions (i.e., OCFs) by their marginalizations computed locally over subsignatures, but also to build up global ranking functions from local ones. In the context of decomposing conditional belief bases Δ containing conditionals over Σ into subbases, i.e., $\Delta = \Delta_1 \cup \ldots \cup \Delta_m$, each subsignature Σ_i will correspond to the signature of a subbase Δ_i. The basic challenge is to ensure that the syntactical splitting $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_m$ corresponds to a semantic splitting of Δ, i.e., the marginalizations of a (global) ranking model κ of Δ on Σ_i yield ranking models $\kappa_{|\Sigma_i}$ of Δ_i, and κ can be computed from the $\kappa_{|\Sigma_i}$ resp. decomposed by the $\kappa_{|\Sigma_i}$. The paper (Eichhorn and Kern-Isberner 2014) provides a solution to this basic challenge by making use of OCF-LEG networks. We recall main techniques and results from that paper, starting with defining hypergraphs and hypertrees.

Definition 1 ((reduced) hypergraph, hypertree (Pearl 1989)). *Let $\mathfrak{S}$ by a system of subsets of a propositional alphabet Σ*

such that $\bigcup_{\Sigma' \in \mathfrak{S}} \Sigma' = \Sigma$. *Then* $\langle \Sigma, \mathfrak{S} \rangle$ *is a* hypergraph. *The elements of* $\mathfrak{S}$ *are called* hyperedges.

$\langle \Sigma, \mathfrak{S} \rangle$ *is a* hypertree *if and only if there is an enumeration of* $\mathfrak{S} = \{\Sigma_1, \ldots, \Sigma_m\}$ *such that* $m = 1$, *or for all* $1 < i \leq m$, *there is* $j = j(i) < i$, *such that*

$$\mathbb{S}_i = \Sigma_i \cap (\Sigma_1 \cup \ldots \cup \Sigma_{i-1}) \subseteq \Sigma_{j(i)}. \tag{1}$$

Condition (1) is called Running Intersection Property (RIP). *We call* $\mathcal{S} = \{\mathbb{S}_1, \ldots, \mathbb{S}_m\}$ *the* separators *of* $\mathfrak{S}$.

A hypergraph (or hypertree, respectively) is called reduced *if no hyperedge is a subset of another hyperedge.*

Note that in (1), the notation $j(i)$ is used in an informal way to express the dependence on i. The index $j(i)$ might be chosen in several ways, but once a choice has been made, $j(i)$ is uniquely determined for each $i \in \{2, \ldots, m\}$. Then Σ_i becomes a child of $\Sigma_{j(i)}$. Hypergraphs provide the basic graphical structure for OCF-LEG networks.

Definition 2 (OCF-LEG network (Eichhorn and Kern-Isberner 2014))**.** *Let* $\Sigma_1, \ldots, \Sigma_m$ *be a set of covering subsets over an alphabet* Σ *such that* $\Sigma_i \subseteq \Sigma$, $1 \leq i \leq m$, *and* $\Sigma = \bigcup_{i=1}^{m} \Sigma_i$. *Let* $\kappa_1, \ldots, \kappa_m$ *be ranking functions with* $\kappa_i : \Omega_i \to \mathbb{N}_0 \cup \{\infty\}$, $1 \leq i \leq m$. *We call a tuple* $\langle \Sigma_i, \kappa_i \rangle$ *of a subset* $\Sigma_i \subseteq \Sigma$ *with a ranking function* κ_i *on* Ω_i *a* local event group (LEG). *The system* $\langle (\Sigma_1, \kappa_1), \ldots, (\Sigma_m, \kappa_m) \rangle$, *abbreviated as* $\langle (\Sigma_i, \kappa_i) \rangle_{i=1}^{m}$, *is a* ranking network of local event groups (OCF-LEG network) *if there is a global ranking function* κ *on* Ω *with* $\kappa(\omega^i) = \kappa_i(\omega^i)$ *for all* $\omega^i \in \Omega_i$ *and all* $1 \leq i \leq n$, *that is,* κ_i *are the marginals of* κ *on* Ω_i.

Note that, in practice, OCF-LEG networks offer considerable computational advantages over a plain representation of ranking functions. The space complexity of representing ranking models of conditional belief bases is $\mathcal{O}(2^{|\Sigma|})$ for the global ranking function, but $\mathcal{O}(\sum_{i=1}^{m} 2^{|\Sigma_i|})$ when an OCF-LEG network is used. This is particularly beneficial for answering inference queries (expressed as conditionals) that mention only atoms from one (or few) of the Σ_i's, which can be answered by just looking up the respective local κ_i instead of having to compute the whole global ranking model.

For a system $\langle (\Sigma_i, \kappa_i) \rangle_{i=1}^{m}$ to have a global ranking function as defined above, obviously the following consistency condition (2) has to be fulfilled.

Proposition 1 (consistency condition (Eichhorn and Kern-Isberner 2014))**.** *There is a global ranking function* $\kappa : \Omega \to \mathbb{N}_0 \cup \{\infty\}$ *for the system* $\langle (\Sigma_i, \kappa_i) \rangle_{i=1}^{m}$ *only if*

$$\kappa_i(\omega^{i,j}) = \kappa_j(\omega^{i,j}) \tag{2}$$

for all pairs $1 \leq i, j \leq m$ *and all worlds* $\omega \in \Omega$.

So, the consistency condition (2) is a necessary condition to set up an OCF-LEG network but not a sufficient one in general. However, if the system $\langle \Sigma, \{\Sigma_1, \ldots, \Sigma_m\} \rangle$ is a hypertree, then the consistency condition is also sufficient.

Theorem 2 ((Eichhorn and Kern-Isberner 2014))**.** *Let* $\Sigma_1, \ldots, \Sigma_m$ *be a set of covering subsets* $\Sigma_i \subseteq \Sigma$ *with separators* $\mathbb{S}_1, \ldots, \mathbb{S}_m$ *for all* $1 \leq i \leq m$ *such that the RIP is satisfied, that is,* $\langle \Sigma, \{\Sigma_1, \ldots, \Sigma_m\} \rangle$ *is a hypertree according to Definition 1. Let* $\kappa_1, \ldots, \kappa_m$ *be a set of OCFs on* $\Omega_1, \ldots, \Omega_m$, *respectively, that satisfy the consistency condition (2). Then the function*

$$\kappa(\omega) = \sum_{i=1}^{m} \kappa_i(\omega^i) - \sum_{i=1}^{m} \kappa_i(\omega^{\mathbb{S}_i}) \tag{3}$$

is a global ranking function for the system $\langle (\Sigma_i, \kappa_i) \rangle_{i=1}^{m}$, *that is,* $\kappa(\omega^i) = \kappa_i(\omega^i)$ *for all* $\omega^i \in \Omega_i$, *for all* $1 \leq i \leq m$. *Hence,* $\langle \Sigma_i, \kappa_i \rangle_{i=1}^{m}$ *is an OCF-LEG network with a global ranking function* κ *according to (3).*

Moreover, local inferences coincide with global ones, i.e., if A, B *are formulas whose atoms are contained in the same* Σ_i, *then* $\kappa \models (B|A)$ *iff* $\kappa_i \models (B|A)$.

This theorem holds in analogy to what we have in probabilistics (Lemmer 1982). Decompositions of the form (3) are often called *factorizations*[1] in probabilistics.

Theorem 2 allows for building up local ranking models of the respective subbases via an inductive reasoning method like System Z (Pearl 1990) or c-representations (Kern-Isberner 2004), and then combining them to a global ranking model of the whole conditional belief base. However, the techniques from (Eichhorn and Kern-Isberner 2014) could not ensure a coherence with respect to the chosen inductive reasoning methodology, i.e., that all global and local ranking models can be induced by the respective conditional belief bases by the same methodology, e.g., System Z or c-representations. This advanced problem is solved in the present paper for c-representations.

3.2 Generalized Safe Conditional Syntax Splitting

Syntax splitting was first introduced for belief revision (Parikh 1999) and has also been applied to the context of inductive inference (Kern-Isberner, Beierle, and Brewka 2020). Syntax splitting properties demand that two syntactically independent parts of a belief base are also independent with respect to other properties. Syntax splitting has been extended to conditional syntax splitting, allowing also for a shared signature between the subbases (Heyninck et al. 2023). Formally, a conditional belief base Δ over a signature Σ splits into subbases Δ_1 and Δ_2 conditional on Σ_3 if there is a partition $\{\Sigma_1, \Sigma_2, \Sigma_3\}$ of Σ such that $\Delta = \Delta_1 \cup \Delta_2$ and, for $i \in \{1, 2\}$, $\Delta_i \subseteq (\mathcal{L}(\Sigma_i \cup \Sigma_3) | \mathcal{L}(\Sigma_i \cup \Sigma_3))$, denoted as $\Delta = \Delta_1 \bigcup_{\Sigma_1, \Sigma_2} \Delta_2 \mid \Sigma_3$. We will follow the notation introduced in the following proposition:

Proposition 3 ((Beierle et al. 2024b))**.** *Let* $\Delta = \Delta_1 \bigcup_{\Sigma_1, \Sigma_2} \Delta_2 \mid \Sigma_3$, *i.e.,* $\Delta = \Delta_1 \cup \Delta_2$, $\Delta_i = \Delta \cap (\mathcal{L}(\Sigma_i \cup \Sigma_3) \mid \mathcal{L}(\Sigma_i \cup \Sigma_3)), i \in \{1, 2\}$, *the signatures* $\Sigma_1, \Sigma_2, \Sigma_3$ *are pairwise disjoint, and* $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. *Let* $\Delta_3 = \Delta_1 \cap \Delta_2 \subseteq (\mathcal{L}(\Sigma_3) \mid \mathcal{L}(\Sigma_3))$, $\Delta_{1\setminus 3} = \Delta_1 \setminus \Delta_3$, $\Delta_{2\setminus 3} = \Delta_2 \setminus \Delta_3$. *Then* $\Delta_{1\setminus 3}, \Delta_{2\setminus 3}, \Delta_3$ *are pairwise disjoint and* $\Delta = \Delta_{1\setminus 3} \cup \Delta_{2\setminus 3} \cup \Delta_3$.

Because purely syntactical conditional syntax splitting is not sufficient to guarantee independence of the subbases, a safety condition was introduced to guarantee proper independence (Heyninck et al. 2023). Later it was shown that

[1]Note that ranks are related to logarithmic probabilities, so the sum in (3) corresponds to products in probabilistic factorizations.

this safety condition does not allow for non-trivial conditionals in the intersection of the subbases (Beierle et al. 2024b). To remedy this issue, the safety property was generalized in recent work to that of generalized safety.

Definition 3 (generalized safe splitting (Spiegel et al. 2025))**.** *A conditional belief base* $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ *can be* generalized safely split into subbases Δ_1, Δ_2 conditional on a subsignature Σ_3*, writing*

$$\Delta = \Delta_1 \bigcup^{\mathsf{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$$

if the following generalized safety property *holds for* $i, j \in \{1, 2\}, i \neq j$*: for every* $\omega^i\omega^3 \in \Omega(\Sigma_i \cup \Sigma_3)$*, there is* $\omega^j \in \Omega(\Sigma_j)$ *s.t.* $\omega^i\omega^3\omega^j \not\models \bigvee_{(F|E)\in\Delta_{j\setminus 3}} E \wedge \neg F$.

Example 1. *Let* $\Delta = \{(f|b), (\overline{f}|p), (b|p), (w|b)\}$ *be a belief base. Then* Δ *has the generalized safe conditional syntax splitting* $\Delta = \Delta_1 \bigcup^{\mathsf{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3 = \{(f|b), (\overline{f}|p), (b|p)\} \bigcup^{\mathsf{gs}}_{\{f,p\},\{w\}} \{(w|b)\} \mid \{b\}$.

Generalized safe splitting implies the following property which ensures that the conditionals in $\Delta_1 \cap \Delta_2$ do not interact with those in $\Delta_1 \setminus \Delta_3$ and $\Delta_2 \setminus \Delta_3$:

Proposition 4 ((Spiegel et al. 2026), adapted)**.** *Let* $\Delta = \Delta_1 \bigcup^{\mathsf{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ *be given. Then it holds that for all* $\omega^3 \in \Omega(\Sigma_3)$*, for* $i \in \{1, 2\}$*, there is* $\omega^i \in \Omega(\Sigma_i)$ *such that* $\omega^1\omega^2\omega^3 \not\models E \wedge \neg F$ *for any* $(F|E) \in \Delta \setminus \Delta_3$.

The generalized safety property is one of the main ingredients we will use in this paper to guarantee that subbases do not interfere with one another while still allowing information to be passed on through their intersections. In the next section we will look at how generalized safe splittings interact with c-representations.

4 Generalized Safe Splittings and c-Representations

Based on the concept of generalized safe splittings, in this section we will show how global c-representations determine local c-representations for the subbases, and conversely, how a local c-representation of a subbase extends to a global c-representation. For proving this, we first look at the constraint systems determining c-representations and show that corresponding splitting properties hold for these constraint systems.

For inductive reasoning from a conditional belief base $\Delta = \{\delta_1 = (B_1|A_1), \ldots, \delta_n = (B_n|A_n)\}$ over $\mathcal{L}$, a *c-representation* of Δ is defined via

$$\kappa_{(\Delta,\vec{\eta})}(\omega) = \sum_{\substack{1\leq i\leq n\\ \omega\models A_i\overline{B_i}}} \eta_i \tag{4}$$

such that $\vec{\eta} = (\eta_1, \ldots, \eta_n) \in \mathbb{N}_0^n$ solves the following system of constraints $CR(\Delta) = \{cr_\Delta(\delta_i)\}_{1\leq i\leq n}$:

$$cr_\Delta(\delta_i): \quad \begin{aligned} \eta_i \quad > \quad & \min_{\omega\models A_iB_i} \Big(\sum_{\substack{j\neq i\\ \omega\models A_j\overline{B}_j}} \eta_j \Big) \\ & - \min_{\omega\models A_i\overline{B}_i} \Big(\sum_{\substack{j\neq i\\ \omega\models A_j\overline{B}_j}} \eta_j \Big) \end{aligned} \tag{5}$$

The η_i's associated with each conditional $(B_i|A_i)$, $i \in \{1, \ldots, n\}$, are called *impact factors*, and $\vec{\eta} = (\eta_1, \ldots, \eta_n)$ is called an *impact vector*. The satisfaction of the constraint system $CR(\Delta)$ ensures that $\kappa_{(\Delta,\vec{\eta})}$ as given by (4) is a model of Δ. When Δ is clear from context, we omit it and simply write $\kappa_{\vec{\eta}}$. The set of constraint variables η_i in $CR(\Delta)$ is denoted by $CVar(\Delta)$, and with $Sol(CR(\Delta))$, we denote the solutions of the constraint system $CR(\Delta)$. The constraint system $CR(\Delta)$ is crucial for c-representations. Any solution of $CR(\Delta)$ defines a c-representation with high-quality properties, e.g., c-representations can be made compliant with syntax splitting (Kern-Isberner and Brewka 2017; Kern-Isberner, Beierle, and Brewka 2020) and conditional syntax and semantic splittings (Beierle et al. 2024b).

For $\delta_i \in \Delta$ as in (5), the *constraint inducing sets* are:

$$\mathcal{V}_i = \{\, \{\eta_j \mid j \in \{1, \ldots, n\}\setminus\{i\},\ \omega \models A_j\overline{B_j}\} \mid \omega \models A_iB_i\}$$

$$\mathcal{F}_i = \{\, \{\eta_j \mid j \in \{1, \ldots, n\}\setminus\{i\},\ \omega \models A_j\overline{B_j}\} \mid \omega \models A_i\overline{B_i}\}$$

where the V_k and F_k in $\mathcal{V}_i = \{V_1, \ldots, V_p\}$ and $\mathcal{F}_i = \{F_1, \ldots, F_q\}$ are subsets of $CVar(\Delta)$. Hereby, the V_k and F_k represent the set of falsified conditionals in Δ with respect to a specific possible world that verifies or falsifies $\delta_i = (B_i|A_i)$. In this way, $\mathcal{V}_i$ and $\mathcal{F}_i$ cover all possible worlds verifying or falsifying δ_i. For any set $\mathcal{S} = \{S_1, \ldots, S_r\}$ of sets S_k of constraint variables let the expression $\min\Sigma(\mathcal{S}) = \min\Sigma(\{S_1, \ldots, S_r\})$ be defined by:

$$\min\Sigma(\{S_1, \ldots, S_p\}) = \min(\sum_{\eta\in S_1} \eta, \ldots, \sum_{\eta\in S_n} \eta) \tag{6}$$

Then the constraint $cr_\Delta(\delta_i)$ in (5) can also be written as

$$vf(\delta_i): \eta_i > \min\Sigma(\{V_1, \ldots, V_p\}) - \min\Sigma(\{F_1, \ldots F_q\})$$

or equivalently as $\eta_i > \min\Sigma(\mathcal{V}_i) - \min\Sigma(\mathcal{F}_i)$. Generally, we call any constraint $vf(\delta_i)$ of this form a *V/F-constraint (with respect to* Δ*)* if $\delta_i \in \Delta$ and the sets $V_1, \ldots, V_p$ and $F_1, \ldots, F_q$ are subsets of $CVar(\Delta)$. Thus, for any Δ, $CR(\Delta)$ is a set of V/F-constraints with respect to Δ.

For subbases $\Delta' \subseteq \Delta$, the set $CR(\Delta)_{\Delta'} = \{cr_\Delta(\delta_i) | \delta_i \in \Delta'\}$ is called the *restriction of* $CR(\Delta)$ *to* Δ'. Note that the constraints in a restriction focus on impact factors of conditionals in Δ', but may mention impact factors of conditionals in $\Delta\setminus\Delta'$. Hence, every constraint in $CR(\Delta')$ is a V/F-constraint with respect to Δ', but also a V/F-constraint with respect to Δ. On the other hand, a V/F-constraint in $CR(\Delta)_{\Delta'}$ is not a V/F-constraint with respect to Δ' if it contains a constraint variable from $CVar(\Delta) \setminus CVar(\Delta')$.

Example 2. *Recall the generalized safe splitting* $\Delta = \{(f|b), (\overline{f}|p), (b|p)\} \bigcup^{\mathsf{gs}}_{\{f,p\},\{w\}} \{(w|b)\} \mid \{b\}$ *from Example 1. Then*

$$\begin{aligned} cr_\Delta(f|b) : \eta_1 > \min\Sigma(\{\eta_2\}, \{\eta_2, \eta_4\}, \{\emptyset\}, \{\eta_4\}) \\ - \min\Sigma(\{\{\emptyset\}, \{\eta_4\}) \end{aligned}$$

$$cr_{\Delta_1}(f|b) : \eta_1 > \min\Sigma(\{\eta_2\}, \{\emptyset\}) - \min\Sigma(\{\emptyset\})$$

which according to the definition of $\min\Sigma(\_)$ *correspond to:*

$$cr_\Delta(f|b) : \eta_1 > \min\{\eta_2, \eta_2 + \eta_4, 0, \eta_4\} - \min\{0, \eta_4\}$$

$$cr_{\Delta_1}(f|b) : \eta_1 > \min\{0, \eta_2\} - \min\{0\}$$

Thus, $cr_\Delta(f|b) \neq cr_{\Delta_1}(f|b)$ *and* $CR(\Delta)_{\Delta_1} \neq CR(\Delta_1)$.

While $CR(\Delta_1)$ and $CR(\Delta)_{\Delta_1}$ in Example 2 are different, they are equivalent in the sense that they coincide after applying the basic algebraic property that the sum of a set of integers cannot decrease if additional non-negative integers are added. The following *subset reduction* rule θ_{sr}, introduced along with some further reduction rules for the compilation of conditional belief bases (Beierle, Kutsch, and Sauerwald 2019), implements this basic algebraic reduction:

$$\theta_{sr}: \quad \frac{\min\Sigma(\mathcal{S} \uplus \{S\} \uplus S')\}}{\min\Sigma(\mathcal{S} \uplus \{S\})} \qquad S \subsetneq S' \tag{7}$$

Applying θ_{sr} exhaustively to a miniumum expression $\min\Sigma(\mathcal{S})$ as in (6) terminates and yields a unique result $\theta_{sr}(\min\Sigma(\mathcal{S}))$ that does not depend on the ordering of the sets θ_{sr} is applied to.

Definition 4 ($\equiv_{sr}$). *Let Δ be a conditional belief base and let $\delta_i \in \Delta$.*

*Two V/F-constraints $vf(\delta_i) : \eta_i > \mathcal{V}_i - \mathcal{F}_i$ and $vf(\delta_i) : \eta_i > \mathcal{V}'_i - \mathcal{F}'_i$ are sr-*equivalent*, denoted by $vf(\delta) \equiv_{sr} vf'(\delta)$, if $\theta_{sr}(\mathcal{V}_i) = \theta_{sr}(\mathcal{V}'_i)$ and $\theta_{sr}(\mathcal{F}'_i) = \theta_{sr}(\mathcal{F}'_i)$.*

*For $\Delta' \subseteq \Delta$ with $\Delta' = \{\delta_1, \ldots, \delta_m\}$, two sets of V/F-constraints $VF = \{vf(\delta_1), \ldots, vf(\delta_m)\}$ and $VF' = \{vf'(\delta_1), \ldots, vf'(\delta_m)\}$ are sr-*equivalent*, denoted by $VF \equiv_{sr} VF'$, if, for all $\delta_i \in \Delta'$, we have $vf(\delta_i) \equiv_{sr} vf'(\delta_i)$.*

Thus, $VF \equiv_{sr} VF'$ denotes that the sets of V/F-constraints VF and VF' coincide after exhaustively applying the rule θ_{sr} implementing subset reduction according to (7) to all V/F-constraints occurring in VF and VF'.

Because the subset reduction rule θ_{sr} only exploits basic algebraic properties of arithmetics and the minimum, the set of solutions of sr-equivalent constraint systems coincide.

Proposition 5. *Let $\Delta' \subseteq \Delta$, and let VF and VF$'$ be two sets of V/F-constraints for Δ' with respect to Δ. Then $VF \equiv_{sr} VF'$ implies $Sol(VF) = Sol(VF')$.*

We now present a technical lemma regarding the modularity and sr-equivalence of the constraint systems in case of generalized safe conditional splittings (Spiegel et al. 2025) that will be very helpful for this paper.

Lemma 6. *If $\Delta = \Delta_1 \bigcup^{\mathsf{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$, then the following statements hold for the corresponding constraint systems:*

- *Global vs. local constraints are sr-equivalent:*

$$CR(\Delta)_{\Delta_1} \equiv_{sr} CR(\Delta_1) \tag{8}$$
$$CR(\Delta)_{\Delta_2} \equiv_{sr} CR(\Delta_2) \tag{9}$$
$$CR(\Delta)_{\Delta_3} \equiv_{sr} CR(\Delta_3) \tag{10}$$

- *All restrictions to Δ_3 are sr-equivalent, i.e.:,*

$$CR(\Delta)_{\Delta_3} \equiv_{sr} CR(\Delta_1)_{\Delta_3} \equiv_{sr} CR(\Delta_2)_{\Delta_3} \equiv_{sr} CR(\Delta_3) \tag{11}$$

Lemma 6 shows that generalized safe splittings ensure quite a strong modularity of the constraint system. The next proposition transfers this to c-representations and summarizes results by Spiegel et al. (2025).

Proposition 7. *We presuppose $\Delta = \Delta_1 \bigcup^{\mathsf{gs}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$.*

1. Global to local: *If κ is a c-representation of Δ, then, for $i \in \{1, 2\}$, $\kappa_i = \kappa_{|\Sigma_i \cup \Sigma_3}$ are c-representations of Δ_i, and it holds that $(\kappa_1)_{|\Sigma_3} = (\kappa_2)_{|\Sigma_3}$. Moreover,*

$$\kappa(\omega) = \kappa_1(\omega^1\omega^3) + \kappa_2(\omega^2\omega^3) - \kappa_1(\omega^3)$$

2. Local to global: *Let $\kappa_1 = \kappa_{\vec{\eta}_1}$ be a c-representation of Δ_1. Then there is a c-representation $\kappa_2 = \kappa_{\vec{\eta}_2}$ of Δ_2 such that $(\vec{\eta}_1)_{|\Delta_3} = (\vec{\eta}_2)_{|\Delta_3}$ and $\kappa_{1|\Sigma_3} = \kappa_{2|\Sigma_3}$, and a c-representation κ of Δ can be defined by*

$$\kappa(\omega) = \kappa_1(\omega^1\omega^3) + \kappa_2(\omega^2\omega^3) - \kappa_1(\omega^3).$$

5 LEG-Networks for c-Representations

In this section, we generalize the idea of generalized safe conditional syntax splitting to a decomposition $\Delta = \Delta_1 \cup \ldots, \cup\Delta_m$ of more than two subbases and combine this with the idea of OCF-LEG networks presented in Sect. 3.1. A major aim is to have a result similar to Theorem 2, but where all ranking functions are also c-representations of the respective (sub)bases. This could not be ensured with the techniques presented in (Eichhorn and Kern-Isberner 2014). Here the safety property will turn out to be the crucial ingredient to guarantee the full compatibility of c-representations with OCF-LEG networks.

First, we define what it means that a conditional belief base can be decomposed safely over a hypertree in Section 5.1. Section 5.2 then exploits these techniques for c-representations, introducing c-LEG networks.

5.1 Safe Decompositions Over Hypertrees

The missing link between a conditional belief base Δ and a hypergraph structure yielding a suitable decomposition of Δ is established in quite a simple, intuitive way and analogous to what is done in (Eichhorn and Kern-Isberner 2014). This hypergraph is usually not a hypertree, but can be decomposed into one via an algorithm that is fully described in (Eichhorn 2018; Meyer 1998), but all relevant steps already appear in (Lauritzen and Spiegelhalter 1988). This algorithm uses a well-known result from graph theory: a hypergraph is a hypertree iff its edges correspond to the cliques of a so-called *triangulated*[2] graph. We recall the main steps of this procedure from (Tarjan and Yannakakis 1984):

1. Let $\Delta = \{\delta_1, \ldots, \delta_n\}$. Initialize a hypergraph $H = \langle \Sigma, \mathfrak{S} \rangle$ with $\mathfrak{S} = \{S_1, \ldots, S_n\}$ via $S_i = \{v \in \Sigma \mid v$ appears in $\delta_i\}$ for all $1 \leq i \leq n$.
2. Form the *primal graph* $G = \langle \Sigma, \mathcal{E} \rangle$ of H by connecting all symbols which are elements of a hyperedge in H.
3. Order the nodes according to *maximum-cardinality search*: Label one random node with 1. Then proceed by labeling the node with the most labeled neighbors, until all nodes are labeled. For each $v \in \Sigma$, let $\mathrm{MCS}(v)$ denote the label obtained in this way.
4. Construct a graph $G' = \langle \Sigma, \mathcal{E}' \rangle$ as follows: For each node v, connect its neighbours that have a lower MCS-label. The resulting graph G' is triangulated.

[2] A graph is *triangulated* if for all cycles of length greater than 3, an edge connecting two non-consecutive nodes on the cycle exists.

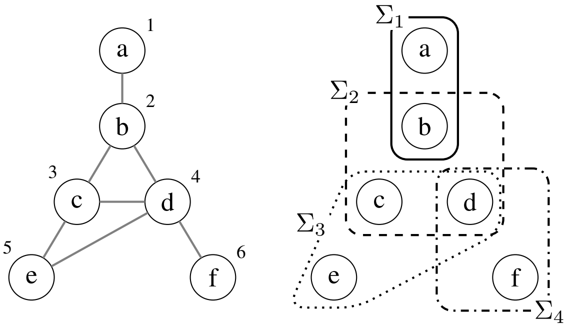

Figure 1: Primal graph and corresponding hypertree for Example 3.

5. Find the set of all inclusion-maximal cliques[3] $\mathfrak{C} = \{C_1, \ldots, C_q\}$ in the graph G'. Then $\langle \Sigma, \mathfrak{C} \rangle$ is a hypertree, because sorting the cliques C_j according to $\max_{v \in C_j} \mathrm{MCS}(v)$ produces an ordering with RIP.

More formal details of this procedure are shown in the algorithm COMPUTE-HYPERTREE in Section 5.2. The runtime complexity of this algorithm was analyzed in (Eichhorn 2018) and found to be $\mathcal{O}(|\Delta| \cdot |\Sigma|^2)$, i.e. polynomial time. Note that the algorithm above computes a hypertree $\langle \Sigma, \mathfrak{C} \rangle$ which is reduced. Each hyperedge C_i in $\langle \Sigma, \mathfrak{C} \rangle$ covers a subset of Σ containing all symbols used within a subset of Δ. Therefore, we can decompose Δ naturally according to the structure of the hypertree via $\Delta_i = \Delta \cap (\mathcal{L}(C_i) \mid \mathcal{L}(C_i))$, for $1 \leq i \leq q$. Moreover, this decomposition covers the original conditional belief base and signature, i.e., $\bigcup_{i=1}^{q} C_i = \Sigma$ and $\bigcup_{i=1}^{q} \Delta_i = \Delta$. The algorithm is illustrated by Example 3.

Example 3. *Let* $\Sigma = \{a, b, c, d, e, f\}$ *and let*

$$\Delta = \{\delta_1 : (b|a), \delta_2 : (c|\overline{b}d), \delta_3 : (d|c)$$
$$\delta_4 : (e|c), \delta_5 : (\overline{d}|c\overline{e}), \delta_6 : (d|f)\}$$

be a conditional belief base over Σ. *For the initial hypergraph* H, *we obtain the hyperedges* $S_1 = \{a, b\}$, $S_2 = \{b, c, d\}$, $S_3 = \{c, d\}$, $S_4 = \{c, e\}$, $S_5 = \{c, d, e\}$, *and* $S_6 = \{d, f\}$. *The corresponding primal graph* G *is shown on the left side of Figure 1 along with a possible MCS numbering. The primal graph is already triangulated. The inclusion-maximal cliques in the primal graph are* $\Sigma_1 = \{a, b\}$, $\Sigma_2 = \{b, c, d\}$, $\Sigma_3 = \{c, d, e\}$, *and* $\Sigma_4 = \{d, f\}$. *Ordered in this way, the cliques also have the RIP, which is quickly checked:* $\mathbb{S}_1 = \emptyset$, $\mathbb{S}_2 = \{b\} \subseteq \Sigma_1$, $\mathbb{S}_3 = \{c, d\} \subseteq \Sigma_2$, $\mathbb{S}_4 = \{d\} \subseteq \Sigma_2$. *The resulting hypertree* $\langle \Sigma, \{\Sigma_1, \ldots, \Sigma_4\} \rangle$ *is visualized on the right side of Figure 1.*

Example 3 also nicely illustrates the computational advantages of OCF-LEG networks mentioned in Section 3.1: When using four local ranking models instead of a single global ranking model, instead of having to represent ranks for $2^6 = 64$ possible worlds, one only needs to represent ranking functions defined on at most $2^3 = 8$ possible worlds.

For the following, we can now assume that the conditional belief base Δ over Σ can be decomposed in the form $\Delta = \Delta_1 \cup \ldots \cup \Delta_m$, where $\Delta_i \subseteq (\mathcal{L}(\Sigma_i) \mid \mathcal{L}(\Sigma_i))$, $\Sigma_i \subseteq \Sigma$, for all subbases, and $\langle \Sigma, \{\Sigma_1, \ldots, \Sigma_m\} \rangle$ is a reduced hypertree.

Now, we are going to generalize the idea of generalized safe conditional syntax splitting, which is defined for just two subbases, to this decomposition. The basic idea of generalized safe conditional syntax splitting is that for all worlds over a subsignature, an extension to the whole signature can be found that falsifies (if at all) only conditionals from Δ_i. We simplify the phrasing a bit in the following definition.

Definition 5 ((safe) decomposition)**.** *A decomposition* $\Delta = (\Delta_1, \ldots, \Delta_m)$, *i.e.,* $\Delta = \Delta_1 \cup \ldots \cup \Delta_m$ *such that* $\Delta_i \subseteq (\mathcal{L}(\Sigma_i) \mid \mathcal{L}(\Sigma_i))$, *is said to be* safe *if for all* $i \in \{1, \ldots, m\}$, *for all* $\omega^i \in \Omega(\Sigma_i)$, *there is* $\omega_0 \in \Omega$ *such that* $\omega_0 \models \omega^i$ *and*

$$\omega_0 \not\models E\overline{F} \text{ for all } (F|E) \in \Delta \setminus \Delta_i.$$

A safe decomposition of Δ ensures that the syntactic decomposition into subbases Δ_i over subsignatures Σ_i complies with a corresponding semantic decomposition in the sense that the falsification behavior with respect to one of the subbases does not interfere with falsifications regarding the other subbases. Definition 5 generalizes Definition 3, covering generalized safe splitting as the case $m = 2$.

Proposition 8. *A decomposition* $\Delta = (\Delta_1, \Delta_2)$ *is safe iff* $\Delta = \Delta_1 \bigcup^{\mathrm{gs}}_{\Sigma_1 \setminus \Sigma_2, \Sigma_2 \setminus \Sigma_1} \Delta_2 \mid \Sigma_1 \cap \Sigma_2$.

Hence Example 1 also provides an example for a safe decomposition. Note that for safe decompositions, neither the subbases nor the underlying subsignatures are presupposed to be disjoint, just to the contrary. The intersection of subbases are meant to play a specific role in transferring information concerning impact factors between the subbases. The following proposition is analogous to Proposition 4, but for general safe decompositions.

Proposition 9. *Let* $\Delta = (\Delta_1, \ldots, \Delta_m)$ *be a safe decomposition, let* Δ_i, Δ_j *be subbases of* Δ. *Then for all* $\omega^{i,j} \in \Omega(\Sigma_i \cap \Sigma_j)$, *there is* $\omega_0 \in \Omega$ *such that* $\omega_0 \models \omega^{i,j}$ *and* $\omega_0 \not\models E\overline{F}$ *for all* $(F|E) \in \Delta \setminus (\Delta_i \cap \Delta_j)$.

Proof sketch. This proposition is analogous to Proposition 4 and proved basically in the same way. ☐

Now we combine safe decompositions with hypertrees.

Definition 6 ((safe) decomposition over hypertree)**.** *Let* $\Delta = \Delta_1 \cup \ldots \cup \Delta_m$ *be a conditional belief base such that* $\Delta_i \subseteq (\mathcal{L}(\Sigma_i) \mid \mathcal{L}(\Sigma_i))$ *is defined over* $\Sigma_i \subseteq \Sigma$, $i \in \{1, \ldots, m\}$. *Moreover, we assume that* $\Sigma_1, \ldots, \Sigma_m$ *is a set of covering subsets of* Σ *that forms a (reduced) hypertree, i.e.,* $\Sigma_i \subseteq \Sigma$, $1 \leq i \leq m$, $\Sigma = \bigcup_{i=1}^{m} \Sigma_i$, *no* Σ_i *is a subset of another* Σ_j, *and for all* $1 \leq i \leq m$, *there is* $j(i) < i$ *s.t.*

$$\mathbb{S}_i = \Sigma_i \cap (\Sigma_1 \cup \ldots \cup \Sigma_{i-1}) \subseteq \Sigma_{j(i)}. \tag{12}$$

If for all $1 \leq i \leq m$, *with the same* $j(i) < i$ *from RIP (12),*

$$(\Delta\text{-RIP}) \quad \Delta_i \cap (\Delta_1 \cup \ldots \cup \Delta_{i-1}) \subseteq \Delta_{j(i)}, \tag{13}$$

then $\Delta = (\Delta_1, \ldots, \Delta_m)$ *is a* decomposition over the hypertree $\langle \Sigma, \{\Sigma_1, \ldots, \Sigma_m\} \rangle$. *We call* $\Delta_{j(i)}$ *the* separator companion *of* Δ_i, *and* $\Delta_i \cap \Delta_{j(i)}$ *the* separator (sub)base of Δ_i.

Furthermore, if this decomposition is safe then $\Delta = (\Delta_1, \ldots, \Delta_m)$ *is called a* safe decomposition over the hypertree $\langle \Sigma, \{\Sigma_1, \ldots, \Sigma_m\} \rangle$.

[3]A *clique* is a set of nodes which is fully connected.

Note that the hypertree structure, i.e., (RIP) (12), is not necessarily also imposed on the decomposition $(\Delta_1, \ldots, \Delta_m)$ of Δ, i.e., we may have that $\Delta_i \cap (\Delta_1 \cup \ldots \cup \Delta_{i-1}) \not\subseteq \Delta_{j(i)}$ in spite of the RIP property. Therefore, we have to presuppose (13) explicitly.

In the following lemma, we make some trivial, but technically relevant remarks regarding decompositions over hypertrees that will be helpful in the following.

Lemma 10. *Let $\Delta = (\Delta_1, \ldots, \Delta_m)$ be a decomposition over the hypertree $(\Sigma, \{\Sigma_1, \ldots, \Sigma_m\})$.*

1. *For all $i \in \{1, \ldots, m\}$ and the corresponding $j(i)$ according to (12), the following statements hold:*
   (i) $\mathbb{S}_i = \Sigma_i \cap \Sigma_{j(i)}$
   (ii) $\Sigma_i \setminus \mathbb{S}_i = \Sigma_i \setminus \Sigma_{j(i)} = \Sigma_i \setminus (\Sigma_1 \cup \ldots \Sigma_{i-1})$
2. *For all $i \in \{1, \ldots, m\}$ and the corresponding $j(i)$ according to (13), the following statements hold:*
   (i) $\Delta_i \cap \Delta_{j(i)} = \Delta_i \cap (\Delta_1 \cup \ldots \cup \Delta_{i-1})$.
   (ii) $\Delta_i \setminus (\Delta_i \cap \Delta_{j(i)}) = \Delta_i \setminus \Delta_{j(i)} = \Delta_i \setminus (\Delta_1 \cup \ldots \cup \Delta_{i-1})$.

This lemma has important consequences for the representation of possible worlds and the conditionals of Δ if Δ is decomposed over a hypertree as defined in Definition 6. While we explicitly allow overlappings among subsignatures and subbases, disjoint splittings can be obtained thanks to the hypertree structure. More precisely, for all $i \in \{1, \ldots, m\}$, $\check{\Sigma}_i = \Sigma_i \setminus \mathbb{S}_i$ is disjoint from $(\Sigma_1 \cup \ldots \cup \Sigma_{i-1})$. This means that $\bigcup_{1 \le i \le m} \check{\Sigma}_i$ is a splitting of Σ in disjoint subsignatures. Therefore, each $\omega \in \Omega$ can be written uniquely in the form

$$\omega = \check{\omega}^1 \ldots \check{\omega}^m \tag{14}$$

where $\check{\omega}^i \in \Omega(\check{\Sigma}_i)$, $i \in \{1, \ldots, m\}$. Note that $\mathbb{S}_1 = \emptyset$, hence $\check{\Sigma}_1 = \Sigma_1$. Similarly regarding the subbases of Δ, Δ splits into the disjoint subbases $\Delta_i \setminus (\Delta_i \cap \Delta_{j(i)})$. Therefore, we can fix the following notation for the technical details in the following: For $i \in \{1, \ldots, m\}$,

$$\begin{aligned} \Delta_i \;=\; \{&\delta_1^i = (A_1^i \mid B_1^i), \ldots, \delta_{s_i}^i = (A_{s_i}^i \mid B_{s_i}^i), \\ &\delta_{s_i+1}^i = (A_{s_i+1}^i \mid B_{s_i+1}^i), \ldots, \delta_{n_i}^i = (A_{n_i}^i \mid B_{n_i}^i)\}, \end{aligned} \tag{15}$$

where $\{\delta_1^i, \ldots, \delta_{s_i}^i\} = \Delta_i \cap \Delta_{j(i)} \subseteq (\mathcal{L}(\mathbb{S}_i) \mid \mathcal{L}(\mathbb{S}_i))$ are the conditionals from $(\mathcal{L}(\mathbb{S}_i) \mid \mathcal{L}(\mathbb{S}_i))$ which Δ_i has in common with $\Delta_{j(i)}$. Note that $s_1 = 0$ because of $\mathbb{S}_1 = \emptyset$, and $s_i \ge 0$ for $i \in \{2, \ldots, m\}$. Then Δ can be written in the form

$$\begin{aligned} \Delta \;=\; \{&\delta_1^1 = (A_1^1 \mid B_1^1), \ldots, \delta_{n_1}^1 = (A_{n_1}^1 \mid B_{n_1}^1), \\ &\delta_{s_2+1}^2 = (A_{s_2+1}^2 \mid B_{s_2+1}^2), \ldots, \delta_{n_2}^2 = (A_{n_2}^2 \mid B_{n_2}^2), \ldots \\ &\delta_{s_m+1}^m = (A_{s_m+1}^m \mid B_{s_m+1}^m), \ldots, \delta_{n_m}^m = (A_{n_m}^m \mid B_{n_m}^m)\}, \end{aligned} \tag{16}$$

with $\delta_k^1 \in (\mathcal{L}(\Sigma_1) \mid \mathcal{L}(\Sigma_1)), k \in \{1, \ldots, n_1\}$ and $\delta_k^i \in (\mathcal{L}(\Sigma_i) \mid \mathcal{L}(\Sigma_i))$, but $\delta_k^i \notin (\mathcal{L}(\Sigma_1 \cup \ldots \cup \Sigma_{i-1}) \mid \mathcal{L}(\Sigma_1 \cup \ldots \cup \Sigma_{i-1}))$ for $i \in \{2, \ldots, m\}$.

Moreover, each c-representation of Δ with impact vector $\vec{\eta}$ can be written in the form

$$\kappa(\omega) = \sum_{\substack{1 \le k \le n_1 \\ \omega \models A_k^1 \overline{B_k^1}}} \eta_k^1 + \sum_{\substack{s_2 < k \le n_2 \\ \omega \models A_k^2 \overline{B_k^2}}} \eta_k^2 \ldots + \sum_{\substack{s_m < k \le n_m \\ \omega \models A_k^m \overline{B_k^m}}} \eta_k^m. \tag{17}$$

In the next subsection, we continue considering the consequences of decompositions of conditional belief bases over hypertrees for the respective c-representations.

5.2 c-LEG Networks

For further elaborating on the technical consequences of hypertree decompositions for c-representations and their constraint systems, we need to assume the safeness of the decomposition. The following lemma shows that under this assumption, the separator subbases play a crucial role in that they establish links between the respective constraint systems. It is analogous to Lemma 6, but for hypertree decompositions.

Lemma 11. *If $(\Delta_1, \ldots, \Delta_m)$ is a safe decomposition of Δ over the hypertree $(\Sigma, \{\Sigma_1, \ldots, \Sigma_m\})$, then the following statements hold for the corresponding constraint systems:*

1. *All restrictions of corresponding constraint systems to the separator bases $\Delta_i \cap \Delta_{j(i)}$ are sr-equivalent globally (i.e., wrt Δ) and locally (i.e., wrt Δ_i and $\Delta_{j(i)}$), and can be solved independently, i.e., for all $i \in \{1, \ldots, m\}$,*
$$\begin{aligned} &CR(\Delta)_{\Delta_i \cap \Delta_{j(i)}} \equiv_{sr} CR(\Delta_i)_{\Delta_i \cap \Delta_{j(i)}} \\ &\equiv_{sr} CR(\Delta_{j(i)})_{\Delta_i \cap \Delta_{j(i)}} \equiv_{sr} CR(\Delta_i \cap \Delta_{j(i)}). \end{aligned} \tag{18}$$
2. *Global constraints are sr-equivalent to local constraints, i.e., for all $i \in \{1, \ldots, m\}$,*
$$CR(\Delta)_{\Delta_i} \equiv_{sr} CR(\Delta_i). \tag{19}$$

Proof sketch. The proof of this technical lemma exploits safeness in the context of Proposition 9 and Lemma 10, making use of the representations (14), (15), (16), and (17). □

Lemma 11 reveals how the structure of the constraint system of a conditional belief base Δ that can be safely decomposed over a hypertree follows the structure of the hypertree. The next proposition exploits this for c-representations of Δ, lifting Proposition 7 to hypertree decompositions. It shows that each c-representation of Δ can be represented by its locally marginalized c-representations. Conversely, local c-representations of the subbases which follow a common strategy can be glued together to yield a global c-representation of Δ. Regarding solutions of the constraint systems, the separator subbases pass information from previous subbases to later ones, following the enumeration of the hypertree.

Proposition 12. *Let $(\Delta_1, \ldots, \Delta_m)$ be a safe decomposition of Δ over the hypertree $(\Sigma, \{\Sigma_1, \ldots, \Sigma_m\})$.*

1. Global to local: *If $\kappa = \kappa_{\vec{\eta}}$ is a c-representation of Δ, then, for $i \in \{1, \ldots, m\}$, $\kappa_i = \kappa_{|\Sigma_i}$ is a c-representation of Δ_i with impact vector $\vec{\eta}_{\Delta_i}$, and it holds that $(\kappa_i)_{|\mathbb{S}_i} = (\kappa_{j(i)})_{|\mathbb{S}_i}$. Moreover,*
$$\kappa(\omega) = \sum_{i=1}^{m} \kappa_i(\omega^i) - \sum_{i=1}^{m} \kappa_i(\omega^{i,j(i)}). \tag{20}$$
2. Local to global: *There is a series of c-representations $\kappa_i = \kappa_{\vec{\eta}_i}$ of Δ_i, $i \in \{1, \ldots, m\}$ such that $(\vec{\eta}_i)_{\Delta_i \cap \Delta_{j(i)}} = (\vec{\eta}_{j(i)})_{\Delta_i \cap \Delta_{j(i)}}$ and $\kappa_{i|\mathbb{S}_i} = \kappa_{j(i)}{}_{|\mathbb{S}_i}$, and κ defined by (20) is a c-representation of Δ such that $\kappa_i = \kappa_{|\Sigma_i}$.*

Proof sketch. Both local and global c-representations are shaped by solutions of the respective constraint systems. Using the insights from Lemma 11, this proposition is a direct consequence of the connections among the constraint systems shown in the lemma. □

Proposition 12 establishes a strong connection between safe decomposition of conditional belief bases over hypertrees and c-representations. This allows us to strengthen the notion of OCF-LEG-networks (see Definition 2) by requiring that all involved ranking functions are c-representations of the corresponding conditional belief bases.

Definition 7 (c-LEG network)**.** *Let $\Sigma_1, \ldots, \Sigma_m$ be a set of covering subsets over an alphabet Σ, i.e., $\Sigma_i \subseteq \Sigma$, $1 \leq i \leq m$ and $\Sigma = \bigcup_{i=1}^{m} \Sigma_i$. Let $\Delta = \Delta_1 \cup \ldots \cup \Delta_m$ be a conditional belief base such that $\Delta_i \subseteq (\mathcal{L}(\Sigma_i) \mid \mathcal{L}(\Sigma_i))$ is defined over Σ_i, $i \in \{1, \ldots, m\}$. Let $\kappa_1, \ldots, \kappa_m$ be ranking functions $\kappa_i : \Omega_i \to \mathbb{N}_0^\infty$, $1 \leq i \leq m$.*

We call a tuple $\langle \Sigma_i, \Delta_i, \kappa_i \rangle$ a c-local event group (c-LEG) *of Δ if $\kappa_i = \kappa_{\vec{\eta}_i}$ is a c-representation of Δ_i, i.e., $\vec{\eta}_i \in Sol(CR(\Delta_i))$. The system $\langle (\Sigma_i, \Delta_i, \kappa_i) \rangle_{i=1}^m = \langle (\Sigma_1, \Delta_1, \kappa_1), \ldots, (\Sigma_m, \Delta_m, \kappa_m) \rangle$ is a* c-LEG network *iff there is a c-representation $\kappa = \kappa_{\vec{\eta}}$ of Δ such that $\kappa_i = \kappa_{|\Sigma_i}$ and $\vec{\eta}_{\Delta_i} = \vec{\eta}_i$ for $i \in \{1, \ldots, m\}$.*

The following theorem is a direct consequence of Proposition 12, now using the notion of c-LEG networks. It states that if a conditional belief base Δ can be safely decomposed over a hypertree, then Proposition 12(1.) implies that any global c-representation of Δ can be decomposed by the help of its local marginalizations to satisfy (20). In this case, the consistency condition (2) is fulfilled in a particularly strong way: Not only the local ranking functions coincide on the intersections of subsignatures, but also the impact factors of conditionals in the intersections of subbases coincide.

Theorem 13. *Let $(\Delta_1, \ldots, \Delta_m)$ be a safe decomposition of Δ over the hypertree $(\Sigma, \{\Sigma_1, \ldots, \Sigma_m\})$ (cf. Def. 6), i.e., $\Delta_i \subseteq (\mathcal{L}(\Sigma_i) \mid \mathcal{L}(\Sigma_i))$ and both (12) and (13) hold. Then any c-representation $\kappa = \kappa_{\vec{\eta}}$ of Δ has a c-LEG-network representation over $(\Sigma, \{\Sigma_1, \ldots, \Sigma_m\})$, i.e., the system $\langle (\Sigma_i, \Delta_i, \kappa_i) \rangle_{i=1}^m = \langle (\Sigma_1, \Delta_1, \kappa_1), \ldots, (\Sigma_m, \Delta_m, \kappa_m) \rangle$ is a c-LEG network such that for $i \in i \in \{1, \ldots, m\}$, $\kappa_i = \kappa_{|\Sigma_i} = \kappa_{\vec{\eta}_i}$ with $\vec{\eta}_i = \vec{\eta}_{\Delta_i}$ are c-representations of Δ_i. Moreover, (20) holds.*

However, Proposition 12 not only offers the possiblity of decomposing c-representations into components of (usually much) lower complexity. In its second part, this proposition proposes to construct a global c-representation step by step by following the structure of the hypertree decomposing the conditional belief base. This means that instead of having to solve a constraint system over all variables η_k representing all conditionals in Δ, one can solve (smaller) constraint systems over the subbases Δ_i, starting with Δ_1. For higher indices $i > 1$, the constraint system over Δ_i will reuse solutions from predecessor subbases. This is possible thanks to the safety of the decomposition of Δ. In this way, it can be ensured that the solution vectors $\vec{\eta}_i$ of the local constraint systems can be merged (following the order of the hypertree) to yield a global impact vector $\vec{\eta}$ for all conditionals in Δ that defines a global c-representation κ of Δ. For this, the local impact vectors have to satisfy $(\vec{\eta}_i)_{\Delta_i \cap \Delta_{j(i)}} = (\vec{\eta}_{j(i)})_{\Delta_i \cap \Delta_{j(i)}}$. Such links among impact vectors referring to different conditional belief bases can be ensured resp. postulated by selection strategies.

Definition 8 (selection strategy σ, strategic c-representations)**.** *A* selection strategy *for c-representations of a set $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ of conditionals is a function $\sigma : \Delta \mapsto \vec{\eta}$, assigning to each such set of conditionals Δ an impact vector $\vec{\eta}$ that solves $CR(\Delta)$. If $\sigma(\Delta) = \vec{\eta}$, the* c-representation of Δ determined by σ *is denoted by $\kappa_{\sigma(\Delta)} = \kappa_{\vec{\eta}}$ and is called a* strategic c-representation*.*

If $\Delta' \subseteq \Delta$ is a subbase of Δ, then the restriction of $\sigma(\Delta)$ to Δ'*, denoted by $\sigma(\Delta)_{\Delta'}$, is the respective subvector consisting of all impact factors belonging to conditionals in Δ'.*

In general, selection strategies for c-representations allow us to establish links between c-representations of different inductive reasoning problems (Kern-Isberner, Beierle, and Brewka 2020; Beierle and Kern-Isberner 2021; Kern-Isberner, Sezgin, and Beierle 2023). Based on Proposition 12 (2.), we can now define a strategy for building up c-representations of conditional belief bases that safely decompose over a hypertree by only making use of local information provided by the hyperedges. The proof of the following theorem is immediate from the proposition.

Theorem 14. *Let $(\Delta_1, \ldots, \Delta_m)$ be a safe decomposition of Δ over the hypertree $(\Sigma, \{\Sigma_1, \ldots, \Sigma_m\})$. Then the following strategy σ computes an impact vector $\vec{\eta}$ for a c-representation $\kappa_{\vec{\eta}}$ of Δ:*

1. *For Δ_1, set $\sigma(\Delta_1) = \vec{\eta}_1$ for some solution $\vec{\eta}_1 \in Sol(CR(\Delta_1))$;*
2. *for $i = 2, \ldots, m$, in this order, choose successively $\sigma(\Delta_i) = \vec{\eta}_i \in Sol(CR(\Delta_i))$ such that $(\vec{\eta}_i)_{\Delta_i \cap \Delta_{j(i)}} = (\vec{\eta}_{j(i)})_{\Delta_i \cap \Delta_{j(i)}}$ is set according to a previously chosen solution (note that $j(i) < i$);*
3. *merge the impact vectors $\vec{\eta}_1, \ldots, \vec{\eta}_m$ by identifying their common subvectors corresponding to the separator subbases to yield the vector $\vec{\eta} = \sigma(\Delta)$ as the impact vector of the global c-representation $\kappa_{\vec{\eta}}$ of Δ.*

We illustrate Theorem 14 by the following example.

Example 4. *Consider again the conditional belief base Δ from Example 3 and the hypertree from Figure 1. A corresponding decomposition of Δ is given by*

$$\begin{aligned}
&\Delta_1 = \{\delta_1 : (b \mid a)\},\\
&\Delta_2 = \{\delta_2 : (c \mid \overline{b}d), \delta_3 : (d \mid c)\},\\
&\Delta_3 = \{\delta_3 : (d \mid c), \delta_4 : (e \mid c), \delta_5 : (\overline{d} \mid c\overline{e})\}\\
&\Delta_4 = \{\delta_6 : (d|f)\},
\end{aligned}$$

i.e., we have $\Delta_i \subseteq (\mathcal{L}(\Sigma_i)|\mathcal{L}(\Sigma_i)), i \in \{1, \ldots, 4\}$, and $\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$. It is straightforward to check that $\Delta = (\Delta_1, \Delta_2, \Delta_3, \Delta_4)$ is a safe decomposition over the hypertree $(\Sigma, \{\Sigma_1, \ldots, \Sigma_4\})$ where all separator subbases are empty except for $\Delta_3 \cap (\Delta_1 \cup \Delta_2) = \{\delta_3 : (d \mid c)\} \subseteq \Delta_2$. Hence also Δ-RIP (13) is satisfied. Note that we built in δ_3 both in Δ_2 and Δ_3 to illustrate how information is passed from one hyperedge to the next via the separator base.

ω	$\kappa_1(\omega)$	ω	$\kappa_1(\omega)$
ab	0	$\overline{a}b$	0
$a\overline{b}$	1	$\overline{a}\overline{b}$	0

ω	$\kappa_4(\omega)$	ω	$\kappa_4(\omega)$
df	0	$\overline{d}f$	1
$d\overline{f}$	0	$\overline{d}\,\overline{f}$	0

ω	$\kappa_2(\omega)$	ω	$\kappa_2(\omega)$
bcd	0	$\overline{b}cd$	0
$bc\overline{d}$	1	$\overline{b}c\overline{d}$	1
$b\overline{c}d$	0	$\overline{b}\overline{c}d$	1
$b\overline{c}\overline{d}$	0	$\overline{b}\overline{c}\overline{d}$	0

ω	$\kappa_3(\omega)$	ω	$\kappa_3(\omega)$
cde	0	$\overline{c}de$	0
$cd\overline{e}$	2	$\overline{c}d\overline{e}$	0
$c\overline{d}e$	1	$\overline{c}\overline{d}e$	0
$c\overline{d}\overline{e}$	1	$\overline{c}\overline{d}\overline{e}$	0

Figure 2: Local c-representations $\kappa_1, \ldots, \kappa_4$ for Example 4

Now we show how a global c-representation of Δ is constructed by following a strategy as described in Theorem 14. We start with Δ_1 and find that on Σ_1, the impact factor η_1 of δ_1 has to satisfy $\eta_1 > 0$. We choose $\eta_1 = 1$ minimally, yielding $\sigma(\Delta_2) = (1)$. The resulting c-representation κ_1 on Σ_1 is shown in Figure 2. We proceed with Δ_2 on Σ_2 and can choose again η_2, η_3 minimally: $\eta_2 = \eta_3 = 1$ (see Figure 2).

Now we have to use $\eta_3 = 1$ for the constraint system of Δ_3. For δ_4 and δ_5, we compute $\eta_4 > 0 - \min\{\eta_3, \eta_5\}$ and $\eta_5 > (\eta_3 + \eta_4) - \eta_4 = \eta_3 = 1$. Again, we choose minimally $\eta_4 = 0$ and $\eta_5 = 2$. The resulting κ_3 is shown in Figure 2. Finally for Δ_4, we can choose $\eta_6 = 1$ minimally to set up κ_4 in Figure 2.

Merging the impact vectors of $\Delta_1, \ldots, \Delta_4$, we obtain a global c-representation κ via the impact vector $(1, 1, 1, 0, 2, 1)$. Its marginals coincide with $\kappa_1, \ldots, \kappa_4$, and (20) holds. E.g., we compute $\kappa(a\overline{b}c\overline{d}\overline{e}f) = \kappa_1(a\overline{b}) + \kappa_2(\overline{b}c\overline{d}) - \kappa_2(\overline{b}) + \kappa_3(c\overline{d}\overline{e}) - \kappa_3(c\overline{d}) + \kappa_4(\overline{d}f) - \kappa_4(\overline{d}) = 3$. Note that inference queries (expressed as conditionals) that mention only atoms from one of the Σ_i's can be answered by just looking up the respective local κ_i. E.g., the query whether $(b|c)$ is accepted by the global κ can be answered by just looking into the local κ_2. Here we find $\kappa_2(bc) = 0 = \kappa_2(\overline{b}c)$, hence $(b|c)$ is not accepted by κ_2, and hence also not accepted by κ.

We summarize the results of this section by providing two high-level algorithms to make the general work flow clear: COMPUTE-HYPERTREE describes the first step of computing an initial hypertree from a conditional belief base as explained in the beginning of this section. After safety has been checked, COMPUTE-C-LEG shows how a c-LEG network is set up from a safe decomposition over a given hypertree; a global c-representation can be computed from this network as specified in Theorem 14.

6 Connection to Safe Covers

Safe decompositions of a conditional belief base Δ are closely linked to so-called *safe covers* of Δ and therewith to *constraint networks* as defined in (Wilhelm and Kern-Isberner 2024). While safe decompositions are motivated by syntactical splittings of Δ, safe covers exploit the dependencies between the impact factors in $CR(\Delta)$, thus are semantic-driven. The basic idea of safe covers is to identify subbases Δ' of Δ for which it holds that $CR(\Delta)_{\Delta'}$ is "closed under the occurrence of impact factors" and which are therefore called *safe* in (Wilhelm and Kern-Isberner 2024). In plain words, if a safe subbase Δ' contains a conditional δ, then Δ' has to also contain all conditionals the impact factors of which are mentioned in $cr_\Delta(\delta)$. A finite set of safe subbases is called a *safe cover* of Δ if the subbases cover Δ. An example of a safe cover is $(\Delta_1, \ldots, \Delta_4)$ from Example 4. We prove that every safe decomposition is a safe cover.

Algorithm 1 COMPUTE-HYPERTREE

Input: conditionals $\Delta = \{\delta_1, \ldots, \delta_n\}$ over signature Σ.
Output: hypertree $\langle \Sigma, \mathfrak{S} \rangle$.

```
for 1 ≤ i ≤ n do
    Set S_i := {v ∈ Σ | v appears in δ_i}.
end for
Set H := ⟨Σ, 𝔖⟩ with 𝔖 := {S_1, ..., S_n}.
Set G := ⟨Σ, ℰ⟩ with ℰ := {{v, v'} ⊆ S | S ∈ 𝔖}.
Perform MCS on Σ.
for all v ∈ Σ do
    Set F(v) := {v' ∈ Σ | MCS(v') < MCS(v)}.
    for all {v', v''} ⊆ F(v) do
        Set ℰ := ℰ ∪ {{v', v''}}.
    end for
end for
Set 𝔖 := {inclusion-maximal cliques in G}.
return ⟨Σ, 𝔖⟩.
```

Algorithm 2 COMPUTE-C-LEG

Input: conditional belief base Δ as safe decomposition $(\Delta_1, \ldots, \Delta_m)$ over hypertree $(\Sigma_1, \ldots, \Sigma_m)$.
Output: c-LEG network of Δ.

```
Compute η⃗_1 ∈ Sol(CR(Δ_1)).
for 2 ≤ i ≤ m do
    Compute η⃗_i ∈ Sol(CR(Δ_i))
        such that (η⃗_i)_{Δ_i ∩ Δ_j(i)} = (η⃗_j(i))_{Δ_i ∩ Δ_j(i)}.
end for
for 1 ≤ i ≤ m do
    Set κ_i := κ_{η⃗_i}.
end for
return ⟨(Σ_i, Δ_i, κ_i)⟩_{i=1}^m.
```

Proposition 15. *Let Δ be a conditional belief base, and let $(\Delta_1, \ldots, \Delta_m)$ be a safe decomposition of Δ. Then $(\Delta_1, \ldots, \Delta_m)$ is a safe cover of Δ.*

Proof sketch. The safeness property of the decomposition $(\Delta_1, \ldots, \Delta_m)$ ensures that, after applying θ_{sr} from Section 4, the constraints of the conditionals in Δ_i for $i = 1, \ldots, m$ no longer mention impact factors of conditionals outside Δ_i. Hence, Δ_i forms a safe subbase, which implies that the decomposition $(\Delta_1, \ldots, \Delta_m)$ is a safe cover. $\square$

Notably, adding any non-empty intersection of subbases from a safe cover to the cover yields a safe cover again.

Hence, we may consider safe covers that are closed under subset formation. Every such safe cover $\mathcal{O}(\Delta)$ that is closed under subset formation can be arranged as a constraint network: The constraint network with respect to $\mathcal{O}(\Delta)$ is the directed acyclic graph $\mathcal{N}(\Delta) = \langle \mathcal{V}, \mathcal{E} \rangle$ with $\mathcal{V} = \mathcal{O}(\Delta)$ and $\mathcal{E} = \{(V, V') \mid V, V' \in \mathcal{V} \mid V \subset V'\}$. A constraint network for Δ from Example 4 is given by $\mathcal{V} = (\Delta_1, \ldots, \Delta_4, \{\delta_3\})$ and $\mathcal{E} = \{(\{\delta_3\}, \Delta_2), (\{\delta_3\}, \Delta_3)\}$.

According to (Wilhelm and Kern-Isberner 2024), c-representations of a consistent conditional belief base Δ can be computed locally with the help of a constraint network $\mathcal{N}(\Delta)$ by solving $CR(\Delta')$ for subbases $\Delta' \in \mathcal{V}$ without incoming edges first, propagating the solutions to the respective child nodes, and solving the constraint systems with respect to the child nodes thereafter. For a detailed example and further technical details see (Wilhelm and Kern-Isberner 2024). Safe decompositions now indicate a possible approach to computing safe covers without having to formulate the global constraint system $CR(\Delta)$.

7 Conclusion and Future Work

In this paper, we brought together techniques from syntax splitting and network-based methodologies to introduce c-LEG networks which are useful to decompose conditional belief bases Δ by hypertrees and set up suitable c-representations via local computations following the structure of the hypertree. The key ingredients to this approach are LEG networks (Lemmer 1982; Eichhorn and Kern-Isberner 2014) and the concept of generalized conditional syntax splitting from (Spiegel et al. 2025). We adapted the notion of safety to the situation of hypertree decompositions of conditional belief bases, which allowed us to also decompose the constraint systems defining c-representations accordingly. In this way, it could be shown that the constraint systems of local c-representations of subbases of Δ encode all relevant information that is needed for the global constraint system regarding Δ.

So, the safety condition for decomposition of conditional belief bases introduced in this paper plays a crucial role for our approach. Since it refers to the existence of a suitable possible world defined over the whole signature, it requires a global check before the algorithms can focus on local information. As part of our ongoing and future work, we investigate more local versions of this safety check that elaborate on the structure of the hypertree in more depth. As a further, more general topic, we plan to apply the methodology presented in the paper also to other approaches to inductive reasoning from conditional belief bases like System Z (Pearl 1990). We also plan to implement our approach within the reasoning system InfOCF-Web (Beierle et al. 2024a), evaluate it in practice, and analyze the complexity of OCF-LEG networks, and c-LEG networks in particular, in terms of fixed-parameter complexity. Moreover, we will extend our approach to also deal with belief bases containing propositional facts in addition to conditionals.

Acknowledgements

We thank the reviewers for their helpful feedback and detailed comments. This work was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation), project number 512363537, grant KE 1413/15-1 awarded to Gabriele Kern-Isberner, and grant BE 1700/12-1 awarded to Christoph Beierle. Alexander Hahn was supported by grant KE 1413/15-1, and Lars-Phillip Spiegel was supported by grant BE 1700/12-1.

AI Declaration

The authors have not employed any Generative AI tools.

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