Probabilistic Abduction in a Fuzzy Logic Framework

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Abstract

We study the problem of explaining observations about the probabilities of events, such as 'it rains 20% of the time', 'rain and snow are equally likely', etc. We explain these statements with a probability distribution or a statement about probabilities of (other) events that are consistent with our knowledge and entail the observation. We formalise this problem in a fuzzy probabilistic logic FP. We define and motivate the notions of abduction problems and their solutions. Our main technical contribution is a comprehensive study of the complexity of solution recognition and existence for a given abduction problem in FP for the case of full language and its disjunctive-clause fragments. We also obtain a translation of classical probabilistic abduction (finding the most likely explanation of a given event) to FP.

1 Introduction

Deduction, induction, and abduction are the main forms of reasoning (Flach and Kakas 2000). Abduction, or inference of *explanations*, has found multiple applications in artificial intelligence. Among the most prominent ones are diagnosis (Pople 1973; Console and Torasso 1990; El Ayeb, Marquis, and Rusinowitch 1993; Josephson and Josephson 1994; Koitz-Hristov and Wotawa 2018), commonsense reasoning (Paul 1993; Bhagavatula et al. 2020), formalisation of scientific reasoning (Magnani 2011), and machine learning (Dai et al. 2019), in particular, explainable AI (Ignatiev, Narodytska, and Marques-Silva 2019).

In logic-based abduction (Eiter and Gottlob 1995), the reasoning is usually conducted in the framework of *abduction problems* that contain a *theory* Γ formalising the background knowledge and an *observation* δ. The goal is to find an *explanation* of δ *relative to* Γ using a pre-defined set of *hypotheses*. That is, a formula η s.t. $\Gamma, \eta \not\models \bot$ and $\Gamma, \eta \models \delta$ (in other words, η and Γ should *consistently entail* δ).

In practice, however, our data may be unreliable, and thus the knowledge might bear a degree of uncertainty. In such contexts, one might search for *probabilistic* explanations of an observation. Hence, to provide explanations in the context of uncertainty, one may employ *probabilistic abduction*.

Probabilistic Abduction Probabilistic abduction was first introduced by Poole (1993) as probabilistic Horn abduction (PHA). This framework computes probabilities of Bayesian networks by definite clauses with probabilistic hypotheses, but does not compute probabilities based on probability measures over possible worlds as defined by Fenstad (1967).

Sato (1995) uses definite clauses with probabilistic hypotheses like PHA, but proposes the distribution semantics, which defines probabilistic measures over possible worlds without assumptions of PHA. Sato et al. (2011) extended this framework to allow arbitrary clauses in the background theory and any ground formulas in observations as constraint-based probabilistic modelling. The distribution semantics were also adopted in probabilistic logic programming. In particular, de Raedt et al. (2007) present ProbLog, which considers a probability for each definite clause, i.e., a probabilistic hypothesis to select each clause, and Azzolini et al. (2022) propose algorithms for computing explanations to probabilistic logic programs.

Other frameworks include a Bayesian view of prior distributions in abduction (Dubois, Gilio, and Kern-Isberner 2008) and abduction in Markov logic networks (Kate and Mooney 2009). They, however, do not strictly provide a logical formalisation of abduction to explain observations.

Probabilistic Reasoning in Fuzzy Logics Probabilistic abduction requires formalisation of reasoning about uncertainty. One of the approaches to this task is via the employment of so-called *probabilistic logics*. Probabilistic logic in the context of AI was first proposed by Nilsson (1986). The propositional fragment of Nilsson's logic was then formalised by Fagin et al. (1990). Hájek et al. (1995; 2000) formalised reasoning about probabilities using *fuzzy logics*, i.e., logics where formulas have values in the real-valued interval $[0, 1]$. The intuitive idea was to consider a (non-nesting) modality Pr such that, for a classical formula ϕ, the modal formula $\mathsf{Pr}(\phi)$ is interpreted as the probability measure of the event corresponding to ϕ.

Connections between probabilities and fuzzy logics have been extensively studied. Hájek and Tulipani (2001) considered the complexity of probabilistic reasoning, in particular, with conditional probabilities. Flaminio (2007) proposed a simpler formalisation of reasoning with conditional probabilities. Baldi et al. (2019; 2020) showed that fuzzy probabilistic logics and the probabilistic logics proposed by Fagin et al. have the same expressive power. Corsi et al. (2023) studied epistemic probabilistic logics and Kozhemiachenko and Sedlár (2025) considered formalisation of reasoning about probabilities and actions.

Abduction in Fuzzy Logics Abduction in various fuzzy logics has long attracted interest. The first formal presentation was given by Yamada and Mukaidono (1995) in the framework of Łukasiewicz logic. Vojtaš (1999) expanded this approach to Gödel and Product logics. Miyata et al. (1998) considered abductive reasoning with observations of different degrees. Solutions to abduction problems in multiple fuzzy logics were further systematised by d'Allones et al. (2007) and Chakraborty et al. (2013). Recently, Inoue and Kozhemiachenko (2025) provided a comprehensive study of the complexity of abduction in Łukasiewicz logic.

Abduction in fuzzy logic has found numerous applications. Namely, Vojtaš (2001) and Ebrahim (2001) proposed fuzzy logic programming. Bergadano et al. (2000) studied fuzzy abduction in the context of machine learning. Among other applications are decision-making and learning in the presence of incomplete information (Mellouli and Bouchon-Meunier 2003; Tsypyschev 2017) and robot perception (Shanahan 2005).

Contributions Although probabilistic abduction and applications of fuzzy logics to probabilistic reasoning have been extensively studied, there has been, to the best of our knowledge, no formalisation of probabilistic abduction in a fuzzy logic setting. Furthermore, in most probabilistic abduction frameworks, the main task is to find *the most likely explanation* of an observed event. On the other hand, in some situations, we are interested in *why the event occurred with the observed frequency*. In such contexts, an explanation can be a *probability distribution* consistent with our knowledge about probabilities of events that explains the frequency of the occurrence of the observed event. Consider the following running example for an illustration.

Example 1. *We study weather records of a town and notice that there was very bad weather (it rained and was cold on the same day) at least* 20% *of the time. The records are incomplete, so we do not know how likely it was to rain or to be cold on each given day. Moreover, there could be cloudy days without rain and days with mild weather (i.e., events 'it rains' and 'it is sunny' as well as 'it is cold' and 'it is warm' are* not complementary*). We know, however, that good weather (sunny and warm days) was twice as rare as bad weather (rainy or cold days). Furthermore, cold days were* 1.5 *times more often than warm days, and sunny days were rarer than rainy days. Now, we aim to* explain the observed frequency *of days with very bad weather.*

We consider probabilistic abductive reasoning within the framework of a fuzzy probabilistic logic FP that extends Łukasiewicz logic. We will formally define these logics in the next section. In our proofs, we will rely on the results for abduction in (propositional) Łukasiewicz logic (Inoue and Kozhemiachenko 2025). Our contribution is threefold. (1) We define and motivate abduction problems and their solutions that adequately represent the contexts, as in the example above. (2) We study the complexity of probabilistic abduction in the full logic and its disjunctive-clause fragments. We consider *solution recognition* (given a solution candidate ζ and a problem $\mathbb{P}$, check if ζ solves $\mathbb{P}$) and *solution existence* (given $\mathbb{P}$, check if it has solutions). (3) We formalise finding most likely explanations in FP.

Omitted proofs can be found in the appendix of the technical report (Flaminio, Inoue, and Kozhemiachenko 2026).

2 Probabilistic Łukasiewicz Logic

In this section, we present the fuzzy probability logic FP — a probabilistic expansion of Łukasiewicz logic Ł introduced by Hájek et al. in (1995). The logic FP has a *two-layered* language made of *probabilistic atoms* of the form $\mathsf{Pr}(\phi)$. E.g., if $\phi = r \vee h$, and r stands for 'it is raining' and h for 'it is hot outside', the value of $\mathsf{Pr}(r \vee h)$ corresponds to the probability of the event 'it is raining, or it is hot outside'. *Compound probability formulas* are connected by operators of Ł. To improve its expressivity, we extend its language with the $\triangle$ operator of Baaz (1996) and truth constants. We fix a countable set Var of propositional variables, let $p \in \mathsf{Var}$, $c \in \mathbb{Q} \cap [0,1]$, $\diamond \in \{\leq, <, >, \geq\}$, and define $\mathcal{L}^{\mathbb{Q}}_{\text{Ł}}$:

$$\phi \coloneqq p \mid p \diamond \overline{\mathsf{c}} \mid \triangle\phi \mid \neg\phi \mid (\phi \odot \phi) \mid (\phi \oplus \phi) \mid (\phi \rightarrow \phi)$$

Convention 1. $\mathcal{L}_{\mathsf{CPL}}$ *denotes the* $\{\neg, \wedge, \vee\}$*-language of the classical propositional logic. For a set of formulas* Γ *and a formula* ϕ, $\mathsf{Var}(\phi)$ *and* $\mathsf{Var}[\Gamma]$ *are the set of all variables occurring in* ϕ *and* Γ*, respectively.*

$\mathbb{R}$ *and* $\mathbb{Q}$ *denote the sets of real and rational numbers, respectively. A square (round) bracket means that the endpoint is included in (excluded from) the interval. Lower index* $_{\mathbb{Q}}$ *means that the interval contains rational numbers only. E.g.,* $(\frac{1}{2}, \frac{2}{3}]_{\mathbb{Q}} = \{x \mid x \in \mathbb{Q}, x > \frac{1}{2}, x \leq \frac{2}{3}\}$.

Definition 1 (Semantics of Łukasiewicz logic)**.** *An* Ł-valuation *is a function* $v : \mathsf{Var} \rightarrow [0,1]$ *extended to the complex formulas as shown in Figure 1 (cf. the top of the next page).*

We say that $\phi \in \mathcal{L}^{\mathbb{Q}}_{\text{Ł}}$ *is* Ł-valid ($\text{Ł} \models \phi$) *if* $v(\phi) = 1$ *in all* Ł*-valuations;* ϕ *is* Ł-satisfiable *if* $v(\phi) = 1$ *in some* Ł*-valuation. Given a finite* $\Phi \subseteq \mathcal{L}^{\mathbb{Q}}_{\text{Ł}}$, Φ entails χ in Ł ($\Phi \models_{\text{Ł}} \chi$) *iff* $v(\chi) = 1$ *in every* v *s.t.* $v(\phi) = 1$ *for all* $\phi \in \Phi$. Φ consistently entails χ in Ł ($\Phi \models^{\mathsf{cons}}_{\text{Ł}} \chi$) *iff* $\Phi \models_{\text{Ł}} \chi$ *and* $\Phi \not\models_{\text{Ł}} \bot$.

The language $\mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$ of FP is defined as follows (here, $\phi \in \mathcal{L}_{\mathsf{CPL}}$, $\heartsuit \in \{\neg, \triangle\}$, $\circ \in \{\odot, \oplus, \rightarrow\}$, and $\diamond \in \{\leq, <, >, \geq\}$).

$$\alpha \coloneqq \mathsf{Pr}(\phi) \mid \mathsf{Pr}(\phi) \diamond \overline{\mathsf{c}} \mid \heartsuit\alpha \mid (\alpha \circ \alpha)$$

We will interpret $\mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$-formulas on *probabilistic* models following (Hájek and Tulipani 2001). The only change is that, to simplify some proofs, instead of evaluating variables at each state, we define states of models as sets of variables.

Definition 2 (Probability measures and distributions)**.** *Given a finite* $W \neq \varnothing$,

- *a map* $\mu : 2^W \rightarrow [0,1]$ *is a* probability measure on 2^W *iff* $\mu(W) = 1$, $\mu(\varnothing) = 0$, *and* $\mu(X \cup Y) = \mu(X) + \mu(Y)$ *for every* $X, Y \subseteq W$ *s.t.* $X \cap Y = \varnothing$*;*

$$v(\neg\phi) = 1{-}v(\phi) \qquad v(p \diamond \overline{\mathsf{c}}) = \begin{cases}1 \text{ if } v(p) \diamond c\\ 0 \text{ otherwise}\end{cases} \qquad v(\triangle\phi) = \begin{cases}1 \text{ if } v(\phi){=}1\\ 0 \text{ otherwise}\end{cases}$$

$$v(\phi{\odot}\chi) = \max(0, v(\phi){+}v(\chi){-}1) \qquad v(\phi{\oplus}\chi) = \min(1, v(\phi){+}v(\chi)) \qquad v(\phi{\rightarrow}\chi) = \min(1, 1{-}v(\phi){+}v(\chi))$$

Figure 1: Semantics of $\text{Ł}^{\mathbb{Q}}_{\triangle}$.

- *a map* $\partial : W \to [0,1]$ *is a* probability distribution on W *iff* $\sum_{w\in W} \partial(w) = 1$.

Definition 3 (Probabilistic models)**.** *Given a finite set* $\mathbf{V} \subseteq \mathsf{Var}$*, a* probabilistic model for $\mathbf{V}$ *is a tuple* $\mathfrak{M} = \langle 2^{\mathbf{V}}, \mu\rangle$ *s.t.* $\mu : 2^{2^{\mathbf{V}}} \to [0,1]$ *is a* probability measure on $2^{2^{\mathbf{V}}}$*. For* $p{\in}\mathbf{V}$*,* $\phi{\in}\mathcal{L}_{\mathsf{CPL}}$ *s.t.* $\mathsf{Var}(\phi) \subseteq \mathbf{V}$*, and a probabilistic model* $\mathfrak{M}{=}\langle 2^{\mathbf{V}}, \mu\rangle$*, we define the* truth set of ϕ *(*$\|\phi\|_{\mathfrak{M}}$*) as follows:*
- $\|p\|_{\mathfrak{M}} = \{X \in 2^{\mathbf{V}} \mid p \in X\}$*;*
- $\|\neg\phi\| = 2^{2^{\mathbf{V}}} \setminus \|\phi\|_{\mathfrak{M}}$*;*
- $\|\phi \wedge \chi\|_{\mathfrak{M}} = \|\phi\|_{\mathfrak{M}} \cap \|\chi\|_{\mathfrak{M}}$.

Definition 4 (Semantics of FP)**.** *The* FP-interpretation *induced by* $\mathfrak{M}$ *is a map* $\mathcal{I}_{\mathfrak{M}} : \mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}} \to [0,1]$ *s.t.* $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(\phi)) = \mu(\|\phi\|_{\mathfrak{M}})$*. The interpretation is extended onto complex* $\mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$*-formulas using the conditions of Definition 1 (cf. Fig. 1).*

The notions of FP-satisfiability, validity, and entailment generalise those in Łukasiewicz logic (cf. Definition 1).

Definition 5. *Let* $\Gamma \cup \{\alpha, \delta\} \subseteq \mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$*,* $\mathbf{V} = \mathsf{Var}(\alpha)$*, and* $\mathbf{V}' = \mathsf{Var}[\Gamma \cup \{\delta\}]$*. We say that*
- α *is* FP-satisfiable *iff* $\mathcal{I}_{\mathfrak{M}}(\alpha) = 1$ *for some* $\mathfrak{M} = \langle 2^{\mathbf{V}}, \mu\rangle$*;*
- α *is* FP-valid *(*$\mathsf{FP}{\models}\alpha$*) iff* $\mathcal{I}_{\mathfrak{M}}(\alpha){=}1$ *in each* $\mathfrak{M}{=}\langle 2^{\mathbf{V}}, \mu\rangle$*;*
- Γ FP-entails δ *(*$\Gamma \models_{\mathsf{FP}} \delta$*) iff* $\mathcal{I}_{\mathfrak{M}'}(\delta) = 1$ *for all* $\mathfrak{M}' = \langle 2^{\mathbf{V}'}, \mu\rangle$ *s.t.* $\mathcal{I}_{\mathfrak{M}'}(\gamma) = 1$ *for all* $\gamma \in \Gamma$*;*
- Γ consistently FP-entails δ *(*$\Gamma \models^{\mathsf{cons}}_{\mathsf{FP}} \delta$*) iff* $\Gamma \models_{\mathsf{FP}} \delta$ *and* $\Gamma \not\models_{\mathsf{FP}} \bot$.

Convention 2. $\mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$*-formulas are called* outer formulas. *Formulas occurring in probabilistic atoms are* inner formulas *or* events. *We denote* $\mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$ *formulas with* α*,* β*, …,* λ*,* $\mathcal{L}_{\mathsf{CPL}}$ *and* $\mathcal{L}^{\mathbb{Q}}_{\text{Ł}}$ *formulas with* π*,* ϱ*, …,* ψ*. We denote* sets *of* $\mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$ *formulas with* Γ*,* Δ*, …,* Ξ*, and* sets *of* $\mathcal{L}_{\mathsf{CPL}}$ *or* $\mathcal{L}^{\mathbb{Q}}_{\text{Ł}}$ *formulas with* Υ*,* Φ*, and* Ψ*. For* $\Gamma{\cup}\{\alpha\}{\subseteq}\mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$*, we set* $\mathcal{E}(\alpha) = \{\phi \in \mathcal{L}_{\mathsf{CPL}} \mid \mathsf{Pr}(\phi)$ *occurs in* $\alpha\}$ *and* $\mathcal{E}[\Gamma]{=}\bigcup_{\gamma\in\Gamma} \mathcal{E}(\gamma)$.

For $\Phi \subseteq \mathcal{L}_{\mathsf{CPL}}$ *and a probabilistic model* $\mathfrak{M} = \langle 2^{\mathbf{V}}, \mu\rangle$*, we will write* $\mu(\|\Phi\|_{\mathfrak{M}})$ *to denote* $\mu(\|\bigwedge_{\phi\in\Phi} \phi\|_{\mathfrak{M}})$*.*

An $\mathcal{L}_{\mathsf{CPL}}$-term *(*$\mathcal{L}_{\mathsf{CPL}}$*-clause) is a formula of the form* $\bigwedge_{i=1}^{n} l_i$ *(*$\bigvee_{i=1}^{n} l_i$*) with* l*'s being literals. We write* $\alpha \leftrightarrow \beta$ *as a shorthand for* $(\alpha{\rightarrow}\beta){\odot}(\beta{\rightarrow}\alpha)$*,* $\mathsf{Pr}(\phi){\blacklozenge}\overline{\mathsf{c}}$ *and* $p{\blacklozenge}\overline{\mathsf{c}}$ *as shorthands for* $\neg(\mathsf{Pr}(\phi){\diamond}\overline{\mathsf{c}})$ *and* $\neg(p{\diamond}\overline{\mathsf{c}})$*, and* $p{\approx}\overline{\mathsf{c}}$ *and* $\mathsf{Pr}(\phi){\approx}\overline{\mathsf{c}}$ *as shorthands for* $(p{\geq}\overline{\mathsf{c}}) \odot (p{\leq}\overline{\mathsf{c}})$ *and* $(\mathsf{Pr}(\phi){\geq}\overline{\mathsf{c}}) \odot (\mathsf{Pr}(\phi){\leq}\overline{\mathsf{c}})$*.*

Let us remark on FP. First, $\mathcal{E}(\alpha)$ contains only the events explicitly mentioned in α (not their combinations or subformulas). E.g., if $\alpha = \mathsf{Pr}(p \wedge q) \to \mathsf{Pr}(r)$, then $\mathcal{E}(\alpha) = \{p \wedge q, r\}$ (i.e., $p, q \notin \mathcal{E}(\alpha)$). Thus, $\mathcal{E}(\alpha)$ is linear in the length of α. Second, $\triangle$ allows for a reduction between validity and satisfiability: α is FP-valid iff $\neg\triangle\alpha$ is FP-unsatisfiable; α is FP-satisfiable iff $\neg\triangle\alpha$ is not FP-valid. Third, we obtain the following statement by a straightforward expansion of (Flaminio 2007, Theorem 6).

Proposition 1. FP*-satisfiability is* NP*-complete, and* FP*-entailment is* coNP*-complete.*

3 Probabilistic Abduction in FP

Let us now present abduction in FP. We begin with the notion of abduction problems.

Definition 6. *An* FP-abduction problem *(*FP *AP) is a tuple* $\mathbb{P} = \langle \Gamma, \delta, \mathsf{H}, \mathcal{V}\rangle$ *s.t.* $\Gamma \cup \{\delta\} \subseteq \mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$*,* H *is a finite set of* $\mathcal{L}_{\mathsf{CPL}}$*-terms s.t.* $\mathsf{Var}[\mathsf{H}] \subseteq \mathsf{Var}[\Gamma \cup \{\delta\}]$*, and* $\mathcal{V} = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ *with* $n \in \mathbb{N}$*. We call* Γ theory*,* δ observation*, members of* H *and* $\mathcal{V}$ hypotheses *and* values.

Let us represent Example 1 as an FP-abduction problem.

Example 2. *Recall the context of Example 1. We use* c*,* r*,* s*, and* w *to denote cold, rainy, sunny, and warm days, respectively, and define* $\mathbb{P}_{\mathsf{rain}} = \langle \Gamma_{\mathsf{rain}}, \delta_{\mathsf{rain}}, \mathsf{H}_{\mathsf{rain}}, \mathcal{V}_{\mathsf{rain}}\rangle$*.*

$$\Gamma_{\mathsf{rain}} = \left\{\begin{array}{c}\mathsf{Pr}(s{\wedge}w){\leq}\overline{\tfrac{1}{2}}, \mathsf{Pr}(w){\leq}\overline{\tfrac{2}{3}}, \neg\triangle(\mathsf{Pr}(r){\rightarrow}\mathsf{Pr}(s)),\\ (\mathsf{Pr}(s \wedge w) \oplus \mathsf{Pr}(s \wedge w)) \leftrightarrow \mathsf{Pr}(c \vee r),\\ (\mathsf{Pr}(x) \oplus \mathsf{Pr}(x)) \leftrightarrow \mathsf{Pr}(w),\\ \mathsf{Pr}(c) \leftrightarrow (\mathsf{Pr}(w) \oplus \mathsf{Pr}(x))\end{array}\right\}$$

$\delta_{\mathsf{rain}} = \mathsf{Pr}(c{\wedge}r){\geq}\overline{\tfrac{1}{5}}$; $\mathsf{H}_{\mathsf{rain}} = \{s, w, s \wedge w\}$;

$\mathcal{V}_{\mathsf{rain}} = \{\frac{i}{100} \mid 0 \leq i {\leq} 100\}$

Let us briefly discuss the formalisation. By Definitions 1 and 4, $\mathcal{I}_{\mathfrak{M}}(\neg\triangle(\alpha \to \beta)) = 1$ *iff* $\mathcal{I}_{\mathfrak{M}}(\alpha) > \mathcal{I}_{\mathfrak{M}}(\beta)$*. Thus,* $\neg\triangle(\mathsf{Pr}(r) \to \mathsf{Pr}(s))$ *means 'the probability of a rainy day is greater than the probability of a sunny day'.* $\mathsf{Pr}(s{\wedge}w){\leq}\overline{\tfrac{1}{2}}$ *and* $(\mathsf{Pr}(s \wedge w) \oplus \mathsf{Pr}(s \wedge w)) \leftrightarrow \mathsf{Pr}(c \vee r)$ *imply* $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(s \wedge r)) = \mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(c \vee r)) \cdot \frac{1}{2}$*, i.e., rainy or cold days are two times likelier than days with warm and sunny weather.*

We also add an auxiliary variable x *to express* $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(w)) = 2 \cdot \mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(x))$ *via* $(\mathsf{Pr}(x) \oplus \mathsf{Pr}(x)) \leftrightarrow \mathsf{Pr}(w)$ *and* $\mathsf{Pr}(w) \leq \overline{\tfrac{2}{3}}$*. Now, 'a cold day is* 1.5 *times likelier than a warm one' can be formalised as* $\mathsf{Pr}(c) \leftrightarrow (\mathsf{Pr}(w){\oplus}\mathsf{Pr}(x))$*. To provide an informative explanation of the frequency of days with cold and rainy weather, we need hypotheses* s*,* w*, and* $s \wedge w$ *allowing us to express probabilities of sunny and warm days. Finally, we choose* $\mathcal{V}$ *expressing percentages.*

Now, to explain the observed frequency, we will need to assume probabilities of sunny and warm days. This means that the solution to an FP-abduction problem will not be a conjunction of hypotheses, but *a statement about probabilities of hypotheses*. Note that we need $s{\wedge}w$ because, in general, probability measures are not truth functional. In particular, $\mu(\|s{\wedge}w\|_{\mathfrak{M}})$ is not uniquely determined by $\mu(\|s\|_{\mathfrak{M}})$ and $\mu(\|w\|_{\mathfrak{M}})$. E.g., consider $\mathbf{V} = \{s, w\}$, $\mathfrak{M}_1 = \langle 2^{\mathbf{V}}, \mu_1\rangle$, and $\mathfrak{M}_2 = \langle 2^{\mathbf{V}}, \mu_2\rangle$ being such that $\mu_1(\{s, w\}) = \mu_1(\{s\}) = \mu_1(\{w\}) = \mu_1(\{\varnothing\}) = \frac{1}{4}$, $\mu_2(\{s, w\}) = \mu_2(\{\varnothing\}) = \frac{1}{2}$, and $\mu_2(\{s\}) = \mu_2(\{w\}) = 0$ (recall that the measure is defined on $2^{2^{\mathbf{V}}}$). One can see that $\mu_1(\|s\|_{\mathfrak{M}_1}){=}\mu_1(\|w\|_{\mathfrak{M}_1}){=}\mu_2(\|s\|_{\mathfrak{M}_2}){=}\mu_2(\|w\|_{\mathfrak{M}_2}) = \frac{1}{2}$, but $\mu_1(\|s \wedge w\|_{\mathfrak{M}_1}) = \frac{1}{4}$ and $\mu_2(\|s \wedge w\|_{\mathfrak{M}_2}) = \frac{1}{2}$.

In $\mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$, statements about probabilities of events can be formalised using formulas of the form $\mathsf{Pr}(\phi) \geq \overline{\mathsf{c}}$ ('probability of ϕ is at least c') or $\mathsf{Pr}(\phi) \rightarrow \mathsf{Pr}(\chi)$ ('probability of ϕ is at least as high as the probability of χ'). Note, however, that explanations are usually expressed as (sets of) facts (i.e., conjunctions of literals). As we are explaining observed frequencies of events, it makes sense to use probabilities of facts as solutions. To do that, we introduce *probabilistic interval literals* and *terms* that generalise (propositional) interval literals and terms proposed by Inoue and Kozhemiachenko (2025) for Łukasiewicz logic.

Definition 7 (Probabilistic interval literals and terms)**.**
- *A* probabilistic interval literal *(PIL) is a formula* $\mathsf{Pr}(\tau) \diamond \overline{\mathsf{c}}$ *with* τ *being an* $\mathcal{L}_{\mathsf{CPL}}$*-term.*
- *A* probabilistic interval term *(PIT) is a formula* $\bigodot_{i=1}^{n} \lambda_i$ *with* λ*'s being PILs.*

For a PIL $\mathsf{Pr}(\tau) \diamond \overline{\mathsf{d}}$*, we define its* set of permitted values*:*

$$\mathsf{V}_{\mathsf{Pr}}(\mathsf{Pr}(\tau) \diamond \overline{\mathsf{d}}) = \{\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(\tau)) \mid \mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(\tau) \diamond \overline{\mathsf{d}}) = 1\}$$

Given $\mathbf{V} = \{p_1, \ldots, p_m\}$ *and* $\mathcal{V} = \{0, \frac{1}{n-1}, \ldots, 1\}$*, we call a PIT* $\theta = \bigodot_{i=1}^{k} \mathsf{Pr}(\tau_i) {\leq} \overline{\mathsf{c}}_i \odot \bigodot_{i=1}^{k} \mathsf{Pr}(\tau_i) {\geq} \overline{\mathsf{c}}_i$ $\langle \mathbf{V}, \mathcal{V}\rangle$-complete *if* $\mathsf{Var}[\tau_i] = \mathbf{V}$ *and* $c_i \in \mathcal{V}$ *for every* $i \in \{1, \ldots, k\}$*, and* $\sum_{i=1}^{k} c_i = 1$*.*

Observe briefly that all $\langle \mathbf{V}, \mathcal{V}\rangle$-complete PITs are FP-satisfiable. Let us now define the notion of a solution to an FP-abduction problem.

Definition 8. *Let* $\mathbb{P} = \langle \Gamma, \delta, \mathsf{H}, \mathcal{V}\rangle$ *be an* FP *AP and let further* $\mathbf{V} = \mathsf{Var}[\mathbb{P}]$*.*
- *A* sufficient solution *is a PIT* $\eta = \bigodot_{i=1}^{n} \mathsf{Pr}(\tau_i) \diamond \overline{\mathsf{c}}_i$ *s.t.* $\mathcal{E}(\eta) \subseteq \mathsf{H}$*,* $\overline{\mathsf{c}}_i \in \mathcal{V}$ *for every* $\overline{\mathsf{c}}_i$*, and* $\Gamma, \eta \models^{\mathsf{cons}}_{\mathsf{FP}} \delta$*.*
- *A* full solution *is a* $\langle \mathbf{V}, \mathcal{V}\rangle$*-complete PIT* θ *s.t.* $\Gamma, \theta \models^{\mathsf{cons}}_{\mathsf{FP}} \delta$*.*

To guarantee the finiteness of the set of solutions, we restrict the set of *values* of probabilistic atoms containing hypotheses. Intuitively, this can be interpreted as an explanation with a given precision. Furthermore, full solutions correspond to *probability distributions* on $2^{\mathbf{V}}$. Hence, there is *only one* probabilistic model that satisfies θ. I.e., if θ is a full solution, there is no full solution θ' s.t. $\theta' \models_{\mathsf{FP}} \theta$ and $\theta \not\models_{\mathsf{FP}} \theta'$. Besides, a full solution θ is not necessarily a sufficient solution since in general, $\mathcal{E}(\theta) \not\subseteq \mathsf{H}$. Moreover, while the length of sufficient solutions is bounded by $\mathcal{E}[\mathbb{P}]$ (i.e., is linear in the length of $\mathbb{P}$), the length of full solutions is not. One could interpret *full solutions* as probabilistic counterparts of *most specific explanations (MSEs)* (Poole 1985; Stickel 1990; Sakama and Inoue 1995). The difference is that MSEs specify as many values of hypotheses as possible, and full solutions specify probabilities of *every event*.

Let us now solve the problem from Example 2.

Example 3. *We begin with a sufficient solution:*

$$\eta_{\mathsf{rain}} = \mathsf{Pr}(w) {\approx} \overline{0.4} \odot \mathsf{Pr}(s) {\approx} \overline{0.4} \odot \mathsf{Pr}(s \wedge w) {\approx} \overline{0.3}$$

By Definition 4, we obtain $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(c)) = \frac{3}{5}$*,* $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(r)) \in (\frac{2}{5}, \frac{3}{5}]$*, and* $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(c \vee r)) = \frac{3}{5}$ *from Example 2. Thus,* $\mathcal{I}(\mathsf{Pr}(c \wedge r)) > \frac{2}{5}$ *in every probabilistic model* $\mathfrak{M}$ *that satisfies* Γ_{rain} *and* η_{rain}*. Note that* η_{rain} *is not a full solution: we cannot infer probabilities of* all *events over* $\{c, r, s, w, x\}$*. But* η_{rain} can be extended to a full solution*, e.g., as follows:*

$$\begin{aligned}\theta_{\mathsf{rain}} = &\mathsf{Pr}(s \wedge w \wedge \neg x \wedge \neg c \wedge \neg r) {\approx} \overline{0.3} \odot\\ &\mathsf{Pr}(\neg s \wedge \neg w \wedge \neg x \wedge c \wedge r) {\approx} \overline{0.5} \odot\\ &\mathsf{Pr}(s \wedge \neg w \wedge x \wedge c \wedge \neg r) {\approx} \overline{0.1} \odot\\ &\mathsf{Pr}(\neg s \wedge w \wedge x \wedge \neg c \wedge r) {\approx} \overline{0.1}\end{aligned}$$

By Definitions 4, 5, and 8, θ_{rain} *is indeed a full solution to* $\mathbb{P}_{\mathsf{rain}}$*. Clearly,* $\Gamma_{\mathsf{rain}}, \theta_{\mathsf{rain}} \models^{\mathsf{cons}}_{\mathsf{FP}} \delta_{\mathsf{rain}}$*. Furthermore, all* $\tau \in \mathcal{E}(\theta_{\mathsf{rain}})$ *correspond to some* $X \subseteq \mathbf{V}$ *and the values of* τ*'s add up to* 1*. Thus* θ_{rain} *is* $\langle \mathbf{V}, \mathcal{V}\rangle$*-complete. Moreover,* $\theta_{\mathsf{rain}} \models_{\mathsf{FP}} \eta_{\mathsf{rain}}$*. Hence,* θ_{rain} *can be interpreted as an extension of* η_{rain}*.*

We finish the section with a short observation. From Definition 8, one can see that if a problem has a sufficient solution, then it has a full solution. On the other hand, the converse is not necessarily true. E.g., if we modify the set of hypotheses in Example 3 as follows: $\mathsf{H}^{-}_{\mathsf{rain}} = \{w\}$, $\mathbb{P}^{-} = \langle \Gamma_{\mathsf{rain}}, \delta_{\mathsf{rain}}, \mathsf{H}^{-}_{\mathsf{rain}}, \mathcal{V}_{\mathsf{rain}}\rangle$ will not have sufficient solutions. Still θ_{rain} remains a valid full solution.

4 Complexity of FP-Abduction

Before proceeding to establish the complexity of abduction in FP, let us first define some preliminary notions. Since most of our hardness results will be obtained by reductions from Ł-abduction, we recall here the definitions of Ł-abduction problems and their solutions from (Inoue and Kozhemiachenko 2025).

Definition 9 (Ł-abduction problems and their solutions)**.**
- *An* Ł-abduction problem *is a tuple* $\mathbb{P} = \langle \Phi, \chi, \mathsf{H}\rangle$ *with* $\Phi \cup \{\chi\} \subseteq \mathcal{L}^{\mathbb{Q}}_{Ł}$ *being finite and* H *a finite set of* interval literals *s.t.* $\mathsf{Var}[\mathsf{H}] \subseteq \mathsf{Var}[\Phi \cup \{\chi\}]$*. We call* Φ *a* theory*,* χ *an* observation*, and members of* H hypotheses*.*
- *An* Ł-solution *(simply called a solution) of* $\mathbb{P}$ *is an* interval term τ *composed of hypotheses s.t.* $\Phi, \tau \models^{\mathsf{cons}}_{Ł} \chi$*.*

We note that $Ł^{\mathbb{Q}}_{\triangle}$ is the fragment of FP with *atomic* events. The following statement is a straightforward generalisation of (Flaminio, Preto, and Ugolini 2023, Proposition 5.6) and (Flaminio and Ugolini 2024, Theorem 5.3).

Definition 10. *For* $\phi \in \mathcal{L}^{\mathbb{Q}}_{Ł}$*,* $\sharp \in \{\neg, \triangle\}$*, and* $\circledast {\in} \{\odot, \oplus, \rightarrow\}$*, we define the* probabilistic counterpart ϕ^{Pr} *as follows:*

$$p^{\mathsf{Pr}} = \mathsf{Pr}(p) \qquad (p \diamond \overline{\mathsf{c}})^{\mathsf{Pr}} = \mathsf{Pr}(p) \diamond \overline{\mathsf{c}}$$
$$(\sharp\chi)^{\mathsf{Pr}} = \sharp(\chi^{\mathsf{Pr}}) \qquad (\chi \circledast \psi)^{\mathsf{Pr}} = \chi^{\mathsf{Pr}} \circledast \psi^{\mathsf{Pr}}$$

For $\Phi \subseteq \mathcal{L}^{\mathbb{Q}}_{Ł}$*, we set* $\Phi^{\mathsf{Pr}} = \{\phi^{\mathsf{Pr}} \mid \phi \in \Phi\}$*.*

Proposition 2. *For all $\Phi \cup \{\chi\} \subseteq \mathcal{L}_{\text{Ł}}^{\mathbb{Q}}$, it holds that $\Phi \models_{\text{Ł}} \chi$ iff $\Phi^{\mathsf{Pr}} \models_{\mathsf{FP}} \chi^{\mathsf{Pr}}$.*

We are now ready to consider the complexity of abductive reasoning in FP. We will deal with two problems:

- *solution recognition* — given a problem $\mathbb{P}$ and a PIT ζ, determine whether ζ is a (sufficient or full) solution to $\mathbb{P}$;
- *solution existence* — given a problem $\mathbb{P}$, determine whether it has (sufficient or full) solutions.

4.1 Solution Recognition and Existence

We begin by determining the complexity of solution recognition. The next two statements show that while recognition of *sufficient* solutions is DP-complete (same as in classical or Łukasiewicz logic), recognising a *full* solution is polynomially tractable.

Theorem 1. *Given an* FP *AP $\mathbb{P} = \langle \Gamma, \delta, \mathsf{H}, \mathcal{V} \rangle$ and a PIT η, it is* DP*-complete to check if η is a sufficient solution to $\mathbb{P}$.*

Theorem 2. *Given an* FP *AP $\mathbb{P} = \langle \Gamma, \delta, \mathsf{H}, \mathcal{V} \rangle$ with $\mathbf{V} = \mathsf{Var}[\mathbb{P}]$ and a $\langle \mathbf{V}, \mathcal{V} \rangle$-complete PIT θ, it requires polynomial time (w.r.t. the total number of symbols in θ and $\mathbb{P}$ together) to check if θ is a full solution to $\mathbb{P}$.*

The reason for this discrepancy is that a *full* solution determines *exactly one* probabilistic model where Γ and δ are true. Moreover, a full solution θ corresponds to a probability distribution on $2^{\mathbf{V}}$. Hence, to calculate the value of $\mathsf{Pr}(\phi)$ using $\theta = \bigodot_{i=1}^{k} \mathsf{Pr}(\tau_i) \leq \overline{\mathsf{c}}_i \odot \bigodot_{i=1}^{k} \mathsf{Pr}(\tau_i) \geq \overline{\mathsf{c}}_i$, one needs to compute $\sum_{\tau_i \models_{\mathsf{CPL}} \phi} c_i$. Since τ_i's determine the values of all variables, $\tau_i \models_{\mathsf{CPL}} \phi$ can be decided in polynomial time, and this sum takes polynomial time to compute. On the other hand, *sufficient* solutions can leave probabilities of some events undefined and thus do not correspond to a unique probability distribution. This means that we need two NP-oracle calls: to verify $\Gamma, \eta \not\models_{\mathsf{FP}} \bot$ and $\Gamma, \eta \models_{\mathsf{FP}} \delta$. Since these calls are independent, DP-membership follows.

Lastly, *full* solutions are $\langle \mathbf{V}, \mathcal{V} \rangle$-complete PITs, i.e., they can be exponentially long w.r.t. $\mathsf{Var}[\mathbb{P}]$ (and $\mathcal{E}[\mathbb{P}]$). Thus, we take into account lengths of both θ and $\mathbb{P}$ in Theorem 2.

Let us now turn to the complexity of *solution existence*. As expected from the results in the previous section, determining the existence of *sufficient* solutions will be harder than determining the existence of *full* solutions. Namely, Σ_2^{P}-hardness of *sufficient* solution existence follows from Proposition 2 and Σ_2^{P}-hardness of solution existence for Ł-abduction problems (Inoue and Kozhemiachenko 2025, Theorem 5). Σ_2^{P}-membership can be obtained from Proposition 1 using a 'guess-and-verify' procedure.

Theorem 3. *Given an* FP *AP $\mathbb{P} = \langle \Gamma, \delta, \mathsf{H}, \mathcal{V} \rangle$, it is Σ_2^{P}-complete to determine whether it has sufficient solutions.*

Full solution existence for an FP AP $\mathbb{P} = \langle \Gamma, \delta, \mathsf{H}, \mathcal{V} \rangle$ can be seen as FP-satisfiability of $\bigodot_{\gamma \in \Gamma} \gamma \odot \delta$ provided that the model satisfying $\Gamma \cup \{\delta\}$ should evaluate all probabilistic atoms over $\mathcal{V}$, not $[0, 1]$. It is known from (Hájek and Tulipani 2001) that $\alpha \in \mathcal{L}_{\mathsf{Pr}}^{\mathbb{Q}}$ is FP-satisfiable iff it is satisfiable in a probabilistic model $\mathfrak{M} = \langle 2^{\mathsf{Var}(\alpha)}, \mu \rangle$ s.t. $\mu(\{w\}) > 0$ holds for at most $|\mathcal{E}(\alpha)| + 1$ states. Thus, we will consider full solutions $\theta = \bigodot_{i=1}^{n} \lambda_i$ s.t. $n \leq 2 \cdot |\mathcal{E}[\mathbb{P}]| + 2$ (recall that $\mathsf{Pr}(\pi) \approx \overline{\mathsf{c}}$ is a shorthand for $(\mathsf{Pr}(\pi) \geq \overline{\mathsf{c}}) \odot (\mathsf{Pr}(\pi) \leq \overline{\mathsf{c}})$). We will call such solutions *concise* to differentiate them from full solutions of arbitrary length.

Definition 11. *Given an* FP *AP $\mathbb{P}$ and its full solution $\theta = \bigodot_{i=1}^{n} \lambda_i$, we say that θ is* concise *iff $n \leq 2 \cdot |\mathcal{E}[\mathbb{P}]| + 2$.*

NP-membership of concise full solution existence for FP APs now follows from Theorem 2. For NP-hardness, we can provide a polynomial reduction from CPL-satisfiability.

Theorem 4. *Given an* FP *AP $\mathbb{P} = \langle \Gamma, \delta, \mathsf{H}, \mathcal{V} \rangle$, it is* NP*-complete to determine whether it has concise full solutions.*

4.2 Prioritisation of Solutions

Up to now, we have been considering *arbitrary* full and sufficient solutions. Note, however, that usually we have some preference relation between different solutions. In particular, a usual requirement is that a solution should be as simple as possible. One of the most straightforward choices to formalise simplicity in logic-based abduction is to utilise entailment. If ζ_1 and ζ_2 are two solutions s.t. $\zeta_1 \models \zeta_2$ and $\zeta_2 \not\models \zeta_1$, then ζ_2 is simpler because it makes fewer assumptions and thus ζ_1 is more preferable than ζ_1. In classical logic, this is usually formulated in terms of $\subseteq$*-minimality* (Eiter and Gottlob 1995) (given $\mathcal{L}_{\mathsf{CPL}}$-terms σ and τ, $\sigma \models_{\mathsf{CPL}} \tau$ iff all literals in τ belong to σ) and interpreted as preferring solutions without *redundant hypotheses*: if both $\tau \wedge h$ and τ are solutions, then h is redundant (Peng and Reggia 2012).

An advantage of *entailment-minimal* solutions is that they can be easily generalised to abduction in non-classical logics (cf., e.g., works by Bienvenu et al. (2024; 2026) and Inoue and Kozhemiachenko (2025)). In the framework of FP-abduction, we can employ $\models_{\mathsf{FP}}$-minimality to prioritise *sufficient* solutions. On the other hand, different *full* solutions are either equivalent or incompatible. Hence, we need another approach to prioritisation. Since full solutions correspond to probability distributions on $2^{\mathbf{V}}$, it makes sense to employ the *principle of maximum entropy* by Janes (1957a; 1957b) (cf. a discussion by Williamson (2010)) to establish a preference relation between them. The principle states that the probability distributions that best represent our knowledge are those with the highest entropy. Thus, given an FP AP, we prefer full solutions with higher entropy. Entropy was used to prioritise probabilistic explanations by Valkovsky et al. (1999) in the context of constraint programming, and Dubois et al. (2008). There, however, the explanation was not a probability distribution itself but an event.

We will thus consider two classes of preferred solutions to FP APs: *entropy-maximal* and $\models_{\mathsf{FP}}$*-minimal* (entailment-minimal). We begin with entropy-maximal full solutions.

Definition 12. *Let $\mathbb{P} = \langle \Gamma, \delta, \mathsf{H}, \mathcal{V} \rangle$ be an* FP *AP with $\mathsf{Var}[\mathbb{P}] = \{p_1, \ldots, p_m\}$ and $\theta = \bigodot_{i=1}^{n} \mathsf{Pr}(\tau_i) \approx \overline{\mathsf{c}}_i$ its full solution s.t. $c_i > 0$ for all $i \in \{1, \ldots, n\}$. The* entropy of θ *($\mathbb{H}(\theta)$) is defined as follows: $\mathbb{H}(\theta) = -\sum_{i=1}^{n} c_i \cdot \log_2 c_i$.*

Definition 13. *Let $\mathbb{P} = \langle \Gamma, \delta, \mathsf{H}, \mathcal{V} \rangle$ be an* FP *AP, that contains k unique probabilistic atoms. We say that θ is a* concise entropy-maximal *(CEM) solution to $\mathbb{P}$ iff θ is a full solution, and there is no full solution θ' containing at most $2 \cdot k + 2$ PITs s.t. $\mathbb{H}(\theta) < \mathbb{H}(\theta')$.*

Note that CEM solutions have bounded length. This is done for the following reason. Intuitively, the higher the entropy of a full solution θ, the closer it is to a uniform probability distribution. In other words, the higher $\mathbb{H}(\theta)$ is, *the more θ contains terms $\mathsf{Pr}(\tau)\approx\overline{\mathsf{c}}$ with $c \neq 0$.* Thus, solutions with higher entropy could be exponentially long w.r.t. the size of the problem. For example, if ϕ is a tautology and $\mathsf{Var}(\phi) = \mathbf{V} = \{p_1, \ldots, p_n\}$, then the (non-concise) entropy-maximal explanation of $\mathsf{Pr}(\phi)\geq\overline{1}$ is the uniform probability distribution on $2^{\mathbf{V}}$ expressed as $\bigodot_{i=1}^{2^n}\mathsf{Pr}(\tau_i)\approx\overline{\frac{1}{2^n}}$. To circumvent this problem, we will only consider 'concise' full solutions (recall Definition 11).

Example 4. *Consider θ_{rain}. It contains 8 PITs (note that $\mathsf{Pr}(\tau)\approx\overline{\mathsf{c}}$'s are shorthands for $\mathsf{Pr}(\tau)\leq\overline{\mathsf{c}} \odot \mathsf{Pr}(\tau)\geq\overline{\mathsf{c}}$) and is concise. By Definition 12, $\mathbb{H}(\theta_{\mathsf{rain}}) \approx 1.685$. Recall from Example 2 that $\mathbb{P}_{\mathsf{rain}}$ contains 8 unique probabilistic atoms. Thus, we can consider full solutions to $\mathbb{P}_{\mathsf{rain}}$ that contain up to 18 terms (i.e., up to 9 events).*

Now, to produce a probability distribution with a higher entropy than θ_{rain}, we 'spread' the probability of $\neg s \wedge \neg w \wedge \neg x \wedge c \wedge r$ to other events. Consider:

$$\begin{aligned}\theta^{\Uparrow}_{\mathsf{rain}} =&\mathsf{Pr}(s \wedge w \wedge \neg x \wedge \neg c \wedge \neg r)\approx\overline{0.3}\odot\\ &\mathsf{Pr}(\neg s \wedge \neg w \wedge \neg x \wedge c \wedge r)\approx\overline{0.35}\odot\\ &\mathsf{Pr}(s \wedge \neg w \wedge \neg x \wedge c \wedge r)\approx\overline{0.15}\odot\\ &\mathsf{Pr}(s \wedge \neg w \wedge x \wedge c \wedge \neg r)\approx\overline{0.1}\odot\\ &\mathsf{Pr}(\neg s \wedge w \wedge x \wedge \neg c \wedge r)\approx\overline{0.1}\end{aligned}$$

Using Definition 12, we obtain that $\mathbb{H}(\theta^{\Uparrow}_{\mathsf{rain}}) \approx 2.126$, i.e., $\mathbb{H}(\theta^{\Uparrow}_{\mathsf{rain}}) > \mathbb{H}(\theta_{\mathsf{rain}})$. Thus, $\eta^{\Uparrow}_{\mathsf{rain}}$ is a more preferable full solution to $\mathbb{P}_{\mathsf{rain}}$ than θ_{rain}.

We can now show that recognising concise entropy-maximal solutions is coNP-complete. Indeed, we can solve the complementary problem in NP time: given a $\langle\mathcal{V}, \mathbf{V}\rangle$-complete PIT θ, we guess another (polynomially large) solution candidate θ' and check whether $\mathbb{H}(\theta) < \mathbb{H}(\theta')$ in polynomial time. If the check succeeds, *θ is not a CEM solution*. To show coNP-hardness, we can construct a polynomial-time reduction from CPL validity to the recognition of concise entropy-maximal solutions.

Theorem 5. *Given an* FP *AP $\mathbb{P} = \langle\Gamma, \delta, \mathsf{H}, \mathcal{V}\rangle$ with $\mathbf{V} = \mathsf{Var}[\mathbb{P}]$ and a $\langle\mathbf{V}, \mathcal{V}\rangle$-complete PIT θ, it is* coNP*-complete to determine whether θ is a CEM solution.*

Let us now consider entailment-minimal solutions. We adapt the analogous notion introduced for Ł-abduction problems in (Inoue and Kozhemiachenko 2025).

Definition 14. *Let $\mathbb{P} = \langle\Gamma, \delta, \mathsf{H}, \mathcal{V}\rangle$ be an* FP *AP. We say that η is an* $\models_{\mathsf{FP}}$-minimal *(entailment-minimal) solution iff it is a sufficient solution, and there is no other sufficient solution ζ s.t. $\eta \models_{\mathsf{FP}} \zeta$ but $\zeta \not\models_{\mathsf{FP}} \eta$.*

To illustrate the notion of $\models_{\mathsf{FP}}$-minimal solutions, we come back to Examples 2 and 3. We show how to construct an $\models_{\mathsf{FP}}$-minimal solution from a given sufficient solution.

Example 5. *Recall first a sufficient solution η_{rain} to $\mathbb{P}_{\mathsf{rain}}$. One can easily check that it is* not *$\models_{\mathsf{FP}}$-minimal. We can, however, transform it into an entailment-minimal solution by increasing intervals of permitted values for each literal.*

In fact, it turns out that $\mathsf{Pr}(s \wedge w)\approx\overline{0.3}$ is redundant and η_0=$\mathsf{Pr}(w)\geq\overline{0.4}\odot\mathsf{Pr}(s)\approx\overline{0.4}$ is a solution. Clearly, Γ_{rain},$\eta_0 \not\models_{\mathsf{FP}} \bot$. Moreover, by Definition 4, we obtain from Γ_{rain} (cf. Example 2) that $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(w))\geq 0.4$ entails $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(c))\geq 0.6$ and $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(s))$=0.4 entails $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(r))$>0.4. Additionally, $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(s \wedge w)) \leq 0.4$ because $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(s))$=0.4, and hence $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(c \vee r)) \in [0.6; 0.8]$. Thus, $\mathcal{I}_{\mathfrak{M}}(\mathsf{Pr}(c \wedge r))$>0.2, i.e., $\Gamma_{\mathsf{rain}}, \eta_0 \models_{\mathsf{FP}} \mathsf{Pr}(c \wedge r)\geq\overline{\frac{1}{5}}$, as required. Now observe that neither of the following PITs solve $\mathbb{P}_{\mathsf{rain}}$:

$$\begin{aligned}\eta_1 &= \mathsf{Pr}(w)\geq\overline{0.39} \odot \mathsf{Pr}(s)\approx\overline{0.4}\\ \eta_2 &= \mathsf{Pr}(w)\geq\overline{0.4} \odot \mathsf{Pr}(s)\geq\overline{0.4} \odot \mathsf{Pr}(s)\leq\overline{0.41}\\ \eta_3 &= \mathsf{Pr}(w)\geq\overline{0.4} \odot \mathsf{Pr}(s)\geq\overline{0.39} \odot \mathsf{Pr}(s)\leq\overline{0.4}\end{aligned}$$

Thus, we cannot increment the sets of permitted values of PILs in η_0. Note, however, that $\mathsf{Pr}(s)\approx\overline{0.4}$ is a shorthand for $\mathsf{Pr}(s)\geq\overline{0.4} \odot \mathsf{Pr}(s)\leq\overline{0.4}$. Hence, we can weaken $\mathsf{Pr}(s)\leq\overline{0.4}$ to $\mathsf{Pr}(s \wedge w)\leq\overline{0.4}$. One can check that the resulting PIT

$$\eta^{\min}_{\mathsf{rain}} = \mathsf{Pr}(w)\geq\overline{0.4} \odot \mathsf{Pr}(s)\geq\overline{0.4} \odot \mathsf{Pr}(s \wedge w)\leq\overline{0.4}$$

is a sufficient solution to $\mathbb{P}_{\mathsf{rain}}$. Moreover, it is indeed an $\models_{\mathsf{FP}}$-minimal solution since we can neither increase intervals nor weaken the remaining events.

Now notice that, in contrast to classical and Łukasiewicz logics, the satisfiability and entailment of PITs are not polynomially decidable. It is easy to see that FP-satisfiability of arbitrary probabilistic atoms is NP-complete. By Definition 4, ϕ is CPL-satisfiable iff $\mathsf{Pr}(\phi)$ is FP-satisfiable. Even the FP-satisfiability of PITs is NP-complete (and entailment is coNP-complete) because we can encode DNF-non-validity in CPL using PITs.

Proposition 3.

1. *Given a PIT $\zeta = \bigodot_{i=1}^{n}(\mathsf{Pr}(\tau_i) \diamond \overline{\mathsf{c}}_i)$, it is* NP*-complete to check whether ζ is* FP*-satisfiable.*
2. *Given two PITs ζ_1 and ζ_2 it is* coNP*-complete to check whether $\zeta_1 \models_{\mathsf{FP}} \zeta_2$.*

Still, even though entailment between PITs is not polynomially tractable, recognising $\models_{\mathsf{FP}}$-minimal solutions is DP-complete. For DP-hardness, we construct a polynomial reduction from the recognition of entailment-minimal solutions of Ł-*abduction problems*, which is DP-complete (Inoue and Kozhemiachenko 2025, Theorem 3) to the recognition of $\models_{\mathsf{FP}}$-minimal solutions. For DP-membership, we show that given a PIT η and an FP AP $\mathbb{P}$, there are only polynomially many 'next weakest' terms that we need to check. If none of them is a sufficient solution to $\mathbb{P}$, but η is, then η is a $\models_{\mathsf{FP}}$-minimal solution. These 'next weakest' PITs can be built as shown in Example 5 where we transformed η_{rain} into $\eta^{\min}_{\mathsf{rain}}$.

Theorem 6. *Given an* FP *AP $\mathbb{P} = \langle\Gamma, \delta, \mathsf{H}, \mathcal{V}\rangle$ and a PIT η, it is* DP*-complete to check whether it is an $\models_{\mathsf{FP}}$-minimal solution to $\mathbb{P}$.*

5 Abduction in Fragments of FP

As we saw in the previous section, abduction in FP is intractable in the general case (especially when it comes to *sufficient solutions*). Thus, it makes sense to consider fragments of FP where abductive reasoning is polynomially decidable or at least simpler than in the general case.

In this section, we will consider two fragments of FP that generalise the *simple clause fragment* of Łukasiewicz logic (Bofill et al. 2019) and the *cover-free fragment* of $\text{Ł}^{\mathbb{Q}}_{\triangle}$ (Inoue and Kozhemiachenko 2025). Note, however, that we need to restrict not only the shape of outer formulas, but also the shape of inner formulas. Indeed, as we noted in Section 3, satisfiability of a single formula $\mathsf{Pr}(\phi)$ or $\mathsf{Pr}(\phi) \diamond \overline{\mathsf{c}}$ is NP-hard if ϕ is arbitrary. Moreover, the events should be syntactically simple, yet expressive enough to reason about probabilities of non-trivial events. The next definition presents such a restriction.

Definition 15 (Chained inner-positive theories)**.**
- $\Phi \subseteq \mathcal{L}_{\mathsf{CPL}}$ *is* chained positive *(CP) iff*
  - $\pi = \bigwedge_{i=1}^{m} p_i$ *or* $\pi = \bigvee_{i=1}^{n} q_i$ *for every* $\pi \in \Phi$*;*
  - *for every* $\pi, \sigma \in \Phi$*, it holds that either* $\pi \models_{\mathsf{CPL}} \sigma$*, or* $\sigma \models_{\mathsf{CPL}} \pi$*, or* $\mathsf{Var}(\pi) \cap \mathsf{Var}(\sigma) = \varnothing$*.*
- $\Gamma \subseteq \mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$ *is* chained inner-positive *(CIP) if* $\mathcal{E}[\Gamma]$ *is chained positive.*

Example 6. *Recall the theory* Γ_{rain} *from Example 2. Observe that it is* not CIP. *Indeed,* $\mathcal{E}[\Gamma]$ *contains* $c \vee r$*,* c*, and* r*. Clearly,* $c \models_{\mathsf{CPL}} c \vee r$*,* $r \models_{\mathsf{CPL}} c \vee r$*, but* $c \not\models_{\mathsf{CPL}} r$ *and* $r \not\models_{\mathsf{CPL}} c$*. However* $\Gamma'_{\mathsf{rain}} \cup \{\delta_{\mathsf{rain}}\}$ *is CIP for* $\Gamma'_{\mathsf{rain}} = \Gamma_{\mathsf{rain}} \setminus \{\neg\triangle(\mathsf{Pr}(r) \rightarrow \mathsf{Pr}(s))\}$*. Indeed, we have two 'chains' of events in* Γ'_{rain}*:* $s \wedge w \models_{\mathsf{CPL}} w$ *and* $c \wedge r \models_{\mathsf{CPL}} c \models_{\mathsf{CPL}} c \vee r$*. Note, however, that* η_{rain} *(cf. Example 3) is* not *a solution to* $\mathbb{P}'_{\mathsf{rain}} = \langle \Gamma'_{\mathsf{rain}}, \delta_{\mathsf{rain}}, \mathsf{H}_{\mathsf{rain}}, \mathcal{V}_{\mathsf{rain}} \rangle$*. In fact, since* $r \notin \mathcal{E}[\Gamma'_{\mathsf{rain}}]$*,* $\mathbb{P}'_{\mathsf{rain}}$ has no sufficient solutions. *Still,* θ_{rain} *is a full solution to* $\mathbb{P}'_{\mathsf{rain}}$*.*

An important property of CIP theories is that they can be translated into $\mathcal{L}^{\mathbb{Q}}_{\text{Ł}}$ in polynomial time only. In fact, given a CIP theory Γ, it suffices to construct an $\mathcal{L}^{\mathbb{Q}}_{\text{Ł}}$ theory Ψ_Γ that encodes (classical) entailment between events in Γ. In this case, an Ł-valuation satisfying Ψ_Γ will induce a map coherent with a probability measure on $2^{2^{\mathsf{Var}[\Gamma]}}$.

The next definition is adapted from (de Finetti 1974).

Definition 16. *Let* $\mathbb{E} \subseteq \mathcal{L}_{\mathsf{CPL}}$ *be finite,* $\mathfrak{p} : \mathbb{E} \rightarrow [0, 1]$*, and* $\mathfrak{M} = \langle 2^{\mathsf{Var}[\mathbb{E}]}, \mu \rangle$ *be a probabilistic model. We say that* μ is coherent with $\mathfrak{p}$ *iff* $\mu(\|\phi\|_{\mathfrak{M}}) = \mathfrak{p}(\phi)$ *for all* $\phi \in \mathbb{E}$*. The map* $\mathfrak{p} : \mathbb{E} \rightarrow [0, 1]$ *is* coherent *if there exists a probabilistic model* $\mathfrak{M} = \langle 2^{\mathsf{Var}[\mathbb{E}]}, \mu \rangle$ *such that* μ *is coherent with* $\mathfrak{p}$*.*

Proposition 4. *For every finite CIP theory* $\Gamma \subseteq \mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$*, there is an* $\mathcal{L}^{\mathbb{Q}}_{\text{Ł}}$*-theory* Ψ_Γ *s.t.*
1. Ψ_Γ *is* Ł*-satisfiable iff* Γ *is* FP*-satisfiable;*
2. Ψ_Γ *is constructible in polynomial time w.r.t. the size of* Γ*.*

5.1 Simple Clause Fragment

Let us now consider the probabilistic simple clause fragment of FP. Recall from (Bofill et al. 2019) that (propositional) simple clauses are formulas of the form $\bigoplus_{i=1}^{n} l_i$ with l_i's being propositional variables and their negations. As one can check from Definition 1, $\bigoplus_{i=1}^{n} l_i$ can be equivalently represented as $\bigodot_{i=1}^{k} \neg l_i \rightarrow \bigoplus_{j=n-k}^{n} l_j$ for every $k \leq n$. Thus, simple clauses can be used to express fuzzy (disjunctive) rules. Moreover, in contrast to classical disjunctive clauses, Ł-satisfiability of sets of simple clauses is *polynomially decidable*, whence, simple-clause abduction turns out to be NP-complete (Inoue and Kozhemiachenko 2025), while classical abduction in theories represented as sets of disjunctive clauses is Σ^{P}_2-hard in general (Eiter and Gottlob 1995).

Thus, in the context of *probabilistic* abduction, it makes sense to consider an FP-counterpart of simple clauses. The following definitions present *probabilistic* simple clauses (PSCs) and PSC abduction problems.

Definition 17 (Probabilistic simple clauses)**.**
- *A* probabilistic simple clause *(PSC) is a formula of the form* $\bigoplus_{i=1}^{m} \mathsf{Pr}(\sigma_i) \oplus \bigoplus_{i=1}^{n} \neg\mathsf{Pr}(\tau_i)$ *with* σ *and* τ *being conjunctions or disjunctions of variables.*
- *A* PSC theory *is a CIP theory* Γ *s.t. every* $\gamma \in \Gamma$ *is either a PSC or has a form* $\mathsf{Pr}(\tau) \diamond \overline{\mathsf{c}}$*.*

Let us illustrate the expressivity of PSC theories with an example. Note, in particular, that we can formalise some statements about probabilities of events containing $\neg$. Moreover, formulas $\mathsf{Pr}(\tau) \diamond \overline{\mathsf{c}}$ *do not need to be PILs.*

Example 7. *Consider the following statements:*

α**:** *the probability of precipitation (rain, snow, or hailstorm) tomorrow is* 70%*;*

β**:** *the probability of the rain tomorrow is at least as high as the probability of neither snow nor rain happening tomorrow.*

We set $\alpha = \alpha_1 \odot \alpha_2$ *with* $\alpha_1 = \mathsf{Pr}(r \vee s \vee h) \geq \overline{0.7}$ *and* $\alpha_2 = \mathsf{Pr}(r \vee s \vee h) \leq \overline{0.7}$ *and* $\beta = \mathsf{Pr}(\neg r \wedge \neg s) \rightarrow \mathsf{Pr}(r)$*. In its current form,* $\{\alpha, \beta\}$ *is not a PSC theory because* $\mathcal{E}[\{\alpha, \beta\}]$ *is not chained positive. Note, however, that by Definition 4,* $\mathsf{Pr}(\neg\phi)$ *and* $\neg\mathsf{Pr}(\phi)$ *are equivalent. Thus,* β *is equivalent to* $\beta' = \mathsf{Pr}(r \vee s) \oplus \mathsf{Pr}(r)$ *and* $\{\alpha_1, \alpha_2, \beta'\}$ *is a PSC theory.*

Definition 18. *A* PSC abduction problem *is an* FP *AP* $\mathbb{P} = \langle \Gamma, \delta, \mathsf{H}, \mathcal{V} \rangle$ *s.t.* $\Gamma \cup \{\delta\}$ *is a PSC theory and* $\mathcal{E}[\mathbb{P}]$ *is CP.*

Note that the observation of a PSC AP can have various types, depending on the constraints we want to explain using the theory. Furthermore, we need to list $\mathcal{L}_{\mathsf{CPL}}$-terms in H and cannot allow arbitrary conjunctions of hypotheses: while $\{a, b\}$ is CP, $\{a, b, a \wedge b\}$ is not. Defining H as a set of terms (not literals) and demanding that $\mathcal{E}[\mathbb{P}]$ be chained positive ensures that given a *sufficient* solution η for $\mathbb{P}$, $\Gamma \cup \{\eta, \delta\}$ is a CIP theory. We can use this together with Propositions 2 and 4 to show the following statement.

Theorem 7. *Given a PSC AP* $\mathbb{P} = \langle \Gamma, \delta, \mathsf{H}, \mathcal{V} \rangle$ *and a PIT* η*, it is* P*-complete under logspace reductions to check if* η *is an (*FP*-minimal) sufficient solution of* $\mathbb{P}$*.*

As expected from the previous statement, determining the existence of sufficient solutions is NP-complete. Somewhat surprisingly, the existence of concise *full solutions* is also NP-complete as in the general case. This is because we can reduce classical satisfiability of CNFs to the existence of full solutions to PSC APs.

Theorem 8. *Given a PSC AP $\mathbb{P} = \langle \Gamma, \delta, \mathsf{H}, \mathcal{V} \rangle$, it is:*
1. NP*-complete to check if it has* sufficient *solutions;*
2. NP*-complete to check if it has* concise full *solutions.*

5.2 Interval Clause Fragment

Let us now consider the probabilistic interval clause fragment of FP. Propositional interval clauses (i.e., formulas of the form $\pi_1 \oplus \ldots \oplus \pi_n$ with π's being *interval literals*) were introduced by Inoue and Kozhemiachenko (2025) to formalise statements such as 'if the value of p is at least 0.5, then the value of q is at most 0.6' — $p \geq \overline{0.5} \to q \leq \overline{0.6}$ or, equivalently $p < \overline{0.5} \oplus q \leq \overline{0.6}$. In this section, we introduce the FP-counterparts of interval clauses and investigate the complexity of abductive reasoning with *probabilistic interval clauses*.

Definition 19. *A* probabilistic interval clause *(PIC) is a formula $\bigoplus_{i=1}^{n} \lambda_i$ with every λ having the form $\mathsf{Pr}(\phi) \diamond \overline{\mathsf{c}}$ with $\phi \in \mathcal{L}_{\mathsf{CPL}}$. A* PIC theory *is a set of PICs.*

It is known from (Inoue and Kozhemiachenko 2025) that the complexity of abductive reasoning with interval-clause theories is the same as with arbitrary $\mathcal{L}_{\text{Ł}}^{\mathbb{Q}}$-theories. Thus, by Proposition 2, the same would hold w.r.t. abductive reasoning with PIC theories. Still, the interval clause fragment of $\text{Ł}_{\triangle}^{\mathbb{Q}}$ can be restricted to the so-called *cover-free* fragment, over which solution recognition was polynomial and solution existence NP-complete.

Definition 20. *Let $\Phi \subseteq \mathcal{L}_{\text{Ł}}^{\mathbb{Q}}$ be a set of interval clauses represented as $\bigodot_{i=1}^{m}(p_i \diamond \overline{\mathsf{c}}_i) \to (q \diamond \overline{\mathsf{d}})$ and $\bigodot_{i=1}^{m}(p_i \diamond \overline{\mathsf{c}}_i) \to \bot$. Φ is* cover-free *(CF) if no pair of interval literals π and π' over the same variable occurs in Φ s.t. $\text{Ł} \models \pi \oplus \pi'$.*

By Proposition 2, it would follow from (Inoue and Kozhemiachenko 2025, Theorem 16) that solution existence is NP-hard for FP APs $\mathbb{P} = \langle \Phi^{\mathsf{Pr}}, \lambda^{\mathsf{Pr}}, \mathsf{H}, \mathcal{V} \rangle$ with Φ being a CF set of *interval clauses* and λ an interval literal. It is thus instructive to see whether we can find a subclass of CF theories where solution existence can be determined in polynomial time. The definition below introduces *short* probabilistic cover-free (SPCF) theories. We use this name to indicate that $n = 1$ in the implications of the form $\bigodot_{i=1}^{n} \lambda_i \to \lambda$ (i.e., clauses with a non-empty head must be *binary*).

Definition 21. *An SPCF theory is a CIP theory Γ s.t.*
- *Γ is a set of PICs represented as $\bigodot_{i=1}^{n} \mathsf{Pr}(\pi_i) \diamond \overline{\mathsf{c}}_i \to \bot$ or $\mathsf{Pr}(\pi) \diamond \overline{\mathsf{c}} \to \mathsf{Pr}(\varrho) \diamond \overline{\mathsf{d}}$;*
- *for every pair of probabilistic atoms λ and λ' occurring in Γ, it holds that $\mathsf{FP} \not\models \lambda \oplus \lambda'$.*

Consider now the following example.

Example 8. *Assume that Γ contains the following relations between the frequencies of different events. If black ice (b) or thick fog (f) occurs at least 40% of the time, then traffic accidents (a) will happen at least every other day. If traffic accidents or snowdrifts (s) happen at least twice a week, then major traffic jams (j) will occur at least once a week.*

We can formalise Γ as follows: $\mathsf{Pr}(b \vee f) \geq \overline{\tfrac{2}{5}} \to \mathsf{Pr}(a) \geq \overline{\tfrac{1}{2}}$, $\mathsf{Pr}(a \vee s) \geq \overline{\tfrac{2}{7}} \to \mathsf{Pr}(j) \geq \overline{\tfrac{1}{7}}$. Now, to explain why traffic jams happen every week, we can, for example, assume that thick fog occurred at least 40% of the time: $\eta = \mathsf{Pr}(f) \geq \overline{\tfrac{2}{5}}$. Note that $\Gamma, \eta \models^{\mathsf{cons}}_{\mathsf{FP}} \mathsf{Pr}(j) \geq \overline{\tfrac{1}{7}}$ and $\Gamma \cup \{\eta, \mathsf{Pr}(j) \geq \overline{\tfrac{1}{7}}\}$ is SPCF.

The next definition introduces the class of FP abduction problems whose theories are SPCF. Note that we demand that the set of *all events* occurring in the problem be CP.

Definition 22. *An* SPCF abduction problem *is an FP AP $\mathbb{P} = \langle \Gamma, \mathsf{Pr}(\chi) \diamond \overline{\mathsf{d}}, \mathsf{H}, \mathcal{V} \rangle$ s.t. $\Gamma \cup \{\mathsf{Pr}(\chi) \diamond \overline{\mathsf{d}}\}$ is an SPCF theory and $\mathcal{E}[\mathbb{P}]$ is chained positive.*

The polynomial decidability of the sufficient solution existence follows from the fact that the problem of determining the existence of solutions to SPCF APs is reducible to determining the existence of solutions to classical abduction problems whose theories contain so-called $\mathsf{IHS\text{-}B}^{-}$-clauses (literals — p and $\neg q$ — and implications of the form $p \to q$ and $\bigwedge_{i=1}^{n} p_i \to \bot$) proposed by Khanna et al. (2002). As determining the existence of solutions to $\mathsf{IHS\text{-}B}^{-}$ abduction problems is *polynomially tractable* (Creignou and Zanuttini 2006, Proposition 12), the existence of sufficient solutions to SPCF APs can also be decided in polynomial time. On the other hand, determining the existence of *full* solutions to SPCF APs is NP-hard because we can construct a reduction from CPL-satisfiability of CNFs. Thus, SPCF theories manifest an unusual behaviour: it is *harder* to check whether a full solution exists than whether a sufficient solution exists.

The intuitive reason behind this is that solutions to CPL abduction problems with $\mathsf{IHS\text{-}B}^{-}$ theories and observations in the form of a single variable can be w.l.o.g. assumed to contain one literal only. Thus (after translating the original problem into its CPL counterpart), it suffices to check hypotheses one by one. Moreover, for any candidate η for a sufficient solution to $\mathbb{P} = \langle \Gamma, \mathsf{Pr}(\chi) \diamond \overline{\mathsf{d}}, \mathsf{H}, \mathcal{V} \rangle$, $\Gamma \cup \{\eta, \mathsf{Pr}(\chi) \diamond \overline{\mathsf{d}}\}$ is a CIP theory. However, to check if a full solution exists, one needs to verify whether there is a suitable $\langle \mathbf{V}, \mathcal{V} \rangle$-complete term, which is independent from H and is not necessarily CP.

Theorem 9. *For a SPCF AP $\mathbb{P} = \langle \Gamma, \mathsf{Pr}(\chi) \diamond \overline{\mathsf{d}}, \mathsf{H}, \mathcal{V} \rangle$, it is*
1. *decidable in polynomial time if it has sufficient solutions;*
2. NP*-complete to check if it has concise full solutions.*

6 Modelling Most Likely Explanations in FP

Until now, we have interpreted probabilistic explanations as statements *about probabilities* that explain the observed frequency of events, and showed how one can use FP to model such explanations. A more traditional perspective on probabilistic explanations is to consider *the most likely explanations* of events. More formally, one can represent this task as follows: given a theory Φ, an observation χ, and probability assignments of (some) events, one has to come up with an explanation τ s.t. there is no other explanation σ whose probability is higher than that of τ. In this section, we will demonstrate how to model this reasoning task in FP.

Definition 23. *A* probabilistic abduction problem *(PrAP) is a tuple $\mathbb{P}_{\mathfrak{p}} = \langle \Phi, \chi, \mathsf{H}, \mathbb{E}, \mathfrak{p} \rangle$ s.t.*
- *$\Phi \cup \{\chi\} \cup \mathbb{E} \subseteq \mathcal{L}_{\mathsf{CPL}}$ is finite and $\mathsf{Var}[\mathbb{E}] \subseteq \mathsf{Var}[\Phi \cup \{\chi\}]$;*
- *H is a finite set of literals s.t. $\mathsf{Var}[\mathsf{H}] \subseteq \mathsf{Var}[\Phi \cup \{\chi\}]$;*
- *$\mathfrak{p} : \mathbb{E} \to [0, 1]_{\mathbb{Q}}$.*

We call Φ, χ, H, $\mathbb{E}$, *and* $\mathfrak{p}$ theory, observation, hypotheses, events, *and* value assignment, *respectively.*

Note that PrAPs extend the (propositional fragments of the) frameworks in (Poole 1993; Sato 1995), as we allow arbitrary (not just Horn) theories. Besides, PrAPs are similar to abduction problems with preferences on hypotheses (Eiter and Gottlob 1995, §4.4). The difference is that $\mathfrak{p}$ can be undefined on some hypotheses or their conjunctions and that we use numerical values, not a qualitative preference.

To determine solutions to PrAPs, we will consider probabilistic models whose measures are coherent with $\mathfrak{p}$. For simplicity, we will represent Φ as $\bigwedge_{\phi\in\Phi}\phi$ and w.l.o.g. assume that theories of PrAPs contain only one formula.

Definition 24 (Solutions to PrAPs)**.** *For a PrAP* $\mathbb{P}_\mathfrak{p} = \langle\{\phi\},\chi,\mathsf{H},\mathbb{E},\mathfrak{p}\rangle$ *and* $\mathbf{V} = \mathsf{Var}[\mathbb{P}_\mathfrak{p}]$*, we define the following.*

- *A* solution *to* $\mathbb{P}_\mathfrak{p}$ *is an* $\mathcal{L}_{\mathsf{CPL}}$*-term* τ *built from* H *s.t.* $\phi,\tau \models_{\mathsf{CPL}} \chi$ *and there is a probabilistic model* $\mathfrak{M} = \langle 2^{\mathbf{V}},\mu\rangle$ *s.t.* μ *is coherent with* $\mathfrak{p}$ *and* $\mu(\|\phi\wedge\tau\|_{\mathfrak{M}})>0$.
- *A solution* τ *is* preferred *iff for every other solution* σ*, there is a probabilistic model* $\mathfrak{M} = \langle 2^{\mathbf{V}},\mu\rangle$ *s.t.* μ *is coherent with* $\mathfrak{p}$ *and* $\mu(\|\tau\|_{\mathfrak{M}}) \geq \mu(\|\sigma\|_{\mathfrak{M}})$.

Note that CPL abduction problems can be seen as PrAPs with $\mathbb{E} = \{\phi\}$ and $\mathfrak{p}(\phi) = 1$ (i.e., every statement in the theory has probability 1). Indeed, there is $\mathfrak{M} = \langle 2^{\mathbf{V}},\mu\rangle$ s.t. $\mu(\|\phi\wedge\tau\|_{\mathfrak{M}}) > 0$ iff $\phi\wedge\tau$ is CPL-satisfiable. Observe also, the notion of preferred solutions refines that of $\subseteq$-minimal solutions: if σ and τ are $\subseteq$-minimal, we can still prefer τ if there is a probabilistic model $\mathfrak{M} = \langle 2^{\mathbf{V}},\mu\rangle$ s.t. $\mu(\|\tau\|_{\mathfrak{M}}) \geq \mu(\|\sigma\|_{\mathfrak{M}})$. Solutions to PrAPs can also be characterised in terms of *conditional probability* as follows.

Definition 25. *Let* $\mathfrak{M}{=}\langle 2^{\mathbf{V}},\mu\rangle$ *be a probabilisitic model,* $\mathsf{Var}[\{\phi,\chi\}]\subseteq\mathbf{V}$*, and* $\mu(\|\chi\|_{\mathfrak{M}})>0$*. The* conditional probability of ϕ given χ *is defined as* $\mathfrak{Pr}_\mu(\phi|\chi) = \frac{\mu(\|\phi\wedge\chi\|_{\mathfrak{M}})}{\mu(\|\chi\|_{\mathfrak{M}})}$.

Proposition 5. *Let* $\mathbb{P}_\mathfrak{p} = \langle\{\phi\},\chi,\mathsf{H},\mathbb{E},\mathfrak{p}\rangle$ *be a PrAP,* $\mathbf{V} = \mathsf{Var}[\mathbb{P}_\mathfrak{p}]$*, and* τ *an* $\mathcal{L}_{\mathsf{CPL}}$*-term. Then* τ *is a solution to* $\mathbb{P}_\mathfrak{p}$ *iff the following conditions hold:*

1. $\mu(\|\phi\wedge\tau\|_{\mathfrak{M}}) \leq \mu(\|\chi\|_{\mathfrak{M}})$ *in all probabilistic models;*
2. *there is a probabilistic model* $\mathfrak{M}' = \langle 2^{\mathbf{V}},\mu'\rangle$ *s.t.* μ' *is coherent with* $\mathfrak{p}$ *and* $\mathfrak{Pr}_{\mu'}(\chi \mid \phi\wedge\tau) = 1$.

Let us now establish a translation of classical probabilistic abduction into FP. As solutions to PrAPs are *events* and not *assertions about their probabilities*, we will show the correspondence between pairs of PrAPs and their solutions and FP-theories that express the probabilities coherent with $\mathfrak{p}$.

Definition 26. *Let* $\mathbb{E}\subseteq\mathcal{L}_{\mathsf{CPL}}$ *be finite and* $\mathfrak{p}:\mathbb{E}\to[0,1]_{\mathbb{Q}}$*. The* FP-counterpart of $\mathfrak{p}$ *is the following set of* $\mathcal{L}^{\mathbb{Q}}_{\mathsf{Pr}}$*-formulas:*

$$\Xi_\mathfrak{p} = \{\mathsf{Pr}(\phi)\approx\overline{\mathsf{c}} \mid \phi\in\mathbb{E}, \mathfrak{p}(\phi) = c\}$$

Theorem 10. *Let* $\mathbb{P}_\mathfrak{p} = \langle\{\phi\},\chi,\mathsf{H},\mathbb{E},\mathfrak{p}\rangle$ *be a PrAP,* τ *an* $\mathcal{L}_{\mathsf{CPL}}$*-term, and* $\mathbf{V} = \mathsf{Var}[\mathbb{P}_\mathfrak{p}]$*. Then it holds that:*

1. τ *is a solution to* $\mathbb{P}_\mathfrak{p}$ *iff* $\Xi_\mathfrak{p},\neg\triangle\neg\mathsf{Pr}(\phi\wedge\tau) \not\models_{\mathsf{FP}} \bot$ *and* $\mathsf{FP}\models\mathsf{Pr}(\phi\wedge\tau)\to\mathsf{Pr}(\chi)$*;*
2. τ *is a preferred solution to* $\mathbb{P}_\mathfrak{p}$ *iff, in addition to that,* $\Xi_\mathfrak{p},\mathsf{Pr}(\sigma)\to\mathsf{Pr}(\tau)\not\models_{\mathsf{FP}}\bot$ *for every solution* σ *to* $\mathbb{P}_\mathfrak{p}$.

Let us illustrate the correspondence from Theorem 10.

Example 9. *Let* $\mathbb{P}_\mathfrak{p} = \langle\Phi,s,\mathsf{H},\mathbb{E},\mathfrak{p}\rangle$ *and, additionally,* $\Phi = \{p\to s,(q\wedge r)\to s\}$, $\mathsf{H}{=}\{p,q,r\}$, $\mathbb{E} = \mathsf{H}$, $\mathfrak{p}(p) = \frac{1}{2}$, $\mathfrak{p}(q) = \frac{1}{3}$*, and* $\mathfrak{p}(r) = \frac{1}{4}$*. One can see that* p *and* $q\wedge r$ *are solutions to* $\mathbb{P}_\mathfrak{p}$*. Moreover,* p *is the preferred solution, and* $q\wedge r$ is not *because* $\mu(\|p\|)\geq\mu(\|q\wedge r\|)$ *in all probabilistic models coherent with* $\mathfrak{p}$.

Now, set $\Xi_\mathfrak{p} = \{\mathsf{Pr}(p)\approx\overline{\tfrac{1}{2}},\mathsf{Pr}(q)\approx\overline{\tfrac{1}{3}},\mathsf{Pr}(r)\approx\overline{\tfrac{1}{4}}\}$ *and let* $\phi_\mathbb{P} = \bigwedge_{\phi\in\Phi}$*. This gives us* $\Xi_\mathfrak{p},\neg\triangle\neg\mathsf{Pr}(\phi_\mathbb{P}\wedge p)\not\models_{\mathsf{FP}}\bot$, $\mathsf{FP}\models\mathsf{Pr}(\phi_\mathbb{P}\wedge p)\to\mathsf{Pr}(s)$*, and* $\Xi_\mathfrak{p},\mathsf{Pr}(q\wedge r)\to\mathsf{Pr}(p)\not\models_{\mathsf{FP}}\bot$*, as required.*

Using Theorem 10, we can obtain the following complexity evaluations of classical probabilistic abduction.

Theorem 11. *Given a PrAP* $\mathbb{P}_\mathfrak{p}$ *and an* $\mathcal{L}_{\mathsf{CPL}}$*-term* τ*, it is:*

1. DP*-complete to check if* τ *is a solution to* $\mathbb{P}_\mathfrak{p}$*;*
2. *in* Π^{P}_2 *to check if* τ *is a preferred solution to* $\mathbb{P}_\mathfrak{p}$*;*
3. Σ^{P}_2*-complete to check if* $\mathbb{P}_\mathfrak{p}$ *has solutions.*

Notice that *cardinality-minimal* ($\leq$-minimal) solutions (solutions with the fewest hypotheses) to classical abduction problems are exactly the preferred solutions w.r.t. a uniform probability distribution. If $\mathfrak{p}$ is a uniform probability distribution, then $\mathfrak{p}(\sigma)\leq\mathfrak{p}(\tau)$ iff $|\sigma|\leq|\tau|$. Recognition of $\leq$-minimal solutions is known to be Π^{P}_2*-complete* (Eiter and Gottlob 1995). Unfortunately, there seems to be no way to construct a polynomial reduction from the recognition of $\leq$-minimal solutions to preferred solutions because a uniform probability distribution cannot be concisely represented.

Furthermore, note from Definition 23 that our PrAPs are very general: a value assignment in $\mathbb{P}_\mathfrak{p}$ is not necessarily a probability distribution on $2^{\mathsf{Var}[\mathbb{P}_\mathfrak{p}]}$. In most approaches to probabilistic abduction, however, it is assumed that $\mathfrak{p}$ is a probability distribution. Formally, this means that $\mathbb{E} = \{\tau_1,\ldots,\tau_n\}$ is a set of $\mathcal{L}_{\mathsf{CPL}}$-terms s.t. $\mathsf{Var}(\tau_i) = \mathsf{Var}[\mathbb{P}_\mathfrak{p}]$ for all $i\in\{1,\ldots,n\}$ and $\sum_{i=1}^{n}\mathfrak{p}(\tau_i) = 1$.

Under this assumption, solution recognition becomes coNP-complete. For coNP-membership, observe that verifying $\phi,\tau\models_{\mathsf{CPL}}\chi$ is coNP-complete, but checking that $\Xi_\mathfrak{p},\neg\triangle\neg\mathsf{Pr}(\phi\wedge\tau)\not\models_{\mathsf{FP}}\bot$ takes polynomial time w.r.t. the size of $\mathfrak{p}$. To show coNP-hardness, we construct the following polynomial reduction from CPL-validity. Given $\chi\in\mathcal{L}_{\mathsf{CPL}}$, $p\notin\mathsf{Var}(\chi)$, $\mathsf{H}{=}\{p\}$, $\mathbb{E}{=}\{p\wedge\bigwedge_{q\in\mathsf{Var}(\chi)}q\}$, and $\mathfrak{p}(p\wedge\bigwedge_{q\in\mathsf{Var}(\chi)}q){=}1$, p is a solution to $\mathbb{P}_\mathfrak{p} = \langle\{p\vee\neg p\},\chi,\mathsf{H},\mathbb{E},\mathfrak{p}\rangle$ iff χ is CPL-valid.

Theorem 12. *Given a PrAP* $\mathbb{P}_\mathfrak{p} = \langle\{\phi\},\chi,\mathsf{H},\mathbb{E},\mathfrak{p}\rangle$ *s.t.*

- $\mathbb{E}\subseteq\{\bigwedge_{p\in X}p\wedge\bigwedge_{q\notin X}\neg q | X\subseteq\mathsf{Var}[\{\phi,\chi\}]\}$*, and*
- $\sum_{\sigma\in\mathbb{E}}\mathfrak{p}(\sigma){=}1$

and an $\mathcal{L}_{\mathsf{CPL}}$*-term* τ*, it is* coNP*-complete to check if* τ *is a solution to* $\mathbb{P}_\mathfrak{p}$.

7 Conclusion and Future Work

We considered probabilistic abduction formalised in a fuzzy probabilistic logic FP. Our study provides a comprehensive picture of the complexity of full and sufficient solution recognition and existence. We also formalised *classical* probabilistic abduction in FP, thus connecting two interpretations of probabilistic explanations: (1) the explanation *why* *the given event has the observed probability* and (2) a most likely explanation *why the given event occurs*.

We also observe that the results in Section 4 can be straightforwardly extended to abductive reasoning with *conditional* probabilities. For that, one can consider abduction in the logic $\mathsf{FP}_k(\mathsf{RŁ}_\triangle)$ (Flaminio 2007) that extends FP with multiplication by a rational number from $[0,1]$. Since $\mathsf{FP}_k(\mathsf{RŁ}_\triangle)$ is NP-complete, the complexity is preserved. Even more expressive propositional ground logics can be employed to deal, for instance, with product operation. In this case, we should move to the probability logic $\mathsf{FP}(\mathsf{Ł}\Pi)$ based on the combination of Łukasiewicz and Product logics by Esteva et al. (2001). Besides the increased expressive power, it is known from (Hájek and Tulipani 2001) that the satisfiability of $\mathsf{FP}(\mathsf{Ł}\Pi)$ is in PSACE, which would make abductive reasoning in it highly intractable.

Several questions, however, remain open. First, we did not consider determining the relevance and necessity of hypotheses. That is, given an abduction problem $\mathbb{P}$ and a hypothesis h, determine if h occurs in some (respectively, all) solutions of $\mathbb{P}$. We conjecture that the complexity of relevance and necessity of hypotheses to solutions of FP APs would coincide with that of $\mathsf{Ł}$-abduction problems. It also makes sense to consider relevance and necessity of PILs w.r.t. *full solutions*. Additionally, we plan to establish the exact complexity of recognising preferred solutions to PrAPs.

Another important task would be to provide a translation of FP-abduction into $\mathsf{Ł}$-abduction. Flaminio (2020) constructed an embedding of *deductive* reasoning in fuzzy probabilistic into deductive reasoning in fuzzy propositional logic. Thus, it makes sense to extend their results and obtain a similar correspondence for abduction.

A further connection to investigate would be the one between abduction in FP and alternative notions of explanations. In particular, *contrastive explanations* (Ignatiev et al. 2021) and *extended abduction* studied by Inoue and Sakama (1995; 1999b; 2000; 2003). Intuitively, a contrastive explanation of ϕ is telling *why* $\neg\phi$ *did not happen*. In extended abduction, an explanation entails the observation but not some undesired consequence. In a similar fashion, in FP-abduction, an explanation why ϕ has probability 20% also tells why ϕ *does not have* probability 25%.

Another question is that of *prioritisation* of full solutions. We considered preference based on the *principle of maximal entropy*, but there are other alternatives discussed by Dubois et al. (2008). One of them is the principle of *maximal likelihood* studied by Inoue et al. (2009) and Ishihata et al. (2011). It would be instructive to consider the prioritisation of full solutions according to the maximal likelihood principle.

Finally, we plan to consider other notions of uncertainty. First, more expressive logics that can reason about independent events and conditional probabilities (Hájek and Tulipani 2001) or belief functions (Godo, Hájek, and Esteva 2003). Second, fuzzy logics for *qualitative* probabilities (Bílková et al. 2023). For the latter, it will make sense to establish a correspondence between qualitative probabilistic abduction and preference abduction (Inoue and Sakama 1999a; Konczak and Vogel 2005) where the explanation is a *comparative* statement about likelihoods of events.

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AI Declaration

The authors have not employed any Generative AI tools.

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