But Not Because You Said So! Implicitly Accepting Information with Abductive Belief-Base Change

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Abstract

Abductive expansion is an AGM-style belief-change operation that accommodates new information by adding explanatory hypotheses rather than incorporating the input outright. In contrast to belief sets, belief bases are finite and non-deductively closed, allowing a distinction between explicit and implicit beliefs. We extend abductive belief-change operations to belief bases relative to a hypothesis space, treating the base as firm beliefs and maintaining a separate space of tentative hypotheses that is conditioned on new inputs. Inspired by partial meet and kernel construction from classical belief revision, we provide concrete instances of three different types of abductive change: expansion, suspension (which corresponds to contraction), and revision. In the form of representation theorems, all these operators are shown to be characterized axiomatically by intuitive postulates. Along the constructions, counterparts of the well-known Levi identity and the Harper identity are exploited.

1 Introduction

Standard models of belief change incorporate new information as is, while also maintaining consistency. By contrast, abductive belief change (Pagnucco 1996) accepts information by hypothesizing underlying reasons that explain the new epistemic input. A motivating example comes from scientific explanation (Schurz 2010). Suppose an agent observes that stones sink while wood floats. While standard belief change may incorporate such observations directly, abductive belief change may accept them indirectly, by adopting an explanatory hypothesis such as Archimedes' principle of buoyancy. Thus, the input is accepted not merely because it was received, but through hypotheses that make it explainable. As in the classical model of belief revision of Alchourrón, Gärdenfors, and Makinson (AGM) (1985), an agent's beliefs are represented by belief sets in Pagnucco's abductive change operators (Pagnucco 1996), i.e., by logically closed sets of sentences. As such, belief sets cannot distinguish between *implied* and *explicit* beliefs. Hence, using a deductive theory of explanation, a belief set cannot distinguish between *explanans* (the explaining sentence) and *explanandum* (the explained sentence) after the change.

A more expressive extension to belief sets was proposed by Hansson (1989), using non-deductively closed sets called *belief bases*. Beliefs can be discerned: derived beliefs are not worth retaining for their own sake, but must be supported by a basic belief, included in the belief base.

In this paper, we extend abductive belief change from belief sets to belief bases. Changes of beliefs are governed by an underlying hypothesis space from which explanations can be drawn, and not solely from the incoming information. As a consequence, abductive belief change on belief bases further distinguishes explicit beliefs that are *firm* (i.e., those in the belief base) from those that are *tentative*, i.e., beliefs that remain hypothetical until new information is revealed.

The two central forms of belief change are *contraction*, which removes undesirable or obsolete information, and *revision*, which incorporates new information while maintaining the consistency of the belief corpus. The basic operation of belief *expansion* becomes central to *abductive belief change* in the form of *abductive expansion*, which also incorporates hypotheses explaining the incoming information.

Belief bases may encode additional information about the history of belief changes or the justifications of the beliefs within the syntax, so, e.g., there might be differences between a change of the belief base $B = \{p, q\}$ and a belief base $B = \{p \wedge q\}$. The same holds for the hypothesis space, which we assume to consist of finite sets of sentences. It is due to this syntax dependency that the abductive change operators defined and investigated in this paper are non-prioritized change operators (Hansson 1999a): observations might not be entailed by the change result even if there is an explanation. However, the change operators contain core sentences (of potential explanations) that can be extended to an explanation of the observation. These properties are captured by postulates for abductive expansion operators. We show with a representation theorem that these postulates even characterize the class of abductive expansion operators based on a partial meet construction on hypotheses.

Abductive expansion operators do not change the belief base if the observation is not explainable. In analogy to the case of classical belief change, where belief expansion does not resolve inconsistencies but belief revision does, abductive revision is expected to handle properly the case of non-explainable observations. In constructing an appropriate abductive revision operator, we give a construction that parallels the construction of belief revision via the so-called *Levi identity* (Alchourrón, Gärdenfors, and Makinson 1985), which defines revision via contraction and expansion. Under the term *abductive belief suspension*, we define a new abductive change operator that corresponds to classical contraction. It relinquishes firm beliefs and moves them into the pool of hypotheses to reconsider as tentative beliefs. We specify a set of desired properties of abductive belief suspension via postulates and show in a representation result that the postulates even characterize a class of concrete abductive suspension operators based on a kernel construction. Also, the class of abductive revision operators resulting from the composition of abductive expansion and abductive suspension is characterized by simple postulates. The so-called *Harper identity* is the dual to the Levi identity and describes the construct of contraction via revision. For classical change operators over belief sets, one can show that these identities are indeed inverses in the sense that a chained application amounts to the identity function. The same can be shown for the abductive suspension and abductive revision operators for the case of applying first the Levi identity and then the Harper identity. The application of the Harper identity and then the Levi identity does not necessarily recover the original (suspension) operation.

2 Notation and Background

Beliefs are formulated in an object language $\mathcal{L}$ that we assume to be closed under the usual connectives: negation ($\neg$), conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$), equivalence ($\leftrightarrow$), and the zero-ary connective for contradiction ($\perp$). We define a *consequence operator* $\mathrm{Cn} : 2^{\mathcal{L}} \rightarrow 2^{\mathcal{L}}$ that derives the logical consequences of a set of sentences. The operator Cn satisfies the Tarskian properties[1] (Tarski 1983). We further assume that Cn is *supraclassical* and *compact*. In particular, we assume that $\mathcal{L}$ contains a propositional fragment generated in the usual inductive way from a set $\mathcal{P}$ of propositional symbols.

For notation, we write $K \vdash \phi$ whenever $\phi \in \mathrm{Cn}(K)$. Lowercase Greek letters are used as metavariables for formulae of $\mathcal{L}$. A set of formulae $K \subseteq \mathcal{L}$ is *consistent* iff its Cn-closure is different from the whole language, $\mathrm{Cn}(K) \neq \mathcal{L}$. With the above assumptions, K is consistent iff $\perp \notin \mathrm{Cn}(K)$.

2.1 Belief Change

The AGM framework (Alchourrón, Gärdenfors, and Makinson 1985) characterizes belief change operations on theories guided by general rationality postulates. A *theory* or, alternatively, a *belief set*, is a *deductively closed set* of formulae $K \subseteq \mathcal{L}$, i.e., $K = \mathrm{Cn}(K)$. It defines three operations: expansion ($+$), contraction ($-$), and revision ($*$), each mapping a theory and a formula to a (possibly) changed theory. AGM describes rationality postulates, which specify desired behavior of the operators, provides construction principles for the operators (such as partial meet constructions), and shows, in the form of representation theorems, that the postulates uniquely characterize the class of operators implemented according to the construction principles.

[1] inclusion: $K \subseteq \mathrm{Cn}(K)$, monotonicity: $K \subseteq K' \Rightarrow \mathrm{Cn}(K) \subseteq \mathrm{Cn}(K')$, iteration: $\mathrm{Cn}(\mathrm{Cn}(K)) = \mathrm{Cn}(K)$

In the AGM framework, every belief in a theory is treated alike, including those that follow from logical closure. When it is useful to distinguish explicit from implicit beliefs, one can instead use Hansson's notion of a *belief base* which is not necessarily deductively closed (Hansson 1999b). Similarly to the AGM framework, Hansson proposes rationality postulates for operations on belief bases.

Expansion ($+$) in the AGM framework, as well as in Hansson's approach, simply incorporates new information as is—in a case of AGM expansion, additionally, by closing the result under Cn i.e. $K + \alpha = \mathrm{Cn}(K \cup \{\alpha\})$.

Contraction operators can be constructed syntactically using remainder sets and selection functions for theories as well as bases, resulting in the class of *partial meet contraction* operators. A remainder set, denoted $K \perp \alpha$, is the set of all maximal consistent subsets of K that do not imply α. A selection function for K picks some subset of the remainder set, where $\gamma(K \perp \alpha)$ is a non-empty subset of $K \perp \alpha$ whenever $K \perp \alpha$ is non-empty; if $K \perp \alpha$ is empty, then $\gamma(K \perp \alpha) = \{K\}$. The partial meet contraction generated by γ yields the operation defined as $K -_{\gamma} \alpha = \bigcap \gamma(K \perp \alpha)$.

As a generalization of partial-meet contraction on belief bases, kernel contraction (Hansson 1994) is based on *kernels*, i.e. minimal subsets of a belief base B that entail the sentence α to be contracted; the set of α-kernels is denoted $B \perp\!\!\!\perp \alpha$. An incision function σ selects sentences to remove, where $\sigma(B \perp\!\!\!\perp \alpha) \subseteq \bigcup(B \perp\!\!\!\perp \alpha)$ and, for every non-empty $X \in B \perp\!\!\!\perp \alpha$, $X \cap \sigma(B \perp\!\!\!\perp \alpha) \neq \emptyset$. The kernel contraction generated by σ is then $B \ominus_{\sigma} \alpha = B \setminus \sigma(B \perp\!\!\!\perp \alpha)$.

Revision operations can be obtained by combining contraction and expansion via the so-called *Levi identity* i.e. $K * \alpha = (K - \neg\alpha) + \alpha$. In turn, contraction operations can be obtained from revision (and intersection) via the so-called *Harper identity*, i.e. $K - \alpha = (K * \neg\alpha) \cap K$.

Well-known semantic constructions for operators on theories are given by Grove's system of spheres (Grove 1988) and faithful pre-orders (Katsuno and Mendelzon 1991).

2.2 Abductive Belief Change

Rather than simply adding new information verbatim, abductive expansion ($\oplus$) incorporates an explanation for that information into current beliefs. Several concrete instances of this operation were proposed by Pagnucco in his dissertation (1996), which also includes a representation theorem. Similar constructions have been proposed in (Aliseda 2006) and (Schurz 2010).

The general definition by Pagnucco and colleagues (1994) of an abductive expansion operator $\oplus$ states that if an observation α can be non-trivially explained by some sentence $\beta \in \mathcal{L}$ of the language, then the expansion result $K \oplus \alpha$ should be the Cn closure of the original belief set K together with some chosen explanation β. If no explanation β exists, nothing is changed. Formally, for a belief set K and formula α, $\oplus$ is called an *abductive expansion operator* according to Pagnucco iff it has the following form:

$$K \oplus \alpha = \begin{cases} \mathrm{Cn}(K \cup \{\beta\}) & \text{for some } \beta \in \mathcal{L} \text{ with} \\ & K \cup \{\beta\} \vdash \alpha \text{ and} \\ & K \cup \{\beta\} \nvdash \bot \\ & \text{if such a } \beta \text{ exists} \\ K & \text{otherwise.} \end{cases} \tag{1}$$

Fermé and Hansson (2018) give a different definition based on the notion of an abduction function. Their definition can be understood as a strengthening of Pagnucco's definition along two lines: (1) choosing an appropriate explanation β is explicated by a function f and (2) the function f is required to fulfill an extensionality condition (called "weak extensionality" by Pagnucco (1996)).

Definition 1 (Abduction function and abductive expansion (Fermé and Hansson 2018))**.** *Let K be a belief set.*

- *An* abduction function *f for K is a function with*
  1. *($K \cup \{f(\alpha)\} \vdash \alpha$ and $K \cup \{f(\alpha)\} \nvdash \bot$) if $K \cup \{\alpha\} \nvdash \bot$.*
  2. *$f(\alpha) = \top$ if $K \cup \{\alpha\} \vdash \bot$.*
  3. *For all α, α', if $\vdash \alpha \leftrightarrow \alpha'$ then $\vdash f(\alpha) \leftrightarrow f(\alpha')$*
- *An operator $\oplus$ on K is an* abductive expansion *iff there exists an abduction function f for K such that*

$$K \oplus \alpha \ = \ K + \{f(\alpha)\}$$

If the third condition of an abduction function is skipped, the two notions of abductive expansion operators can be shown to be equivalent. (Note that Equation (1) does not guarantee weak extensionality: Take $K = \mathrm{Cn}(\emptyset)$, $\alpha = p \vee q$, $\alpha' = p \vee p \vee q$. An operator of the form (1) may choose the non-equivalent explanations $\beta = p$ and $\beta' = q$.)

In the case of our expansion framework, the potential explanation β is not allowed to be an arbitrary sentence but must be (equivalent to) a hypothesis set. Both definitions, that of Pagnucco and colleagues (1994) and that given in Fermé and Hansson's book (2018)—even if adapted to hypotheses—have in common the fact that the results of abductive expansion entail the observation in case an explanation exists at all. This does not hold for our abductive expansion operator defined below. The rationale behind this is that we might have many explanations for the observations. When we intersect/aggregate equally preferred explanations, only their shared part is retained, which may be too weak to imply the observation. Adapting the terminology of Hansson (1999a), this kind of operation can be termed *a non-prioritized abductive expansion*.

Based on ideas from classical belief revision, various constructions for abductive expansion change operators have been proposed in the literature, in particular in the PhD thesis of Pagnucco (1996).

A syntactic construction similar to that of AGM contraction can be formulated by means of selection functions and replacing remainder sets with sets of maximal consistent supersets of K that imply α, denoted $K\top\alpha$. A semantic construction of abductive expansion is given by an adaptation of Grove's system of spheres.

An alternative construction using an *abductive entrenchement ordering*, analogous to epistemic entrenchment, was given (Pagnucco and Rajaratnam 2005).

3 Abductive Belief Base Expansion

A belief base distinguishes between *explicit* beliefs, which are directly contained in the base, and *implicit* beliefs, which are merely logical consequences (Lorini 2020; Fermé, Herzig, and Martinez 2025).

Any hypothesis formed by abduction may compete with other possible alternatives and is therefore uncertain. For abductive expansion on belief bases, we draw a distinction between the *firm* beliefs within the belief base and the set of (competing) hypotheses, which are *tentative*.

Intuitively, we treat the belief base as a background theory during abduction, whilst hypotheses are sets of formulae.

An earlier version of abductive expansion appeared in (Bayerkuhnlein, Özçep, and Wolter 2026). Here, we use the classical belief-change format. The operator returns only the changed base, while the selected hypotheses are captured indirectly by the postulates below (see Appendix B for the correspondence). This format is then used uniformly for expansion, suspension, and revision.

Definition 2. *Let $H \subseteq 2^{\mathcal{L}}$ be a hypothesis space (a set of hypothesis sets). The set of* possible explanations *from the set H for an observation $\alpha \in \mathcal{L}$ relative to a belief base $B \subseteq \mathcal{L}$ is defined as*

$$\mathcal{S}\big(B, H, \alpha\big) \ = \ \big\{ \Delta \in H \ \big| \ B \cup \Delta \vdash \alpha \ \textit{and} \ B \cup \Delta \nvdash \bot \big\}$$

Furthermore, we say that α is *explainable* relative to a belief base B and the set H iff there is at least one possible explanation, i.e. $\mathsf{Exp}\big(B, H, \alpha\big)$ iff $\mathcal{S}\big(B, H, \alpha\big) \neq \emptyset$.

3.1 Abductive Selective Expansion

An abductive selective expansion is defined as an operation $\oplus : 2^{\mathcal{L}} \times 2^{2^{\mathcal{L}}} \times \mathcal{L} \to 2^{\mathcal{L}}$, where $B \oplus_{\mathrm{H}} \alpha$ denotes the base of firm beliefs B expanded by hypotheses from H in order to conclude α. The set of all formula used inside the set H (i.e. $\bigcup H$), form the *abducibles*, the operation can consider.

The operations are characterized by the following postulates.

- (B $\oplus$ 1) If not $\mathsf{Exp}\big(B, H, \alpha\big)$, $B \oplus_{\mathrm{H}} \alpha = B$. (Failure)
- (B $\oplus$ 2) $B \subseteq B \oplus_{\mathrm{H}} \alpha$. (Inclusion)
- (B $\oplus$ 3) If $\mathsf{Exp}\big(B, H, \alpha\big)$, then there is a $\Delta \in \mathcal{S}\big(B, H, \alpha\big)$ such that $\big(B \oplus_{\mathrm{H}} \alpha \setminus B\big) \subseteq \Delta$ (Witness)
- (B $\oplus$ 4) If $\mathsf{Exp}\big(B, H, \alpha\big)$, $\psi \in \bigcup H$, and $\psi \notin B \oplus_{\mathrm{H}} \alpha$, then there is a set of possible hypothesis sets $S \subseteq \mathcal{S}\big(B, H, \alpha\big)$ such that $S \neq \emptyset$ and there is a hypothesis set $\Delta \in S$ with $B \oplus_{\mathrm{H}} \alpha \subseteq B \cup \Delta$ and $\psi \notin \Delta$. (Alternative)
- (B $\oplus$ 5) If $\forall B' \subseteq B.\ \mathcal{S}\big(B', H, \alpha\big) = \mathcal{S}\big(B', H, \beta\big)$, then $B \oplus_{\mathrm{H}} \alpha = B \oplus_{\mathrm{H}} \beta$. (Uniformity)

As with abductive expansion on belief sets, the operation on bases fails if no candidate hypothesis is available (Failure). Here, the failure is due to the lack of admissible hypotheses in the hypothesis space. Postulate (Inclusion) states that no explicit belief is removed during expansion.

Since $B \oplus_{\mathrm{H}} \alpha$ represents the firm beliefs of the agent, the abductive consequence depends on the selected hypothesis. So each formula added to the beliefs has to be part of some hypothesis, capable of explaining the epistemic input (Witness).

An agent can commit to a tentative belief when other alternatives are ruled out or disregarded. This is captured in (Alternative), which requires that omitting an abducible be justified by the presence of an alternative tentative belief that is taken into account. Here, $(S \subseteq \mathcal{S}(B, H, \alpha))$ denotes the subset of hypotheses on which the agent conditions.

Finally, if two observations α and β induce exactly the same set of possible explanations for every subbase, then the operator must treat them identically (Uniformity), where the quantification over subbases ensures invariance under redundant explicit beliefs in B.

Proposition 3.1. *For every belief base B, set of hypotheses H, and formula α, operator $\oplus$ which satisfies postulates* $(\mathrm{B} \oplus 1)$–$(\mathrm{B} \oplus 5)$ *also satisfies*

$$B \oplus_H \alpha \subseteq B \cup \bigcup H \qquad \textit{(H-Inclusion)}$$

The abductive selective expansion operator is indeed selective because, in general, the sets of explanations for the observations chosen, i.e. $(S \subseteq \mathcal{S}(B, H, \alpha))$, are not necessarily closed. Hence, their intersection $\bigcap S$ might not entail the observation.

Syntactic Construction Using a selection function, we select among the possible hypotheses, and define a partial meet construction for abductive expansion.

Definition 3. *A function γ is a* selection function *if, for a belief base B, epistemic input α, and set of hypotheses H, writing $S = \mathcal{S}(B, H, \alpha)$ as shorthand, the following hold: (selection) if $S \neq \emptyset$, then $\gamma(S) \subseteq S$, (non-emptiness) if $S \neq \emptyset$, then $\gamma(S) \neq \emptyset$ and (empty selection) if $S = \emptyset$, then $\gamma(S) = \emptyset$* [2].

Intuitively, we use the selection among all the potential hypotheses to model an agent's preference on which hypotheses are regarded as plausible or can be ignored. This selection is maintained in the belief base as a grounded set of beliefs, as well as in the tentative conditioned hypothesis space.

This notion parallels the hypothesis space and version space frameworks in concept learning (Mitchell 1997). In contrast to concept learning, selection of hypotheses as described in this paper is determined by γ and picks hypotheses according to the agent's *preference*. This construction follows that of (Bayerkuhnlein, Özçep, and Wolter 2026).

Definition 4. *Given a selection function γ as specified in Definition 3, for a belief base B and a formula α, a* partial meet abductive expansion *is defined as follows:*

$$B \oplus^{\gamma}_{\mathrm{H}} \alpha = \begin{cases} B \cup \bigcap \gamma(\mathcal{S}(B, H, \alpha)) & \textit{if } \mathsf{Exp}(B, H, \alpha) \\ B & \textit{otherwise}. \end{cases} \tag{2}$$

Theorem 3.1. *Let $\oplus$ be an abductive expansion operator on belief bases with hypotheses H. For every belief base B, the operator $B \oplus_{\mathrm{H}} \alpha$ satisfies postulates* $(\mathrm{B} \oplus 1)$–$(\mathrm{B} \oplus 5)$ *if and only if it is a partial meet abductive base expansion.*

Special instances of partial meet abductive expansion are the maxichoice and full meet variants. Their construction and interpretive implications mirror those of their counterparts in belief set change.

Example 1. *Consider the belief base $B = \{p \rightarrow r, q \rightarrow \neg p\}$ with the set of abducibles $\mathcal{A} = \{p,\ q,\ q \rightarrow r\}$ from which H is generated as the powerset $\mathcal{P}(\mathcal{A})$. We want to abductively expand by r. There are three possible explanations $\mathcal{S}(B, H, r)$:*

$$\Delta_1 = \{p\} \quad \Delta_2 = \{p, q \rightarrow r\} \quad \Delta_3 = \{q, q \rightarrow r\}$$

Under a maxichoice *strategy, a single hypothesis is immediately incorporated into the belief base, yielding an abductive expansion for each hypothesis, i.e., $A_i = B \cup \Delta_i$ and $A_i \vdash r$ (implicitly), while not $r \in A_i$ (explicitly). Partial- and full-meet variants instead condition on sets of hypotheses: $A_4 = B \cup (\Delta_1 \cap \Delta_2)$, $A_5 = B \cup (\Delta_2 \cap \Delta_3)$, $A_6 = B \cup (\Delta_1 \cap \Delta_3)$, and $A_7 = B \cup (\Delta_1 \cap \Delta_2 \cap \Delta_3)$. The resulting bases may entail the observation (e.g., $A_4 \vdash r$) or commit only to a common part that does not entail the observation (see A_5, A_6, and A_7) leaving the options tentative. Depending on the agent's preference, one can trade explanatory commitment for caution.*

4 Reconsidering Hypotheses

A firm belief can be questioned and suspended to make room for other hypotheses to take its place. Likewise, one may introduce previously ruled out hypotheses or entirely new ones.

In databases, it is common to receive external commands instructing a system to delete an item, a situation that is well captured by belief contraction. In abductive contexts, however, we do not treat contraction primarily as a model of serious doxastic change (Hansson 2011). Rather, we employ a form of contraction in which a firm belief is suspended and thereby made tentative to make room for alternative explanations. We will refer to this as a *suspending contraction*.

4.1 Suspending Contraction

The postulates for suspending contraction are closely related to those for kernel contraction. However, while kernel contraction removes sentences in order to prevent an epistemic input from being derivable as a consequence of the belief base, suspending contraction is used to permit the input as a credulous consequence within the full hypothesis space.

[2] The three clauses are separated for clarity. Equivalently: $\gamma(S) \subseteq S$ and $\gamma(S) = \emptyset$ iff $S = \emptyset$.

Postulates Intuitively, the postulates on suspending contraction ensure that firm beliefs from the belief base are suspended and thereby only remain within the possible alternatives within the set of hypotheses, allowing them to be reconsidered in further steps. The postulates rest on an assumption regarding the sufficient expressiveness of the hypothesis space: The epistemic input α is explainable within the hypothesis space H alone, i.e. $\mathsf{Exp}\big(\emptyset, H, \alpha\big)$. That means, if an observation α is not explainable under the current belief base B, then this is the case not because there are no explanations at all but because the belief base B blocks those explanations.

For a finite belief base B, the following postulates characterize a *suspending contraction*:

- $(B \oslash 1)$ $\mathsf{Exp}\big(B \oslash_{\mathrm{H}} \alpha, H, \alpha\big)$ (S-Success)
- $(B \oslash 2)$ If $\mathsf{Exp}\big(B, H, \alpha\big)$, then $B \oslash_{\mathrm{H}} \alpha = B$. (S-Vacuity)
- $(B \oslash 3)$ $B \oslash_{\mathrm{H}} \alpha \subseteq B$. (S-Inclusion)
- $(B \oslash 4)$ If $\big(\mathsf{Exp}(B', H, \alpha)$ iff $\mathsf{Exp}(B', H, \beta)\big)$ for all $B' \subseteq B$, then $B \oslash_{\mathrm{H}} \alpha = B \oslash_{\mathrm{H}} \beta$. (S-Uniformity)
- $(B \oslash 5)$ If $\psi \in B \setminus B \oslash_{\mathrm{H}} \alpha$, then there is $B' \subseteq B$ such that $\mathsf{Exp}\big(B', H, \alpha\big)$ does not hold but $\mathsf{Exp}\big((B' \setminus \{\psi\}), H, \alpha\big)$ does. (S-Core-Retainment)

Postulate (S-Success) requires that the elimination of beliefs of B unblocks potential explanations for the observation. Postulate (S-Vacuity) states that if B does not block possible explanations of the observation, then no belief should be eliminated. Postulate (S-Inclusion) states that the suspension change results at most in the elimination of some beliefs of the belief base. Postulate (S-Uniformity) is meant to handle a uniform treatment of observations α, β as long as potential eliminations from B lead to the explainability of α iff the same potential eliminations from B leads to the explainability of β. Finally, Postulate (S-Core-Retainment) states that the elimination of some belief ψ from the original belief base must be justified on the grounds of a considerable explanation.

A cautious agent may suspend beliefs preemptively, awaiting further information, whereas bolder agents rely on suspension to reconsider previous commitments

Example 2 (Continued from Example 1)**.** *Assume the hypothesis space has been conditioned to $H' = \{\Delta_2, \Delta_3\}$. We want to abductively expand A_5 by formula p, which results in $A_5 \oplus_{\mathrm{H'}} p = \{p \to r, q \to \neg p, q \to r, p\}$.*

Syntactic Construction We extend the notion of a kernel by using *minimal blocking sets* (rather than minimal entailing sets). As in kernel contraction, the selection of sentences to be removed is determined by an incision function.

Definition 5. *For a belief base B, a set of hypotheses H, and a formula α, define*

$$B \perp\!\!\!\perp^{\mathrm{H}} \alpha = \left\{ K \subseteq B \;\middle|\; \begin{array}{l} \neg\mathsf{Exp}\big(K, H, \alpha\big) \wedge \\ (\forall K' \subset K.\ \mathsf{Exp}\big(K', H, \alpha\big)) \end{array} \right\}. \tag{3}$$

Definition 6. *A function σ is an* incision function *if,* (subset) $\sigma(B \perp\!\!\!\perp^{\mathrm{H}} \alpha) \subseteq \bigcup B \perp\!\!\!\perp^{\mathrm{H}} \alpha$, (hitting) *if $K \in B \perp\!\!\!\perp^{\mathrm{H}} \alpha$ and $K \neq \emptyset$, then $K \cap \sigma(B \perp\!\!\!\perp^{\mathrm{H}} \alpha) \neq \emptyset$.*

Definition 7. *Given an incision function σ as specified in Definition 6, for a belief base B, a set of hypotheses H, and a formula α, the* kernel abductive suspension *is defined as follows:*

$$B \oslash^{\sigma}_{\mathrm{H}} \alpha = \begin{cases} B, & \text{if } \mathsf{Exp}\big(B, H, \alpha\big), \\ B \setminus \sigma\big(B \perp\!\!\!\perp^{\mathrm{H}} \alpha\big), & \text{otherwise.} \end{cases} \tag{4}$$

Example 3 (Continued from Example 1)**.** *Let $B = \{\, p \to r,\ q \to \neg p,\ q,\ q \to r \,\}$. Extend the abducibles to $\mathcal{A}'$ by adding $p \to s$ (and p), and set $H'' = \mathcal{P}(\mathcal{A}')$. We suspend B for s. Although s can be explained from $\emptyset$ (e.g. $\{p, p \to s\} \vdash s$), it is* blocked *in B since $B \vdash \neg p$. Thus,*

$$B \perp\!\!\!\perp^{\mathrm{H''}} s = \big\{\{q, q \to \neg p\}\big\}.$$

Hence an incision may choose $\{q\}$, $\{q \to \neg p\}$, or $\{q, q \to \neg p\}$, and in each case $B \oslash_{\mathrm{H''}} s = B \setminus \sigma(B \perp\!\!\!\perp^{\mathrm{H''}} s)$ satisfies $\mathsf{Exp}\big(B \oslash_{\mathrm{H''}} s, H'', s\big)$.

Proposition 4.1. *Let σ be an incision function as specified in Definition 6, and let $\oslash^{\sigma}$ be the operator defined by Definition 7. Then $\oslash^{\sigma}$ satisfies postulates $(B \oslash 1)$–$(B \oslash 5)$.*

For completeness, assume downward closure of explainability: if $A \subseteq B$ and $\mathsf{Exp}\big(B, H, \alpha\big)$, then $\mathsf{Exp}\big(A, H, \alpha\big)$. Intuitively, the base supplies constraints while hypotheses supply explanations, so passing to a subbase cannot destroy explainability.

Proposition 4.2. *Let $\oslash$ be a suspension operator on belief bases relative to a set of hypotheses H. If $\oslash$ satisfies postulates $(B \oslash 1)$–$(B \oslash 5)$ and explainability is downward closed then there exists an incision function σ (Definition 6) such that $\oslash$ is representable by the kernel construction in Definition 7.*

The downward closure property of explainability can be induced for the construction by saturating the set of hypotheses. We saturate a set of hypotheses by a belief base to obtain a *saturated hypothesis* set H^B where $H^B = \{\Delta \cup B' \mid \Delta \in H \wedge B' \subseteq B\}$.

Proposition 4.3. *For any belief base B, explainability is downward closed on the saturated set of hypotheses i.e. If $(A \subseteq B)$ and $\mathsf{Exp}\big(B, H^B, \alpha\big)$ then $\mathsf{Exp}\big(A, H^B, \alpha\big)$.*

Theorem 4.1. *Let $\oslash$ be a suspension operator on belief bases relative to a set hypotheses H enforcing downward closure on explainability. For every belief base B and formula α, the operator $\oslash$ satisfies postulates $(B \oslash 1)$–$(B \oslash 5)$ if and only if it is representable by the kernel construction in Definition 7.*

4.2 Abductive Revision

Following the proposals of (Pagnucco 1996; Aliseda 2006), abductive revision operators can be defined by contraction and abductive expansion. However, abductive revision on belief bases constructed from abductive expansion by a straightforward application of the Levi identity fails to capture the dynamics of updating a hypothesis, since it resets explanatory commitments, as pointed out by Schurz (2010). The construction of an abductive revision we propose is laid out below and uses suspension followed by abductive expansion. We start again with some desirable properties for abductive revision captured by the following postulates.

Postulates The postulates are inspired by revision postulates for belief bases (Hansson 1999b) and can be considered as a (compositional) merge of the postulates for expansion and suspension.

($B \otimes 1$) If $\mathsf{Exp}\big(\emptyset, H, \alpha\big)$, then $\mathsf{Exp}\big(B \otimes_{\mathrm{H}} \alpha, H, \alpha\big)$. (R-Success)

($B \otimes 2$) $B \otimes_{\mathrm{H}} \alpha \subseteq B \cup \bigcup H$. (R-Inclusion)

($B \otimes 3$) If $\mathcal{S}\big(B, H, \alpha\big) \neq \emptyset$, then there exists $\Delta \in \mathcal{S}\big(B, H, \alpha\big)$ such that $B \otimes_{\mathrm{H}} \alpha \setminus B \subseteq \Delta$. (R-Witness)

($B \otimes 4$) If $\psi \in B \setminus B \otimes_{\mathrm{H}} \alpha$, then there is $B' \subseteq B$ such that $\mathsf{Exp}\big(B', H, \alpha\big)$ does not hold but $\mathsf{Exp}\big((B' \setminus \{\psi\}), H, \alpha\big)$ does. (R-Relevance)

($B \otimes 5$) If $\big(\mathsf{Exp}\big(B', H, \alpha\big)$ iff $\mathsf{Exp}\big(B', H, \beta\big)\big)$ for all $B' \subseteq B$, then $B \cap B \otimes_{\mathrm{H}} \alpha = B \cap B \otimes_{\mathrm{H}} \beta$. (R-Uniformity)

Construction To define the Levi-like abductive base revision operator, we need an auxiliary concept, keeping suspended beliefs from being reintroduced.

Definition 8. *Given a belief base B and a set of hypotheses H, the* fresh part *of H w.r.t. B is*

$$\mathit{fresh}(H, B) = \{\Delta \in H \mid \Delta \cap B = \emptyset\}.$$

Lemma 4.1. *Let B be a belief base, H a set of hypotheses, and α a formula. If $B_0 \subseteq B$, then $B \cap \big(B_0 \oplus_{\mathit{fresh}(H,B)} \alpha\big) = B_0$.*

The Levi-like abductive base revision operator now becomes the composition of suspension with expansion—with a change of the hypothesis space in between.

Definition 9 (Levi-like abductive base revision)**.** *Let $\oslash$ be a suspension operator and $\oplus$ an abductive expansion operator. Define the* Levi-like revision *operator $\otimes$ by*

$$B \otimes_{\mathrm{H}} \alpha = \ (B \oslash_{\mathrm{H}} \alpha) \oplus_{\mathit{fresh}(\mathrm{H},\mathrm{B})} \alpha.$$

Proposition 4.4. *Let $\oslash$ be a suspension operator satisfying postulates ($B \oslash 1$)–($B \oslash 5$), and let $\oplus$ be an abductive expansion operator satisfying postulates ($B \oplus 1$)–($B \oplus 5$). Let $\otimes$ be the Levi-like revision operator defined in Definition 9. Then $\otimes$ satisfies the revision postulates ($B \otimes 1$)–($B \otimes 5$).*

Definition 10. *Let $\otimes$ be an abductive base revision operator. The* Harper-like suspension *induced by $\otimes$ is defined by*

$$B \oslash_{\mathrm{H}}^{\otimes} \alpha = (B \otimes_{\mathrm{H}} \alpha) \cap B.$$

Proposition 4.5. *Let $\oslash$ be an abductive suspension operator and $\oplus$ an abductive expansion operator. Let $\otimes$ be the Levi-like revision defined from them by*

$$B \otimes_{\mathrm{H}} \alpha \ = \ B \oslash_{\mathrm{H}} \alpha \oplus_{\mathit{fresh}(\mathrm{H},\mathrm{B})} \alpha.$$

Then the Harper-like suspension induced by $\otimes$ coincides with the original suspension:

$$(B \otimes_{\mathrm{H}} \alpha) \cap B \ = \ B \oslash_{\mathrm{H}} \alpha.$$

Equivalently, $B \oslash_{\mathrm{H}}^{\otimes} \alpha = B \oslash_{\mathrm{H}} \alpha$.

Negative Result The previous result shows that for Levi-like revision, the Harper-like construction recovers the underlying suspension. The converse direction fails in general: the retained firm part $C = (B \otimes_{\mathrm{H}} \alpha) \cap B$ does not determine the revision outcome, because abductive choice information is lost.

Lemma 4.2. *Let B and C be belief bases with $C \subseteq B$, let H be a set of hypotheses, and let α be a formula. Let $H_f := \mathit{fresh}(H, B)$. If there exist two distinct possible explanations $\Delta_1, \Delta_2 \in \mathcal{S}\big(C, H_f, \alpha\big)$ with $\Delta_1 \neq \Delta_2$, then the two bases $R_1 := C \cup \Delta_1$ and $R_2 := C \cup \Delta_2$ satisfy $R_1 \cap B = C$, $R_2 \cap B = C$, and $R_1 \neq R_2$.*

Hence revision cannot, in general, be uniquely reconstructed from the retained part of the base alone without recording which possible explanation was selected. This aligns with the view that abductive revision is inherently iterated and requires a richer belief state than a base alone (Pagnucco 1996).

5 Related Work

Much of the work in this paper is based on Pagnucco's pioneering work on abduction in a belief-revision centered framework as laid out in his PhD thesis (1996). The main differences of our approach are the switch from belief sets to belief bases and the explicit account for a hypothesis space. Though Pagnucco discusses the general framework of abduction based on hypotheses, these have no active role in his abductive change operators. However, in later work (Pagnucco and Rajaratnam 2005), Pagnucco puts hypotheses into focus: he describes how to model inverse resolution (i.e., finding potential hypotheses that in a resolution step might allow one to deduce the resolvent) as abductive change operators.

In a similar AGM-oriented spirit, Lobo and Uzcátegui (1996) develop a general framework for abductive change operators, including abductive revision operators in the sense of Gärdenfors. In their framework, the notion of explanation is embedded in the revision process itself: a formula counts as a potential explanation for an observation only if revising the knowledge base by that formula, together with a domain theory, leads to a revised state in which the observation follows. Walliser, Zwirn, and Zwirn (2005) study four schemes of abduction based on belief revision and relate them to revision operators characterized by postulates. Their presentation is semantic, using subsets of possible worlds, and therefore remains closer to belief set revision than to syntactic belief base change. These approaches do not focus on the distinction between explicit and implicit beliefs or on the role of an explicitly given hypothesis space. Extending such AGM-oriented accounts of abductive change to belief base revision and corresponding base-level postulates would be an interesting topic for future work.

The hypothesis space is modeled in this paper explicitly as a set of finite sets. This simple data structure has been introduced under the term of a *flock* by Fagin and colleagues (1986) to model database updates. The theory of flocks has been taken on by Bochman (1999) (with a slight twist leading to different induced change operators) to the general setting of belief change. In both approaches, the flocks are used to give a fine-grained representation of a database or a belief base which allows to encode more information than just a set of sentences. In our approach, the main object of potential change is the belief base and the hypothesis space (alias the flock) is the auxiliary data structure that guides the change.

Classical belief revision is triggered by (potential) logical inconsistencies between the belief set (belief base) and the observed new information. In the framework of abduction it is rather the inability of the belief set/belief base to explain an observation which triggers a change – and the result is not only guaranteed to be consistent but also to entail the triggering observation. In that property, abductive change operators can be considered as a special form of belief merge under integrity constraints (Konieczny and Pérez 1999). An integrity constraint is given as a logical formula. A belief merge operator w.r.t. an IC merges (in the simple case) two belief bases into a new belief base that entails the IC. Now, abductive expansion (and revision) operators can be considered as belief merge operators where the observation takes both roles, that of the second belief base and the integrity constraint.

The descriptor revision framework (Hansson 2017) also generalizes the idea of consistency preservation. Descriptors are (possibly) complex constraints on the belief change result that, e.g., might express which sentences the agent in the resulting state should belief or not belief (or any other Boolean combinations of such statements). If we consider belief set based abduction operators, the relevant descriptor says that the given observation should be entailed. A belief based abduction operator rather would require that the explanations do not block the observation which amounts to believing that at least one of the hypothesis must be compatible with the current beliefs. The technical details building the bridge to descriptor revision is part of future work.

As abduction and explanation play a prominent role in general science practice, many applications for abductive change operators could be expected to be found in the dynamics of scientific knowledge. Hansson (2011) points out the commonalities but also the differences of belief dynamics according to belief revision vs. dynamics of scientific knowledge. He sketches a new approach for belief revision to give a better fit of the dynamics of scientific knowledge. In that approach, Hansson treats the distinction between explicit and implicit beliefs not as a static distinction but proposes a closure operator (which might be a classical consequence operator) that is applied on demand on a long-term (iterated) change of scientific corpora. It is w.r.t. such an approach that we consider our non-prioritized abduction operators as useful operators. The change operators may not lead to a successful explanation of the observation in the first step; but under later observations parts of the explanation will be put together in the iterative process so that then a closure of the belief base might lead to an explanation of the observations made in earlier steps.

6 Conclusion and Outlook

The basic rationale for abductive change is to make observations part of the firm beliefs only if the observation is explainable. Hence, instead of believing something (and acting) just because someone said so (as Simon Peter did following Jesus' commandment according to Luke 5:5), the agent seeks for a reason to do so. The two operators of abductive expansion and abductive revision follow this rationale in different ways, expansion rather conservatively and revision more creatively by eliminating (via suspension) blocking beliefs to regain possible explanations. The paper contributed to the area of abductive change by moving from the more well-behaved belief sets to belief bases. The resulting abductive change operators can be shown to be characterized by some intuitive postulates that account for the underlying hypothesis structure.

One main topic for future work is to define abductive change operators that are situated between abduction with belief sets and abduction with belief bases, in particular searching for operators that fulfill the (weak) success postulate (as with abductive revision for belief sets) and are still feasible (as with abductive belief base revision). One of the ideas to reach such a class is to close up belief bases and/or the hypotheses disjunctively (Hansson 1993). Another open topic is that of iterated abduction that goes beyond the simple two-step application of abduction for defining abductive revision as suspension followed by expansion. In this context, the concrete change of the hypothesis space in the steps will become of interest. Connections to learning on the hypothesis space and properties such as reliability will have to be investigated (Kelly 1998).

A Proofs

A.1 Proofs of Section 3.1

Proof of ***Proposition 3.1***. We distinguish two cases.

Case 1: $\mathcal{S}(B, H, \alpha) = \emptyset$. Then the expansion is vacuous, hence $B \oplus_H \alpha = B$, and the claim follows immediately since $B \subseteq B \cup \bigcup H$.

Case 2: $\mathcal{S}(B, H, \alpha) \neq \emptyset$. By (Witness), there exists some $\Delta \in \mathcal{S}(B, H, \alpha)$ such that all formulas newly added by the expansion are contained in Δ, i.e., $(B \oplus_H \alpha) \setminus B \subseteq \Delta$. Moreover, $\Delta \in \mathcal{S}(B, H, \alpha)$ entails $\Delta \in H$, hence $\Delta \subseteq \bigcup H$. Now every element of $B \oplus_H \alpha$ is either already in B or is newly added: $B \oplus_H \alpha \subseteq B \cup \big((B \oplus_H \alpha) \setminus B\big) \subseteq B \cup \Delta \subseteq B \cup \bigcup H$. Thus $B \oplus_H \alpha \subseteq B \cup \bigcup H$, as required. $\square$

Proof of ***Theorem 3.1***.
If part

(Failure): Follows directly from the definition.
(Inclusion): Follows directly from the definition.
(Witness): Let $T := \gamma(\mathcal{S}(B, H, \alpha))$. Since $\mathsf{Exp}(B, H, \alpha)$, we have $T \neq \emptyset$. Pick $\Delta \in T$ (hence $\Delta \in \mathcal{S}(B, H, \alpha)$). By Definition 4, $B \oplus^{\gamma}_{\mathrm{H}} \alpha = B \cup \bigcap T$. Therefore $B \oplus^{\gamma}_{\mathrm{H}} \alpha \setminus B \subseteq \bigcap T \subseteq \Delta$, where $\bigcap T \subseteq \Delta$ holds because $\Delta \in T$ and $\bigcap T$ contains exactly the formulas common to all members of T.
(Alternative): Let $T := \gamma(\mathcal{S}(B, H, \alpha))$ and $E := B \oplus^{\gamma}_{\mathrm{H}} \alpha$. Since $\mathsf{Exp}(B, H, \alpha)$, we have $T \neq \emptyset$ and, by Definition 4, $E = B \cup \bigcap T$. If $p \notin E$, then $p \notin \bigcap T$, hence there exists $\Delta \in T$ such that $p \notin \Delta$ (otherwise p would belong to the intersection). Moreover, $\Delta \in T$ implies $\bigcap T \subseteq \Delta$, so $E = B \cup \bigcap T \subseteq B \cup \Delta$. Finally, $T \subseteq \mathcal{S}(H, B, \alpha)$. Taking $S := T$, we obtain $S \subseteq \mathcal{S}(H, B, \alpha)$, $S \neq \emptyset$, and $\Delta \in S$ with $E \subseteq B \cup \Delta$ and $p \notin \Delta$.
(Uniformity): Follows from the definition.

Only if part

Given any *abductive selective expansion* operator $\oplus$ satisfying postulates ($B\oplus$ 1) –($B\oplus$ 5), construct the selection function:

$$\gamma_{\oplus}(H, B, \alpha) := \{\Delta \in \mathcal{S}(H, B, \alpha) \mid B \oplus_{\mathrm{H}} \alpha \setminus B \subseteq \Delta\}.$$

We want to show that $\gamma_{\oplus}$ is a selection function by Definition 3: (*selection*) holds by definition, (*empty-selection*) holds by definition (from set comprehension), (*non-emptiness*) if $\mathsf{Exp}(B, H, \alpha)$, (Witness) provides some $\Delta \in \mathcal{S}(H, B, \alpha)$ with $B \oplus_{\mathrm{H}} \alpha \setminus B \subseteq \Delta$; hence $\Delta \in \gamma_{\oplus}(H, B, \alpha)$, so $\gamma_{\oplus}(H, B, \alpha) \neq \emptyset$.

Finally, for $B \oplus_{\mathrm{H}} \alpha = B \cup \bigcap \gamma_{\oplus}(H, B, \alpha)$, we prove equality by showing both inclusions. Let $E := B \oplus_{\mathrm{H}} \alpha$ and $T := \gamma_{\oplus}(H, B, \alpha)$. (Under $\mathsf{Exp}(B, H, \alpha)$, we have $T \neq \emptyset$ by the argument above)

(i) $E \subseteq B \cup \bigcap T$. Take any $p \in E$. If $p \in B$, then $p \in B \cup \bigcap T$. Otherwise $p \notin B$, hence $p \in E \setminus B$. By definition of T, for every $\Delta \in T$ we have $E \setminus B \subseteq \Delta$, so $p \in \Delta$ for all $\Delta \in T$. Therefore $p \in \bigcap T$, and thus $p \in B \cup \bigcap T$.

(ii) $B \cup \bigcap T \subseteq E$. By (Inclusion), we have $B \subseteq E$. It remains to show $\bigcap T \subseteq E$. Assume for contradiction that there exists $p \in \bigcap T$ with $p \notin E$. By (Alternative), there exist a nonempty $S \subseteq \mathcal{S}(H, B, \alpha)$ and some $\Delta \in S$ such that $E \subseteq B \cup \Delta$ and $p \notin \Delta$.

From $E \subseteq B \cup \Delta$ it follows that $E \setminus B \subseteq \Delta$; together with $\Delta \in \mathcal{S}(H, B, \alpha)$ this yields $\Delta \in T$ by definition of T. But then $p \in \bigcap T$ implies $p \in \Delta$, contradicting $p \notin \Delta$. Hence $\bigcap T \subseteq E$, and therefore $B \cup \bigcap T \subseteq E$.

□

A.2 Proofs of Section 4.1

The proofs follow the kernel-contraction argument of Hansson (1994), adapted here to the abductive suspension setting.

Proof of ***Proposition 4.1***.
(S-Success): If $\mathsf{Exp}(B, H, \alpha)$, then by definition $B \oslash^{\sigma}_{\mathrm{H}} \alpha = B$, and we are done. Otherwise let $\mathcal{K} = B \perp\!\!\!\perp^{\mathrm{H}} \alpha$, let $S = \sigma(B \perp\!\!\!\perp^{\mathrm{H}} \alpha)$, and put $B_0 = B \setminus S$, so $B \oslash^{\sigma}_{\mathrm{H}} \alpha = B_0$. Suppose for contradiction that $\neg\mathsf{Exp}(B_0, H, \alpha)$. Since B_0 is finite and $\mathsf{Exp}(\emptyset, H, \alpha)$ (hence $\emptyset$ admits an explanation), there exists a nonempty $\subset$-minimal non-explanation admitting set $K \subseteq B_0$. By definition of kernel sets, $K \in \mathcal{K}$. Because σ hits every nonempty kernel, $K \cap S \neq \emptyset$, contradicting $K \subseteq B_0 = B \setminus S$. Hence $\mathsf{Exp}(B_0, H, \alpha)$, as required.
(S-Vacuity): trivial by definition.
(S-Inclusion): trivial by definition.
(S-Uniformity): From the assumption, for every $K \subseteq B$ we have $\mathsf{Exp}(K, H, \alpha)$ iff $\mathsf{Exp}(K, H, \beta)$. Equivalently, K blocks (i.e. does not admit an explanation of) α iff it blocks β. Hence the corresponding kernel families coincide, $B \perp\!\!\!\perp^{\mathrm{H}} \alpha = B \perp\!\!\!\perp^{\mathrm{H}} \beta$, since membership in a kernel family is defined purely in terms of being a $\subset$-minimal blocking subset of B.

Now consider two cases. If $\mathsf{Exp}(B, H, \alpha)$ holds, then by the standing assumption also $\mathsf{Exp}(B, H, \beta)$ holds (take $B' = B$). By definition of kernel suspension, both contractions are vacuous, so $B \oslash^{\sigma}_{\mathrm{H}} \alpha = B = B \oslash^{\sigma}_{\mathrm{H}} \beta$.

Otherwise, neither α nor β is explainable from B, and by the equality of kernel families above the same incision set is removed in both cases. Thus again $B \oslash^{\sigma}_{\mathrm{H}} \alpha = B \oslash^{\sigma}_{\mathrm{H}} \beta$.
(S-Core-Retainment): If $\mathsf{Exp}(B, H, \alpha)$, then by definition $B \oslash^{\sigma}_{\mathrm{H}} \alpha = B$, contradicting $\psi \in B \setminus B \oslash^{\sigma}_{\mathrm{H}} \alpha$. Hence assume $\neg\mathsf{Exp}(B, H, \alpha)$. Let $\mathcal{K} = B \perp\!\!\!\perp^{\mathrm{H}} \alpha$. By Definition 7, $B \oslash^{\sigma}_{\mathrm{H}} \alpha = B \setminus \sigma(\mathcal{K})$. Thus $\psi \in B \setminus B \oslash^{\sigma}_{\mathrm{H}} \alpha$ implies $\psi \in \sigma(\mathcal{K})$. Since $\sigma(\mathcal{K}) \subseteq \bigcup \mathcal{K}$, there is some $K \in \mathcal{K}$ with $\psi \in K$. By definition of $\mathcal{K}$, we have $K \subseteq B$, K blocks α (i.e. $\neg\mathsf{Exp}(K, H, \alpha)$), and K is $\subset$-minimal with this property, so every strict subset $K' \subset K$ admits an explanation of α.

Since $\psi \in K$, we have $K \setminus \{\psi\} \subset K$, hence by minimality $\mathsf{Exp}(K \setminus \{\psi\}, H, \alpha)$. Taking $B' := K$ yields $B' \subseteq B$, $\neg\mathsf{Exp}(B', H, \alpha)$, and $\mathsf{Exp}(B' \setminus \{\psi\}, H, \alpha)$, as required. □

Proof of ***Proposition 4.2***. Let $\oslash$ be any suspension operator satisfying postulates ($B \oslash 1$)–($B \oslash 5$) (and assume, as in the construction section, that blocking is monotone in the base: if $A \subseteq B$ and A does not admit an explanation of α under H, then neither does B). We define the *induced incision* by

$$\sigma_{\oslash}(B, H, \alpha) := \begin{cases} \emptyset, & \text{if } \mathsf{Exp}(B, H, \alpha), \\ B \setminus (B \oslash_{\mathrm{H}} \alpha), & \text{otherwise.} \end{cases}$$

Intuitively, $\sigma_{\oslash}(B, H, \alpha)$ is exactly the set of beliefs removed by $\oslash$ when α is unexplainable from B.

We want to show that $\sigma_{\oslash}$ is incision-like (for kernels). Let B, H, α be given, and assume B is finite and the hypothesis space is *rich enough*, i.e. $\mathsf{Exp}(\emptyset, H, \alpha)$. We show that $\sigma_{\oslash}(B, H, \alpha)$ behaves like an incision for the kernel family $B \perp\!\!\!\perp^{\mathrm{H}} \alpha$.

(i) Subset. We have $\sigma_{\oslash}(B, H, \alpha) \subseteq \bigcup B \perp\!\!\!\perp^{\mathrm{H}} \alpha$. To see this, let $\psi \in \sigma_{\oslash}(B, H, \alpha)$. Then $\psi \in B \setminus (B \oslash_{\mathrm{H}} \alpha)$, so by (S-Core-Retainment) there exists $B' \subseteq B$ such that B' does not admit an explanation of α but $B' \setminus \{\psi\}$ does admit an explanation of α. By finiteness, choose a $\subset$-minimal $K \subseteq B'$ such that $\psi \in K$ and K does not admit an explanation of α. Then $K \subseteq B$ and K is $\subset$-minimal among the sets that do not admit an explanation, hence $K \in B \perp\!\!\!\perp^{\mathrm{H}} \alpha$ and $\psi \in K$, which entails $\psi \in \bigcup B \perp\!\!\!\perp^{\mathrm{H}} \alpha$.

(ii) Hitting. For every $K \in B \perp\!\!\!\perp^{\mathrm{H}} \alpha$ with $K \neq \emptyset$, $K \cap \sigma_{\oslash}(B, H, \alpha) \neq \emptyset$. Otherwise, if some nonempty kernel K were disjoint from $\sigma_{\oslash}(B, H, \alpha)$, then $K \subseteq B \setminus \sigma_{\oslash}(B, H, \alpha) = B \oslash_{\mathrm{H}} \alpha$. By monotonicity of blocking, $B \oslash_{\mathrm{H}} \alpha$ would still not admit an explanation of α (since K does not), contradicting (S-Success), since it yields $\mathsf{Exp}\big(B \oslash_{\mathrm{H}} \alpha, H, \alpha\big)$. Hence every nonempty kernel is hit.

Thus $\sigma_{\oslash}$ is an incision function for $B \perp\!\!\!\perp^{\mathrm{H}} \alpha$.

Define the kernel construction induced by $\sigma_{\oslash}$ by

$$B \oslash_H^{\sigma} \alpha \;:=\; \begin{cases} B, & \text{if } \mathsf{Exp}\big(B, H, \alpha\big), \\ B \setminus \sigma_{\oslash}(B, H, \alpha), & \text{otherwise.} \end{cases}$$

By the definition of $\sigma_{\oslash}$, in the second case we have $B \setminus \sigma_{\oslash}(B, H, \alpha) = B \setminus (B \setminus (B \oslash_{\mathrm{H}} \alpha)) = B \oslash_{\mathrm{H}} \alpha$ (using $B \oslash_{\mathrm{H}} \alpha \subseteq B$ from S-Inclusion). In the first case, (S-Vacuity) gives $B \oslash_{\mathrm{H}} \alpha = B$. Hence, in all cases, $B \oslash_{\mathrm{H}} \alpha = B \oslash_H^{\sigma} \alpha$, so $\oslash$ is representable by the kernel construction in Definition 7. ☐

Proof of ***Proposition 4.3***. Assume $\mathsf{Exp}\big(B, H^B, \alpha\big)$. Then there exists $\Delta \in H^B$ such that $\Delta \cup B$ is consistent and $\Delta \cup B \vdash \alpha$. By definition of saturation, $\Delta = \Delta_0 \cup S$ for some $\Delta_0 \in H$ and $S \subseteq B$. Let $S' = B \setminus A$ and define $\Delta' = \Delta_0 \cup (S \cup S')$. Since $S \cup S' \subseteq B$, we have $\Delta' \in H^B$.

Moreover, using $A \subseteq B$ we get $A \cup S' = B$, hence

$$\Delta' \cup A = \Delta_0 \cup (S \cup S') \cup A = \Delta_0 \cup S \cup B = \Delta \cup B.$$

Thus $\Delta' \cup A$ is consistent and $\Delta' \cup A \vdash \alpha$. Therefore $\mathcal{S}\big(A, H^B, \alpha\big) \neq \emptyset$, and hence $\mathsf{Exp}\big(A, H^B, \alpha\big)$. ☐

Proof of ***Theorem 4.1***.
If part shown in Proposition 4.1.
Only if part shown in Proposition 4.2. ☐

A.3 Proofs of Section 4.2

Proof of ***Lemma 4.1***. Let $H_f := \textit{fresh}(H, B)$ and assume $B_0 \subseteq B$. We distinguish two cases.

Case 1: $\mathcal{S}(H_f, B_0, \alpha) = \emptyset$. Then the abductive expansion is vacuous, so $B_0 \oplus_{H_f} \alpha = B_0$. Hence $B \cap (B_0 \oplus_{H_f} \alpha) = B \cap B_0 = B_0$.

Case 2: $\mathcal{S}(H_f, B_0, \alpha) \neq \emptyset$. By (Witness), there exists some $\Delta \in \mathcal{S}\big(B_0, H_f, \alpha\big)$ such that $(B_0 \oplus_{H_f} \alpha) \setminus B_0 \subseteq \Delta$. Moreover, $\Delta \in \mathcal{S}\big(B_0, H_f, \alpha\big)$ entails $\Delta \in H_f$, hence $\Delta \cap B = \emptyset$. So, $\Big(\big(B_0 \oplus_{H_f} \alpha\big) \setminus B_0\Big) \cap B = \emptyset$, since any element added to B_0 is in Δ and thus cannot lie in B.

Now decompose the expansion result into its old and new parts: $B_0 \oplus_{H_f} \alpha \subseteq B_0 \cup \big((B_0 \oplus_{H_f} \alpha) \setminus B_0\big)$. Intersecting with B yields

$$B \cap (B_0 \oplus_{H_f} \alpha) = (B \cap B_0) \cup \Big(B \cap \big((B_0 \oplus_{H_f} \alpha) \setminus B_0\big)\Big).$$

$B \cap B_0 = B_0$ (since $B_0 \subseteq B$), and the second term is empty as shown above. Hence $B \cap (B_0 \oplus_{H_f} \alpha) = B_0$. ☐

Proof of ***Proposition 4.4***.
(R-Success): Let H be a set of hypotheses, α a formula, and B a finite belief base. Assume $\mathsf{Exp}\big(\emptyset, H, \alpha\big)$. Let $B_0 := B \oslash_{\mathrm{H}} \alpha$, $H_f := \textit{fresh}(H, B)$ and $E := B_0 \oplus_{H_f} \alpha$.

By (S-Success) (using finiteness of B and $\mathsf{Exp}\big(\emptyset, H, \alpha\big)$) we obtain $\mathsf{Exp}\big(B_0, H, \alpha\big)$. Since $B \otimes_{\mathrm{H}} \alpha = E$ by Definition 9, it suffices to show $\mathsf{Exp}\big(E, H, \alpha\big)$.

If $\mathcal{S}\big(B_0, H_f, \alpha\big) = \emptyset$, then expansion is vacuous and $E = B_0$, so $\mathsf{Exp}\big(E, H, \alpha\big)$ follows immediately.

Otherwise, choose $\Delta \in \mathcal{S}\big(B_0, H_f, \alpha\big)$ that witnesses the expansion, i.e. $E \setminus B_0 \subseteq \Delta$ (Witness). Then $\Delta \in H_f \subseteq H$ and, by the Definition 2, $(\Delta \cup B_0) \not\vdash \perp$ and $\Delta \cup B_0 \vdash \alpha$. Moreover, since E extends B_0 only by formulas already in Δ, we have $\Delta \cup E = \Delta \cup B_0$. Hence $(\Delta \cup E) \not\vdash \perp$ and $\Delta \cup E \vdash \alpha$, so $\Delta \in \mathcal{S}\big(E, H, \alpha\big)$ and therefore $\mathsf{Exp}\big(E, H, \alpha\big)$. Consequently, $\mathsf{Exp}\big(B \otimes_{\mathrm{H}} \alpha, H, \alpha\big)$.
(R-Inclusion): Let $B_0 := B \oslash_{\mathrm{H}} \alpha$ $H_f := \textit{fresh}(H, B)$.

From (H-Inclusion) of abductive expansion (Proposition 3.1), follows $B_0 \oplus_{H_f} \alpha \subseteq B_0 \cup \bigcup H_f$.

Moreover, because suspension (S-Inclusion), have $B_0 \subseteq B$, and freshness only restricts hypotheses, so $\bigcup H_f \subseteq \bigcup H$. Therefore, $B_0 \cup \bigcup H_f \subseteq B \cup \bigcup H$, and by transitivity, $B \otimes_H \alpha = B_0 \oplus_{H_f} \alpha \subseteq B \cup \bigcup H$, as required.
(R-Witness): Assume $\mathcal{S}(H, B, \alpha) \neq \emptyset$. Then $\mathsf{Exp}\big(B, H, \alpha\big)$, and hence, by (S-Vacuity), $B \oslash_H \alpha = B$. We distinguish two cases.

Case 1: $\mathcal{S}(H_f, B, \alpha) = \emptyset$. Then expansion is vacuous, so $B \oplus_{H_f} \alpha = B$, hence no formula is newly added: $(B \otimes_H \alpha) \setminus B = \emptyset$. Since $\mathcal{S}(H, B, \alpha) \neq \emptyset$, pick any $\Delta \in \mathcal{S}(H, B, \alpha)$; then trivially $(B \otimes_H \alpha) \setminus B \subseteq \Delta$.

Case 2: $\mathcal{S}(H_f, B, \alpha) \neq \emptyset$. By (Witness), there is $\Delta \in \mathcal{S}(H_f, B, \alpha)$ s.t. the newly added formulas satisfy $(B \oplus_{H_f} \alpha) \setminus B \subseteq \Delta$. Moreover, $\Delta \in H_f$ implies $\Delta \in H$, and since $\Delta \in \mathcal{S}(H_f, B, \alpha)$ we have that $\Delta \cup B$ is consistent and entails α; hence $\Delta \in \mathcal{S}(H, B, \alpha)$. Using $B \otimes_H \alpha = B \oplus_{H_f} \alpha$, we obtain $(B \otimes_H \alpha) \setminus B \subseteq \Delta$ as required.
(R-Relevance): To show the implication we assume $\beta \in B \setminus (B \otimes_H \alpha)$, i.e., that β is removed from the original base by revision. Let $B_0 := B \oslash_H \alpha$ and $H_f := \textit{fresh}(H, B)$.

By (Inclusion), $B_0 \subseteq B_0 \oplus_{H_f} \alpha$. Hence $B_0 \subseteq (B \otimes_H \alpha)$, and therefore $\beta \notin B_0$ (since otherwise β would belong to $B \otimes_H \alpha$). Together with $\beta \in B$, this yields $\beta \in B \setminus (B \oslash_H \alpha)$. Now apply (S-Core-Retainment) to obtain some $B' \subseteq B$ such that $\neg\mathsf{Exp}\big(B', H, \alpha\big)$ while $\mathsf{Exp}\big((B' \setminus \{\beta\}), H, \alpha\big)$.
(R-Uniformity): To show the implication we assume that for all $B' \subseteq B$ we have $\mathsf{Exp}\big(B', H, \alpha\big)$ iff $\mathsf{Exp}\big(B', H, \beta\big)$.

Let $B_0^{\alpha} := B \oslash_H \alpha$ and $B_0^{\beta} := B \oslash_H \beta$. By (S-Uniformity), this assumption implies $B_0^{\alpha} = B_0^{\beta}$ and by Definition 9, we have: $B \otimes_H \alpha = B_0^{\alpha} \oplus_{\textit{fresh}(H,B)} \alpha$ and $B \otimes_H \beta = B_0^{\beta} \oplus_{\textit{fresh}(H,B)} \beta$.

Since suspension satisfies (S-Inclusion), both B_0^{α} and B_0^{β} are subsets of B. Hence Lemma 4.1 applies and yields $B \cap (B \otimes_H \alpha) = B_0^{\alpha}$ and $B \cap (B \otimes_H \beta) = B_0^{\beta}$. Using $B_0^{\alpha} = B_0^{\beta}$, we get $B \cap (B \otimes_H \alpha) = B \cap (B \otimes_H \beta)$. ☐

The following proofs use the interdefinability argument between revision and contraction via the Harper and Levi identities (Alchourrón, Gärdenfors, and Makinson 1985; Harper 1976; Levi 1977); see also (Schulte 2006).

Proof of ***Proposition 4.5***. Let $C := B \oslash_{\mathrm{H}} \alpha$ and $H' := \mathit{fresh}(H, B) = \{\Delta \in H \mid \Delta \cap B = \emptyset\}$. By the Levi definition, $B \otimes_{\mathrm{H}} \alpha = C \oplus_{\mathrm{H'}} \alpha$.

We show $(C \oplus_{\mathrm{H'}} \alpha) \cap B = C$. Since $\oslash$ satisfies inclusion, $C \subseteq B$. Since $\oplus$ satisfies inclusion, $C \subseteq C \oplus_{\mathrm{H'}} \alpha$. Hence $C \subseteq (C \oplus_{\mathrm{H'}} \alpha) \cap B$.

For the converse, let $p \in (C \oplus_{\mathrm{H'}} \alpha) \cap B$. If $p \in C$ we are done. Otherwise $p \notin C$, so $p \in C \oplus_{\mathrm{H'}} \alpha \setminus C$ is newly added by the expansion. If $\mathcal{S}(C, H', \alpha) = \emptyset$, then by failure $C \oplus_{\mathrm{H'}} \alpha = C$, contradiction. Thus $\mathcal{S}(C, H', \alpha) \neq \emptyset$, and by the witness postulate for $\oplus$ there exists $\Delta \in \mathcal{S}(C, H', \alpha)$ such that $C \oplus_{\mathrm{H'}} \alpha \setminus C \subseteq \Delta$, hence $p \in \Delta$. But $\Delta \in \mathcal{S}(C, H', \alpha)$ implies $\Delta \in H'$, so $\Delta \cap B = \emptyset$. This contradicts $p \in \Delta$ and $p \in B$. Therefore no such p exists, and $(C \oplus_{\mathrm{H'}} \alpha) \cap B \subseteq C$.

So $(B \otimes_{\mathrm{H}} \alpha) \cap B = (C \oplus_{\mathrm{H'}} \alpha) \cap B = C = B \oslash_{\mathrm{H}} \alpha$. $\square$

Proof of ***Lemma 4.2***. Since $\Delta_i \in \mathcal{S}(H_f, C, \alpha)$, we have $\Delta_i \in H_f$ for $i \in \{1, 2\}$. By Definition 8, $\Delta_i \cap B = \emptyset$.

We show $R_1 \cap B = C$ (the case of R_2 is identical). Because $C \subseteq B$, we have $C \subseteq R_1 \cap B$. Conversely, let $\psi \in R_1 \cap B$. Then $\psi \in C \cup \Delta_1$ and $\psi \in B$. If $\psi \in \Delta_1$, then $\psi \in \Delta_1 \cap B = \emptyset$, contradiction. Hence $\psi \in C$, so $R_1 \cap B \subseteq C$. Thus $R_1 \cap B = C$. Finally, $\Delta_1 \neq \Delta_2$ implies $C \cup \Delta_1 \neq C \cup \Delta_2$, so $R_1 \neq R_2$. $\square$

B Bridge to the Previous Formulation

The formulation of (Bayerkuhnlein, Özçep, and Wolter 2026) returns both the changed base $B \oplus_{\mathrm{H}} \alpha$ and a conditioned hypothesis set $B \oplus_H^{\uparrow} \alpha$, whereas Section 3 uses only the belief base. The formulations correspond when $B \oplus_H^{\uparrow} \alpha \subseteq \mathcal{S}(B, H, \alpha)$, it is non-empty whenever $\mathsf{Exp}(B, H, \alpha)$, the base component is uniform in the sense of (Uniformity), B is returned when no hypothesis is selected, and $B \oplus_{\mathrm{H}} \alpha = B \cup \bigcap B \oplus_H^{\uparrow} \alpha$ whenever $B \oplus_H^{\uparrow} \alpha \neq \emptyset$.

Conversely, any operator satisfying $(B \oplus 1)$–$(B \oplus 5)$ induces this form by $B \oplus_H^{\uparrow} \alpha = \{\Delta \in \mathcal{S}(B, H, \alpha) \mid (B \oplus_{\mathrm{H}} \alpha) \setminus B \subseteq \Delta\}$. (Witness) gives non-emptiness and (Alternative) ensures that the original expansion result $B \oplus_{\mathrm{H}} \alpha$ is recovered as $B \cup \bigcap (B \oplus_H^{\uparrow} \alpha)$.

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