

Explanations for Unrealizability of Infinite-State Safety Shields

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Abstract

Safe Reinforcement Learning focuses on developing optimal policies while ensuring safety. A popular method to address such task is *shielding*, in which a correct-by-construction safety component is synthesized from logical specifications. Recently, shield synthesis has been extended to infinite-state domains, such as continuous environments. This makes shielding more applicable to realistic scenarios. However, often shields might be unrealizable because the specification is inconsistent (e.g., contradictory). In order to address this gap, we present a method to obtain simple unconditional and conditional explanations that witness unrealizability, which goes by temporal formula unrolling. In this paper, we show different variants of the technique and its applicability.

1 Introduction

Deep Reinforcement Learning (DRL) has been shown to successfully control reactive systems of high complexity (Marchesini and Farinelli 2020). However, despite their success, DRL controllers can cause even state-of-the-art agents to react unexpectedly (Goodfellow, Shlens, and Szegedy 2014). This issue raises severe concerns regarding the deployment of DRL-based agents in safety-critical reactive systems. Therefore, techniques that rigorously ensure the safe behaviour of DRL-controlled reactive systems have recently been proposed. One of them is *shielding* (Bloem et al. 2015; Alshiekh et al. 2018; Corsi et al. 2025), which incorporates an external component (a “shield”) that *enforces* an agent to behave safely according to a given specification φ specified in temporal logic. In this paper, we use post-shielding, where the shield does not interrupt the agent unless it violates a safety constraint.

Regularly, shields are built from specifications using reactive synthesis (Pnueli and Rosner 1989b; Pnueli and Rosner 1989a), in which, given a specification φ , a system is crafted that is guaranteed to satisfy φ for all possible behaviors of its environment. Realizability is the related decision problem of deciding whether such a system exists. This problem is well-studied for specifications written in Linear temporal logic (LTL) (Pnueli 1977; Jacobs et al. 2017). However, many realistic specifications use complex data, whereas LTL is inherently propositional. Alternatively, such realistic specifications can be expressed in LTL modulo theories (LTL $\mathcal{T}$), which replaces

atomic propositions with literals from a first-order theory $\mathcal{T}$ (Geatti, Gianola, and Gigante 2022), whose domain might be infinite (e.g., numbers). In (Rodríguez and Sánchez 2023; Rodríguez and Sánchez 2024b) an LTL $\mathcal{T}$ specification is translated into an equi-realizable Boolean LTL specification by (1) substituting theory literals by fresh Boolean variables, and (2) computing, using theory reasoning, an additional Boolean formula that captures the dependencies between the new Boolean variables imposed by the literals. This approach is called *Boolean abstraction* and paves the way to produce infinite-state shields, in which the input and output of the shield is no longer Boolean, but belonging to infinite data like numbers (Wu et al. 2019; Corsi et al. 2024; Rodríguez et al. 2025a; Kim et al. 2025).

However, it is usually the case that the shield is not realizable, which means that the uncontrollable environment has a way to violate the specification and the shield no longer provides safety guarantees. Thus, we want to understand why this happens: i.e., have an explanation. Recently, the problem of explainability in AI has gained traction, which has had a major impact on the interest in explaining the intricacies of reactive systems (Schewe and Finkbeiner 2007; Baier et al. 2021; Bassan et al. 2023). These approaches focus on either building simple reactive systems, tuning their behaviour or explaining, in each timestep, why the system produces an output to an environment input. However, these approaches (1) have not been used for infinite-state systems and (2) have not addressed the particular problem of understanding shields that are to be synthesised from specifications. Thus, in this paper, we want to open an alternative approach to address explainability: to analyze the LTL $\mathcal{T}$ specification φ of a shield itself in order to discover why φ is unrealizable (in case it is) via producing a simple witness of an environment strategy such that φ is violated.

Example 1. Consider the following LTL $\mathcal{T}$ specification:

$$\varphi = \Box[(x < 2) \rightarrow \bigcirc(y > 1)] \wedge [(x \geq 2) \rightarrow (y < x)],$$

where $x, y \in \mathbb{Z}$ and where x is uncontrollable (i.e., belongs to an environment player) and y is controllable (i.e., belongs to a system player).

Specification φ is unrealizable, which means that the environment player has a strategy to assign valuations to x such that φ is violated. Therefore, it can be argued that we can synthesise (e.g., using (Rodríguez and Sánchez 2024a;

Rodríguez, Gorostiaga, and Sánchez 2024)) the environment strategy for φ and analyze it in order to understand why φ is unrealizable. However, this strategy might be complex to interpret and extract simple explanations. Moreover, this extraction is not automatic and the procedure might not scale.

In this paper, we address these problems. We propose a general, reusable and an automatic framework to find unrealizability explanations: a bounded unrealizability checking via generating finite *unconditional* and *conditional strategies* as explanations for unrealizable $\text{LTL}_{\mathcal{T}}$ specifications. The idea with (1) unconditional explanations is to provide a finite set of environment assignments that shows why a specification is unrealizable. Alternatively, (2) conditional explanations mean that the environment observes the stateful memory in order to take a decision, which means that the explanation is a function over previous moves. Our general solution works as follows: (1) given φ , we construct an *unrolled* formula φ^k of length k that creates k copies of variables and instances of φ ; and (2) we use φ^k to find witnesses of plays that make the environment player reach a violation of φ . This search is described as a logical formula with different quantifier alternations, depending on the conditionality of the strategy sought (meaning that the environment is *conditioned* to a greater or lesser extent on previous moves). If the strategy is completely unconditional, then the formula is quantified with the form $\exists^* \forall^* \cdot \varphi$, whereas more conditional strategies involve some quantifier alternations such as $\exists^* \forall^* \exists^* \forall^* \cdot \varphi$, which can be understood as split points where the environment *observes* some prefix of the play. Our method explores all possible paths of length k and finds whether there is at least one violation of $\varphi_{\mathcal{T}}$ that can be achieved by the environment.

The resulting approach is sound, incomplete for unconditional explanations and complete up to k for conditional ones. In summary, the contributions of this paper are:

1. An approach to solve unconditional strategy search of $\text{LTL}_{\mathcal{T}}$ specifications using satisfiability solvers for quantified first-order theory formulae.
2. An alternative method that is based on a combination of a Boolean abstraction, followed by quantified Boolean formulae and a technique to produce values in $\mathcal{T}$. Unlike the first method, this alternative provides termination guarantees in each iteration of method. Both techniques perform an unrolling similar to bounded model checking where we use a k -length unrolling of the original specification.
3. An analogous technique to 1-2 to produce conditional explanations.
4. A case study to discuss the tradeoffs between different explanations and techniques to obtain them.
5. Empirical evaluation showcasing the scalability of the approach.

To the best of our knowledge, this is the first work that focuses on explaining $\text{LTL}_{\mathcal{T}}$ unrealizability and (infinite-state) shield unrealizability¹.

¹Due to space limitations, we refer the reader to the extended version (Rodríguez et al. 2025b) for further information about the encodings, examples with figures and empirical evaluation.

2 Preliminaries

2.1 Temporal Logic and Synthesis

LTL. We consider LTL (Pnueli 1977; Manna and Pnueli 1995), whose formulae contain atomic propositions, $\wedge$ and $\neg$ (the usual Boolean conjunction and negation², respectively), and $\bigcirc$ and $\mathcal{U}$ (the *next* and *until* temporal operators). The semantics of LTL formulae associates traces $\sigma \in \Sigma^\omega$ with LTL formulae as follows:

$$\begin{aligned} \sigma &\models \top && \text{always holds} \\ \sigma &\models a && \text{iff } a \in \sigma(0) \\ \sigma &\models \varphi_1 \vee \varphi_2 && \text{iff } \sigma \models \varphi_1 \text{ or } \sigma \models \varphi_2 \\ \sigma &\models \bigcirc \varphi && \text{iff } \sigma^1 \models \varphi \\ \sigma &\models \varphi_1 \mathcal{U} \varphi_2 && \text{iff for some } i \geq 0 \ \sigma^i \models \varphi_2, \text{ and} \\ &&& \text{for all } 0 \leq j < i, \sigma^j \models \varphi_1 \end{aligned}$$

where $\sigma \models \top$ always holds and from which we can also derive common operators like $\wedge$ and $\Box$ (which means *always*), etc. A safety formula φ is such that for every failing trace $\sigma \not\models \varphi$ there is a finite prefix u of σ , such that all σ' extending u also falsify φ (i.e. $\sigma' \not\models \varphi$).

Synthesis. Reactive synthesis is the problem of producing a system from an LTL specification, where the atomic propositions are split into propositions that are controlled by the environment and those that are controlled by the system. Realizability is the related decision problem of deciding whether such a system exists. Realizability corresponds to a turn-based game in a finite arena where, in each turn, the environment produces values of its variables (inputs) and the system responds with values of its variables (outputs).

We revise now conventions of reactive synthesis research: (1) A play is an infinite sequence of turns. (2) The system player wins a play according to an LTL formula φ if the trace of the play satisfies φ . (3) A strategy ρ of a player is a map from positions into a move for the player. (4) A play is played according to ρ if all the moves of the corresponding player are played according to ρ . Also, (5) ρ is winning for a player if all the possible plays played according to ρ are winning. If (6) ρ is winning for the system, the specification is said to be realizable (resp. unrealizable otherwise).

LTL Modulo Theories. A first-order theory $\mathcal{T}$ (Bradley and Manna 2007) consists of a finite set of functions and constants, a set of variables and a domain (which is the sort of its variables). Popular first-order theories are e.g., Presburger arithmetic or real arithmetic.

LTL Modulo Theories ($\text{LTL}_{\mathcal{T}}$) is the extension of LTL where propositions are replaced by literals from a given first-order theory $\mathcal{T}$ (a finite-trace version is given in (Geatti, Gianola, and Gigante 2022) and a infinite-trace version is given in (Rodríguez and Sánchez 2023)). The semantics of $\text{LTL}_{\mathcal{T}}$ associate traces $\sigma \in \Sigma_{\mathcal{T}}^\omega$ with formulae, where for atomic propositions $\sigma \models l$ holds iff $\sigma(0) \models_{\mathcal{T}} l$, that is, if the valuation $\sigma(0)$ makes the literal l true.

$\text{LTL}_{\mathcal{T}}$ realizability is analogous to LTL realizability, but it corresponds to a game in an arena where positions may have infinitely many successors if ranges of variables are infinite.

²We will use $\neg\varphi$ to represent negation of a formula and $\bar{a}$ to represent negation of an atom a .

2.2 Boolean Abstraction for LTL_T

Recently, the field of reactive synthesis beyond Booleans gained traction, and solutions were proposed for different fragments of LTL_T; e.g., very new works like (Heim and Dimitrova 2025; Rodríguez, Gorostiaga, and Sánchez 2025; Azzopardi et al. 2025). In this paper, we want our solution to be complete (up to a bound) and sound.

Therefore, we build upon (Rodríguez and Sánchez 2023), which showed that some fragments of reactive LTL_T specifications can be translated into equi-realizable purely Boolean LTL specifications via a procedure. The procedure is called Boolean abstraction or Booleanization, and it works as follows: given φ with literals l_i , we get a new specification $\varphi_{\mathbb{B}} = \varphi[l_i \leftarrow s_i] \wedge \varphi^{extra}$, where s_i are fresh Boolean variables, controlled by the system, that replace the literals. The additional subformula φ^{extra} uses s_i as well as additional Boolean variables e_k controlled by the environment, and captures that, for each possible e_k , the system has the *power* to select a response among a collection of choices, where each choice is a truth valuation of each the variables s_i that represent the literals. That is, a *choice* is a concrete valuation of the s_i variables (and hence of the literals l_i), and a *reaction* is a collection of choices. Pairs $(e_k, \bigvee_i (c_i))$ denote that a decision e_k of the environment can be responded by choosing one of the choices c_i in the disjunction. Then, the set of reactions captures precisely the finite collection of decisions of the environment and the resulting finite responses of the system. The set of valid reactions is determined by the literals in a specific theory $\mathcal{T}$.

Example 2. Consider φ from Ex. 1, for which the abstraction is as follows: $\varphi_{\mathbb{B}} = (\varphi'' \wedge \square[(e_0 \leftrightarrow \bar{e}_1) \wedge (e_0 \leftrightarrow \bar{e}_2) \wedge (e_1 \leftrightarrow \bar{e}_2) \rightarrow \varphi^{Extra}])$, where $\varphi'' = (s_0 \rightarrow \bigcirc s_1) \wedge (\bar{s}_0 \rightarrow s_2)$ is a direct translation of φ (s_0 abstracts $(x < 2)$, s_1 abstracts $(y > 1)$ and s_2 abstracts $(y < x)$) that over-approximates the power of the system. The additional subformula φ^{Extra} corrects the over-approximation and makes $\varphi_{\mathbb{B}}$ preserve the original decision power of each player:

$$\begin{aligned} e_0 &\rightarrow (s_0 \wedge s_1 \wedge \bar{s}_2) \vee (s_0 \wedge \bar{s}_1 \wedge s_2) \vee (s_0 \wedge \bar{s}_1 \wedge \bar{s}_2) \\ &\wedge \\ e_1 &\rightarrow (\bar{s}_0 \wedge s_1 \wedge \bar{s}_2) \vee (\bar{s}_0 \wedge \bar{s}_1 \wedge s_2) \\ &\wedge \\ e_2 &\rightarrow (\bar{s}_0 \wedge s_1 \wedge s_2) \vee (\bar{s}_0 \wedge s_1 \wedge \bar{s}_2) \vee (\bar{s}_0 \wedge \bar{s}_1 \wedge s_2) \end{aligned}$$

where e_0, e_1, e_2 belong to the environment and s_0, s_1 belong to the system (and all of them are Boolean). Intuitively, e_0 represents $(x < 2)$, e_1 represents $(x = 2)$ and e_2 represents $(x > 2)$. Choices are playable valuations in control of the system: $c_0 = \{s_0 \wedge s_1 \wedge s_2\}$, $c_1 = \{s_0 \wedge s_1 \wedge \bar{s}_2\}$, $\dots$ $c_7 = \{\bar{s}_0 \wedge \bar{s}_1 \wedge \bar{s}_2\}$. In other words, the system can respond to e_0 with either c_1 , c_2 or c_3 ; to e_1 with either c_5 or c_6 ; and to e_2 with either c_4 , c_5 or c_6 .

Note that e_1 results in a strictly more restrictive set of choices for the system than e_2 , which allows the system to choose one more valuation, specifically $c_4 = (\bar{s}_0 \wedge s_1 \wedge s_2)$. Thus, a “clever” environment will never play e_2 and it will play e_1 instead. Therefore, for simplicity in the paper, we will consider the simplified (equi-realizable) specification $\varphi_{\mathbb{B}} = [(s_0 \rightarrow s_1) \wedge (\bar{s}_0 \rightarrow s_2)] \wedge \square[(e_0 \leftrightarrow \bar{e}_1) \rightarrow \varphi^{Extra'}]$, where $\varphi^{Extra'}$ is a version of φ^{Extra} where e_2 is ignored.

3 Unconditional Explanations

3.1 Unconditional Encoding

The goal of this paper is to generate simple and automatic explanations for unrealizability in safety LTL_T specifications. We soon explain how we perform this automatization, but first introduce a notion of *simplicity*.

Since, in a reactive system, the interplay between the players can be very complex, it is appealing to find a strategy that reduces the noise of this interplay as far as possible: in other words, ignoring some moves of the (potentially infinite) interplay in order to get an explanation that is as close as possible to a prefix. This is particularly interesting in shields, because we provide a simple explanation regardless of how sophisticated the shielded policy is, and also regardless of how complex the environment is. In this paper, we designed these explanations in the form of *unconditional* and *conditional* strategies. Let us begin with the first ones:

Definition 1. Given a specification φ and a length k , we call *unconditional strategies* ρ_k to strategies in which environment can reach a violation of φ in $i \leq k$ timesteps with a sequence of moves that is independent of the system.

More formally, ρ_k is a constant function that assigns valuations to the environment variables up to length k .

Example 3. In φ of Ex. 1 it suffices for the environment to (1) play a value for x such that $(x < 2)$, for example $x : 1$, in timestep $i = 0$; and (2) play $x : 2$ in $i = 1$. This is a winning strategy for the environment in two steps, no matter what the system plays. Thus, we say that the environment has a two-step unconditional strategy (or explanation) in φ and we denote it $\rho_{k:2} = \{x^0 : 1, x^1 : 2\}$.

Note that, from this point onwards, we use the terms *explanation* and *strategy* interchangeably, whenever the meaning is clear from the context. Now, in order to generate unconditional explanations/strategies from φ of a length up to k , we propose to use *unrollings* of φ to formalize the existence of this statement in some logic. Unconditional properties are of the form *there is a sequence of moves of the environment such that, for all moves of the system, the formula is violated*, which corresponds to a prefix $\exists^* \forall^*$ formula.

Definition 2. Given a specification φ and a length k , we call *unconditional unrolling formula* to an encoding $\psi = \exists a^0, a^1, \dots, a^{k-1}. \forall b^0, b^1, \dots, b^{k-1}. \neg[\varphi_0 \wedge \varphi_1 \wedge \dots \wedge \varphi_{k-1}]$, where variables a_i are controlled by the environment and b_j belong to the system, and formulae φ_i correspond to instantiations of the specification φ at instant i .

Note that we negate $[\varphi_0 \dots]$ in our query, because the objective of the environment is to find a witness of the negation of φ . Also, note that copies φ_i resemble classical bounded model checking (Biere et al. 1999; Clarke et al. 2001).

Remark 1. Before the encoding, φ is transformed into negation normal form, and then specialized as $\varphi^{i,k}$ for every step i (and the maximum unrolling k), using the fix-point expansions of the temporal operators. When an appropriate solver for ψ is searching to make a formula $\top$, then every attempt to expand a sub-formula beyond k is replaced by $\perp$ (and vice-versa when seeking to make a formula $\perp$).

3.2 Method #1: A Q-SMT Encoding

For a specification φ in $\text{LTL}_{\mathcal{T}}$, ψ of Def. 2 is a quantified formula in some first-order theory $\mathcal{T}$, which is a natural encoding that can be solved using quantified satisfiability modulo theories (Q-SMT) procedures; for instance, (Cooper 1972) for integers or (Collins 1975) for reals.

Now, we detail our method and we start with $k = 1$ in order to find whether there is an environment one-step unconditional winning strategy $\rho_{k=1}$ for $\varphi_{\mathcal{T}}$.

Example 4. The 1-unrolling for φ in Ex. 1 is: $\exists x^0. \forall y^0. \neg \varphi^{0,1}$, where

$$\varphi^{0,1} = [((x^0 < 2) \rightarrow \top) \wedge ((x^0 \geq 2) \rightarrow (y^0 < x^0))],$$

and where $x^0, y^0 \in \mathbb{Z}$ are variables x and y instantiated in timestep $i = 0$. Note that the sub-formula $\bigcirc(y > 1)$ is replaced by $\top$ because it falls beyond the end of the unrolling, as in standard bounded model checking (as described in remark 1). Thus, the definitive encoding is:

$$\exists x^0. \forall y^0. \neg (((x^0 < 2) \rightarrow \top) \wedge ((x^0 \geq 2) \rightarrow (y^0 < x^0))),$$

which is unsatisfiable (i.e., *unsat*). Thus, there is no strategy $\rho_{k=1}$.

However, as seen in Ex. 3, there exists a two-step unconditional strategy $\rho_{k=2}$. Moreover, the following Q-SMT encoding (for unrollings depth $k = 2$) finds this witness.

$$\exists x^0, x^1. \forall y^0, y^1. \neg [\varphi^{0,2} \wedge \varphi^{1,2}], \text{ where}$$

$$\varphi^{0,2} = [((x^0 < 2) \rightarrow (y^1 > 1)) \wedge ((x^0 \geq 2) \rightarrow (y^0 < x^0))]$$

and

$$\varphi^{1,2} = [((x^1 < 2) \rightarrow \top) \wedge ((x^1 \geq 2) \rightarrow (y^1 < x^1))]$$

In this case, a Q-SMT solver will respond that the formula is satisfiable and will output a model such as $m = \{x^0 := 1, x^1 := 2\}$ as an assignment that satisfies it. We can see that m is precisely $\rho_{k=2}$ and it is easier to understand than inspecting the synthesized environment strategy.

In summary, this approach relies on incremental calls to Alg. 1, which receives the $\text{LTL}_{\mathcal{T}}$ specification $\varphi_{\mathcal{T}}$, an unrolling depth $k = \text{max}$, environment variable set X and system variable set Y . Alg. 1 also uses sub-procedures: (1) *copies*(A, n), which performs n timestep copies of the set A of variables; (2) *unroll*(φ, n), which performs m unrollings of φ ; (3) *QElim*(X, φ), which performs quantifier-elimination (QE) of the set Y from formula φ (which must be the inner set of variables); and (4) *witness*(φ), which returns a model of a satisfiable formula φ . Note that the unrolling formula F (line 5) has $\{x_0, \dots, x_n, y_0, \dots, y_n\}$ as free variables, G (line 6) quantifies universally over $\{y_0, \dots, y_m\}$ so it has $\{x_0, \dots, x_n\}$ as free variables, and therefore $\varphi_{\mathcal{T}}^{\text{smt}}$ (line 8) is quantifier-free with $\{x_0, \dots, x_n\}$ as free variables. Also, note that, for each of the Q-SMT queries to terminate, we require $\mathcal{T}$ to be decidable in the $\exists^* \forall^*$ -fragment.

Soundness of Alg. 1 is due to the following:

Theorem 1. If there is some unrolling depth k such that $\varphi_{\mathcal{T}}^{\text{smt}}$ is sat, then $\varphi_{\mathcal{T}}$ is unrealizable.

Algorithm 1 Unconditional bounded unrealizability check

Require: $\varphi_{\mathcal{T}}, \text{max}, X, Y$

- 1: **for** $n = 1$ to max **do**
- 2: $[y_0, \dots, y_n] \leftarrow \text{copies}(Y, n)$
- 3: $[x_0, \dots, x_n] \leftarrow \text{copies}(X, n)$
- 4: $F \leftarrow \text{unroll}(\varphi_{\mathcal{T}}, n)$
- 5: $G \leftarrow \forall y_0 \forall y_1 \dots \forall y_n. F$
- 6: $\varphi_{\mathcal{T}}^{\text{smt}} \leftarrow \text{QElim}([y_0, \dots, y_n], G)$
- 7: **if** $\neg \varphi_{\mathcal{T}}^{\text{smt}}$ is SAT **then**
- 8: **return** ($\text{true}, \text{witness}(\varphi_{\mathcal{T}}^{\text{smt}})$)
- 9: **end if**
- 10: **end for**
- 11: **return** uncertain

Proof. Let m be such that $\varphi_{\mathcal{T}}^{\text{smt}}$ is SAT and let $e_0, \dots, e_m$ be a model of $\varphi_{\mathcal{T}}^{\text{smt}}$. The unconditional strategy $\rho^{\mathbb{E}}$ that plays first e_0 , then e_1 , etc up to e_m is winning for the environment because it falsifies $\varphi_{\mathcal{T}}$. Note that the universal quantifiers for $y_0, \dots, y_m$ guarantee that all moves of the system for the first m steps are considered and in all cases $\varphi_{\mathcal{T}}$ is falsified. Since $\rho^{\mathbb{E}}$ is winning, then $\varphi_{\mathcal{T}}$ is unrealizable. $\square$

Note that Thm. 2, Thm. 4 and Thm. 6 follow analogously.

Remark 2. The Q-SMT encoding described in Ex. 4 and Alg. 1 is also suitable for theories whose satisfiability problems are semi-decidable: if m is obtained, then it is a legal witness of the desired strategy. For instance, we can use general-purpose SMT solvers such as Z3 (de Moura and Bjørner 2008) and encode unrollings for the theory of non-linear integer arithmetic.

Algorithm 2 Uncond. bounded unrealiz. check for LTL.

Require: $\varphi_{\mathbb{B}}, \text{max}, E, S$

- 1: **for** $n = 1$ to max **do**
- 2: $[s_0, \dots, s_n] \leftarrow \text{copies}(S, n)$
- 3: $[e_0, \dots, e_n] \leftarrow \text{copies}(E, n)$
- 4: $F \leftarrow \text{unroll}(\varphi_{\mathbb{B}}, n)$
- 5: $G \leftarrow \forall s_0 \forall s_1 \dots \forall s_n. F$
- 6: $\varphi_{\mathbb{B}}^{\text{qbf}} \leftarrow \text{QElim}([s_0, \dots, s_n], G)$
- 7: **if** $\neg \varphi_{\mathbb{B}}^{\text{qbf}}$ is SAT **then**
- 8: **return** ($\text{true}, \text{witness}(\varphi_{\mathbb{B}}^{\text{qbf}})$)
- 9: **end if**
- 10: **end for**
- 11: **return** uncertain

3.3 Method #2: Boolean Abstraction to QBF

Although Alg. 1 is sound, each Q-SMT query lacks termination guarantees (Bjørner and Janota 2015). Therefore, we propose an alternative method based on Boolean abstractions for which the evaluation of each unrolling is guaranteed to terminate. In this method, we generate unrolled formulae in QBF. This second method follows these steps: (1) we compute an equi-realizable purely Boolean LTL $\varphi_{\mathbb{B}}$ from φ following the Boolean abstraction method (Rodríguez and Sánchez 2023); (2) find an unconditional strategy $\rho^{\mathbb{B}}$ in $\varphi_{\mathbb{B}}$

using QBF solvers; and (3) translate $\rho^{\mathbb{B}}$ from Booleans to $\mathcal{T}$ using existential theory queries, generating a strategy for φ .

Example 5. Consider the search for an unconditional $\rho^{\mathbb{B}}$ for $\varphi_{\mathbb{B}}$ in Ex. 2. Let α_i^j , where i refers to the variable and j to the timestep. Again, k is the unrolling depth. We now encode the problem in QBF with unrolling $k = 1$:

$$\exists e_0^0, e_1^0. \forall s_0^0, s_1^0, s_2^0. \neg \varphi_{\mathbb{B}}^{0,1},$$

where $\varphi_{\mathbb{B}}^{0,1} = [(s_0^0 \rightarrow \top) \wedge (\overline{s_0^0} \rightarrow s_2^0)] \rightarrow [(e_0^0 \leftrightarrow e_1^0) \rightarrow \varphi_{\text{Extra}}^{0,1}]$ and where $\varphi_{\text{Extra}}^{0,1}$:

$$\begin{aligned} e_0^0 &\rightarrow (s_0^0 \wedge s_1^0 \wedge \overline{s_2^0}) \vee (s_0^0 \wedge \overline{s_1^0} \wedge s_2^0) \vee (s_0^0 \wedge \overline{s_1^0} \wedge \overline{s_2^0}) \\ &\wedge \\ e_1^0 &\rightarrow (\overline{s_0^0} \wedge s_1^0 \wedge \overline{s_2^0}) \vee (\overline{s_0^0} \wedge \overline{s_1^0} \wedge s_2^0) \end{aligned}$$

Note, again like in remark 1, that since $k = 1$, every attempt to access a proposition at time step greater than or equal k (for example, s_1^1) is replaced by $\top$. Hence, the formula $(s_0^0 \rightarrow \top)$ does not impose an actual constraint. As we saw in Subsec. 3.2, there is no strategy ρ for φ . Similarly, the QBF unrolling cannot find a strategy $\rho^{\mathbb{B}}$ for $\varphi_{\mathbb{B}}$, as the system only has to satisfy $(\overline{s_0^0} \rightarrow s_2^0)$, which happens if environment plays e_0^0 . This is as expected, since φ and $\varphi_{\mathbb{B}}$ should be equi-realizable and, thus, strategies should be mutually imitable (in this case, have the same length).

Analogously, since there is $\rho_{k=2}$ for φ , then there is a corresponding $\rho_{k=2}^{\mathbb{B}}$ for $\varphi_{\mathbb{B}}$ via a $k = 2$ QBF unrolling:

$$\exists e_0^0, e_1^0, e_0^1, e_1^1. \forall s_0^0, s_1^0, s_2^0, s_0^1, s_1^1, s_2^1. \neg (\varphi_{\mathbb{B}}^{0,2} \wedge \varphi_{\mathbb{B}}^{1,2}),$$

where $\varphi_{\mathbb{B}}^{0,1} = [(s_0^0 \rightarrow s_1^0) \wedge (\overline{s_0^0} \rightarrow s_2^0)] \rightarrow [(e_0^0 \leftrightarrow \neg e_1^0) \rightarrow \varphi_{\text{Extra}}^{0,1}]$, with $\varphi_{\text{Extra}}^{0,1} = \varphi_{\text{Extra}}^{0,2}$ and where $\varphi_{\mathbb{B}}^{1,2} = [(s_0^1 \rightarrow \top) \wedge (\overline{s_0^1} \rightarrow s_2^1)] \rightarrow [(e_0^1 \leftrightarrow e_1^1) \rightarrow \varphi_{\text{Extra}}^{1,2}]$, with $\varphi_{\text{Extra}}^{1,2}$:

$$\begin{aligned} e_0^1 &\rightarrow (s_0^1 \wedge s_1^1 \wedge \overline{s_2^1}) \vee (s_0^1 \wedge \overline{s_1^1} \wedge s_2^1) \vee (s_0^1 \wedge \overline{s_1^1} \wedge \overline{s_2^1}) \\ &\wedge \\ e_1^1 &\rightarrow (\overline{s_0^1} \wedge s_1^1 \wedge \overline{s_2^1}) \vee (\overline{s_0^1} \wedge \overline{s_1^1} \wedge s_2^1) \end{aligned}$$

Note again that an attempt to generate s_2^1 is replaced by $\top$, so the formula $(s_0^1 \rightarrow \top)$ again imposes no constraint. This time, the environment has a winning strategy if it plays e_0^0 and e_1^1 : playing e_0^0 in timestep $i = 0$ forces s_1^1 to hold in timestep $i = 1$; and e_1^1 in $i = 1$ forces s_2^1 in $i = 1$. Then, only $(\overline{s_0^1} \wedge s_1^1 \wedge \overline{s_2^1})$ and $(\overline{s_0^1} \wedge \overline{s_1^1} \wedge s_2^1)$ are valid responses of the system, but none of them satisfied both s_2^1 forced in timestep 1 and s_1^1 forced by the previous timestep, so the specification is inevitably violated. Thus, in $\varphi_{\mathbb{B}}$, it exists $\rho_{k=2}^{\mathbb{B}} = \{e_0^0, e_1^1\}$. Last, if we recall that e_0 represents $(x < 2)$ and e_1 represents $(x = 2)$, the strategy $\rho^{\mathbb{B}}$ in $\varphi_{\mathbb{B}}$ obtained with QBF encoding is coherent with $\rho_{k=2} = \{x^0 : 1, x^1 : 2\}$ obtained with Q-SMT $\varphi_{\mathcal{T}}$ in Ex. 4.

In summary, this approach starts by φ , performs abstraction $\varphi_{\mathbb{B}}$, and then again relies on incremental calls to an algorithm that is identical to Alg. 1, except for the fact that it receives $\varphi_{\mathbb{B}}$, Boolean environment variable set E and Boolean system variable set S , and does not solve an SMT query, but a QBF query (see Alg. 2). Soundness of this method follows analogously to Thm. 1, given that the Boolean abstraction method in use ensures equi-realizability of φ and $\varphi_{\mathbb{B}}$, which holds with (Rodríguez and Sánchez 2023).

Theorem 2. If there is some unrolling depth k such that $\varphi_{\mathbb{B}}^{qbf}$ is sat, then $\varphi_{\mathbb{B}}$ is unrealizable.

Remark 3. Even though the performance of QBF solvers degrades with quantifier alternations, modern solvers scale efficiently even for large formulae with $\exists^* \forall^*$ prefixes. Although scalability is not the goal of methods presented in this paper, QBF is a more mature technology than Q-SMT, which suggests that performance might also be gained regularly (see Sec. 6 for experiments in scalability).

Remark 4. Similar to remark 2, note that our method is agnostic to the abstraction method in use (even semi-decidable methods), as long as it preserves equi-realizability.

Remark 5. Similar to remark 3, note that $\rho^{\mathbb{B}}$ of Ex. 5, could also be related to other strategies in $\varphi_{\mathcal{T}}$ rather than ρ : indeed, $\rho' = \{x^0 : 0, x^1 : 2\}$, to $\rho'' = \{x^0 : -1, x^1 : 2\}$, to $\rho''' = \{x^0 : -2, x^1 : 2\}$ and an infinite amount of strategies that make the literal $(x^0 < 2)$ true.

Remark 6. One might wonder when is a Q-SMT encoding preferable to the QBF encoding. We believe there are at least three situations to consider: (1) If the abstraction is not terminating, then we can try a Q-SMT encoding. (2) If the theory $\mathcal{T}$ is undecidable, then still Q-SMT solvers have heuristics for semi-decidability that may produce explanations (which are correct by construction). (3) In very small instances, abstraction might consume most of the time of a abstraction+QBF query, so Q-SMT might be faster.

3.4 Strategy Deabstraction

Since the explanation of Alg. 2 is given in terms of Boolean variables, we need a technique to produce proper values in the domain of the theory $\mathcal{T}$. We describe now how to craft a strategy ρ from $\rho^{\mathbb{B}}$.

Definition 3. Consider an LTL $_{\mathcal{T}}$ specification φ and its equi-realizable abstraction $\varphi_{\mathbb{B}}$. Then, if $\rho^{\mathbb{B}}$ is winning for the environment in $\varphi_{\mathbb{B}}$, we call a strategy deabstraction function to a function $d : \rho^{\mathbb{B}} \rightarrow \rho$ such that if $\rho^{\mathbb{B}}$ is winning in $\varphi_{\mathbb{B}}$ then ρ is winning in φ .

In the case of unconditional strategies, these deabstracted strategies can be obtained leveraging the partitions used during the Boolean abstraction.

Algorithm 3 Conditional bounded unrealizability check

Require: $\varphi_{\mathcal{T}}$, $\max$, X , Y

```

1: for  $n = 1$  to  $\max$  do
2:    $[y_0, \dots, y_n] \leftarrow \text{copies}(Y, n)$ 
3:    $[x_0, \dots, x_n] \leftarrow \text{copies}(X, n)$ 
4:    $F \leftarrow \text{unroll}(\varphi_{\mathcal{T}}, n)$ 
5:    $G \leftarrow \text{alternats}([y_0, \dots, y_n], [x_0, \dots, x_n]).F$ 
6:    $\varphi_{\mathcal{T}}^{\text{smt}} \leftarrow \text{QElim}([y_0, x_1, y_1, x_2, \dots, x_n, y_n], G)$ 
7:   if  $\neg \varphi_{\mathcal{T}}^{\text{smt}}$  is SAT then
8:     return  $(\text{true}, \text{witness}(\varphi_{\mathcal{T}}^{\text{smt}}))$ 
9:   end if
10: end for
11: return uncertain

```

Example 6. Consider Ex. 2 and the two environment variables e_0 and e_1 , which are discrete partitions of the infinite input space of for variables of the environment. Recall from Sec. 2 that this partition comes from the fact that the abstraction algorithm found reactions $r_0 = (x < 2)$ and $r_1 = (x = 2)$ to be valid.

In other words, in order to get a literal in $\mathcal{T}$ from an environment variable e_k in $\varphi_{\mathbb{B}}$, it suffices to associate each e_k to the reaction formula in $\mathcal{T}$ that is valid.

Definition 4. Let function conv produce a formula in $\mathcal{T}$ from each e_k . Then, our deabstraction procedure is:

$$d_{\rho^{\mathbb{B}} \text{ to } \rho} : \bigwedge_{e_k^i \in \rho^{\mathbb{B}}} (\text{conv}(e_k^i))$$

This way, we only need to produce a $\mathcal{T}$ -valuation of x that satisfies the reaction associated to e_k .

Example 7. For the reactions in Ex 6, assignments $\exists x.r_0(x) \leftarrow 1$ and $\exists x.r_1(x) \leftarrow 2$ mean that e_0 , which represents, $(x < 2)$ has a witness $x : 1$; and e_1 , which represents $(x = 2)$, has the (only possible) witness $x : 2$. This allows to characterize the strategy of Ex. 2 as expected. Since the Boolean strategy of the environment is $\rho^{\mathbb{B}} = \{e_0^0, e_1^1\}$, then a possible $\mathcal{T}$ -strategy is $\rho = \{(x^0 : 1), (x^1 : 2)\}$, as predicted in Ex. 5.

We now formalize that any Boolean strategy $\rho_{\mathbb{B}} = \{e_i^0, e_j^1, \dots\}$ of the environment is related to a $\rho = \{(x^0 : a), (x^1 : b), \dots\}$, where $a, b \in \mathcal{T}$, in the sense that the outcomes of $\varphi_{\mathbb{B}}$ and $\varphi_{\mathcal{T}}$ are related in terms of literals they satisfy.

Theorem 3. Let φ and $\varphi_{\mathbb{B}}$ it Boolean abstraction. Then, an unconditional strategy ρ of length k exists if and only if $\rho^{\mathbb{B}}$ of length k exists.

Proof. The main idea is to create the corresponding sequence using Lemma 5 of (Rodríguez and Sánchez 2024b). It follows that if the system in $\varphi_{\mathcal{T}}$ can make a collection of literals at some point then the system can make the corresponding Boolean literal hold at the same point (and vice-versa). By structural induction, if the atoms have the same valuation then all sub-formulae have the same valuation. Therefore, given an arbitrary length k , if the unconditional strategy ρ_k in $\varphi_{\mathcal{T}}$ is winning for a player the strategy ρ_k in $\varphi_{\mathbb{B}}$ is winning for a player as well, and vice-versa. $\square$

Note that Thm. 8 follows analogously.

Remark 7. Thm. 3 is different to correctness theorems of (Rodríguez and Sánchez 2023), because we prove that a strategy of **same length** exists in both φ and $\varphi_{\mathbb{B}}$.

4 Conditional Explanations

In Sec. 3 we showed how to discover unconditional explanations for unrealizability, which are simple and preferable for the solvers (because of the lack of quantifier alternations). However, they carry two fundamental problems: (1) incompleteness (unsatisfiability of a k -step unrolling does not mean the formula is unrealizable in k timesteps); and

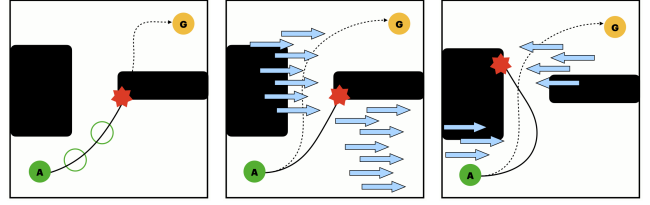


Figure 1: Illustration of agent behavior under different conditions.

(2) lack of practical applicability, due to the fact that often an unrealizable formula does not contain an unconditional strategy that explains it. Both problems require that the environment movements are no longer independent to the system, but instead has the ability to look at them in order to choose the move most promising to win.

Definition 5. Given a specification φ and a length k , we call conditional strategies ρ_k to strategies in which environment can reach a violation of φ in i timesteps with a sequence of moves that depends on previous moves of the system.

Example 8. We illustrate conditionality with φ' :

$$\begin{aligned} (x < 5) &\rightarrow \bigcirc^3(y > 9) \\ &\wedge \\ (5 \leq x < 10) &\rightarrow [((y < 0) \rightarrow \bigcirc(y > 9)) \\ &\quad \wedge ((y \geq 0) \rightarrow \bigcirc(y \leq 9))] \\ &\wedge \\ (10 \leq x \leq 15) &\rightarrow (y < x) \\ (x > 15) &\rightarrow (y > x), \end{aligned}$$

which is unrealizable. Again, we can synthesise an environment automata for φ' and again the corresponding automata is not easy to interpret. Instead, we can get both unconditional and conditional explanations.

First, φ' is unconditionally unrealizable in 4 timesteps: if the environment plays $x : (x < 5)$ in t_k , then $y > 9$ has to hold in t_{k+3} ; and if $x : 10$ in t_{k+3} , then $(y < x)$ has to hold (i.e., $y < 10$), which contradicts $(y > 9)$. Note that, if the first line of φ' did not exist, then there would not exist an unconditional strategy.

Additionally, φ does contain a shorter strategy of the environment to win, but this strategy is conditional (i.e., looks at the previous play of the system): concretely, the environment can play $x : (5 \leq x < 10)$ in t_k , then looks at whether the system played $(y < 0)$ or $(y \geq 0)$ in t_k , which imposes $(y > 0)$ or $(y \leq 9)$ respectively in t_{k+1} and thus the environment reacts accordingly in $k + 1$ by playing $(10 \leq x \leq 15)$ or $(x > 15)$ respectively in order to violate φ . Note how this strategy is shorter, but more complex.

In order to automatically produce these explanations, we now present conditional strategy search, for which we need to solve $(\exists^* \forall^*)^*$ formulae.

Definition 6. Given a specification φ and a length k , we call conditional unrolling formula to an encoding

$$\exists a^0. \forall b^0. \exists a^1. \forall b^1, \dots, \exists a^{k-1}. \forall b^{k-1}. \neg[\varphi_0 \wedge \dots \wedge \varphi_{k-1}],$$

where again variables a_i are controlled by the environment and b_j belong to the system and formulae φ_i correspond to instantiations of φ at instant i .

Example 9. Consider φ' from Ex. 8. Unfortunately, due to space limitations, we cannot show the 1-step unrolling (only note that its verdict is *unsat* and that both conditional and unconditional formulae are the same). Let us denote with Q an arbitrary quantification. Then, the 2-step unrolling is $Q.\neg[\varphi^{0,1} \wedge \varphi^{1,2}]$, where $\varphi^{0,1}$:

$$\begin{array}{ll} (x^0 < 5) \rightarrow & \top \\ (5 \leq x^0 < 10) \rightarrow & [((y^0 < 0) \rightarrow (y^1 > 9)) \\ & \wedge ((y^0 \geq 0) \rightarrow (y^1 \leq 9))] \\ (10 \leq x^0 \leq 15) \rightarrow & (y^0 < x) \\ (x^0 > 15) \rightarrow & (y^0 > x), \end{array}$$

and where $\varphi^{1,2}$:

$$\begin{array}{ll} (x^1 < 5) \rightarrow & \top \\ (5 \leq x^1 < 10) \rightarrow & \top \\ (10 \leq x^1 \leq 15) \rightarrow & (y^1 < x) \\ (x^1 > 15) \rightarrow & (y^1 > x) \end{array}$$

As we can see, an unconditional instantiation, $Q = \exists x^0.\forall y^0.\exists x^1.\forall y^1$, results in an *unsat* verdict (because there is no way the environment can win in 2 steps unconditionally, although it can do it in 3 steps as shown in Ex. 8). However, if the environment is allowed to observe the prefix of the play, then the conditional instantiation $Q' = \exists x^0.\forall y^0.\exists x^1.\forall y^1$ is *sat* with a model m' .

Moreover, since a conditional formula captures the environment's ability to look at the trace and adapt the behaviour, a model m' is no longer an assignment of environment variables to valuations; instead, this only happens in instant $i = 0$, whereas for any $i > 0$ the valuation of x^i is decided using a Skolem function that depends on the previous value of the system. For instance, $\{x^0 : 7, x^1 : f_{x^1}(y^0)\}$, where:

$$f_{x^1}(y^0) = \begin{cases} 10 & \text{if } (y^0 < 5) \\ 18 & \text{otherwise} \end{cases}$$

In summary, we can construct Alg. 3 where changes with respect to Alg. 1 are that (1) G from line 6 now represents quantifier alternations, (2) witness from line 8 does not produce constant Skolem functions and (3) line 11 returns *unreal* (see Thm. 5 below). Soundness of Alg. 3 for safety is due to the following:

Theorem 4. If there is some unrolling depth k such that $\varphi_{\mathcal{T}}^{smt}$ of Alg. 3 is *sat*, then $\varphi_{\mathcal{T}}$ is unrealizable.

Moreover, for arbitrary k , Alg. 3 is guaranteed to find a conditional strategy of length k , whenever φ is an unrealizable safety specifications. This provides k -completeness.

Theorem 5. If φ is an unrealizable safety formula in $\text{LTL}_{\mathcal{T}}$, then there is an unrolling k such that $\varphi_{\mathcal{T}}^{smt}$ of Alg. 3 is *sat*.

Proof. Since $\varphi_{\mathcal{T}}$ is unrealizable, there is some reachability goal that the environment satisfies. Moreover, since $\varphi_{\mathcal{T}}$ is in

safety, then this goal is reached in a finite number of steps. $\square$

Note that Thm. 7 follows analogously.

However, as in Sec. 3, extracting simple conditional explanations is a challenge and using Q-SMT for $(\exists^*\forall^*)^*$ can be intractable or even undecidable for some theories. On the other side, Boolean abstraction for $\text{LTL}_{\mathcal{T}}$ is decidable for $\exists^*\forall^*$ decidable theories, so whenever the abstraction is obtained, we can perform a QBF encoding that is analogous to the one in Subsec. 3.3: this approach starts by $\varphi_{\mathcal{T}}$, performs abstraction $\varphi_{\mathbb{B}}$ then again relies on incremental calls to an algorithm that is identical to Alg. 3, except for the fact that it receives $\varphi_{\mathbb{B}}$, Boolean environment variable set E and Boolean system variable set S , and does not solve an SMT query, but a QBF query. Soundness of and completeness of this method follow analogously.

Theorem 6. If there is some unrolling depth k such that conditional $\varphi_{\mathbb{B}}^{qbf}$ is *sat*, then $\varphi_{\mathbb{B}}$ is unrealizable.

Theorem 7. If φ is an unrealizable safety formula in $\text{LTL}_{\mathcal{T}}$, then there is an unrolling k such that cond. $\varphi_{\mathbb{B}}^{qbf}$ is *sat*.

However, again we need a deabstraction procedure, because the explanations that we will get will be made up of Boolean Skolem functions. Moreover, leaning on Subsec. 3.4, the deabstraction procedure for the Boolean Skolem function is trivial: (1) in timestep $i = 0$ it substitutes the environment assignments by reactions (exactly as in the unconditional case); whereas (2) in $i > 0$ it takes each Boolean Skolem function and performs a term substitution by replacing the domain (variables s^i by their literals in $\mathcal{T}$) and the co-domain (variables e^i by the reactions). Thus:

Theorem 8. Let φ and abstraction $\varphi_{\mathbb{B}}$. Then, an conditional strategy ρ of length k exists iff $\rho^{\mathbb{B}}$ of length k exists.

Remark 8. We can construct arbitrary versions of the algorithms presented in this paper if we consider semi-conditional strategies that are between conditional and unconditional. For instance, for $k = 10$, it can be the case that an unconditional unrolling (which has $a = 1$ quantifier alternations) is *unsat*, a conditional encoding (for which $a = 10$) is *sat* and some semi-conditional encoding (for which $1 < a < 10$) is *sat*.

Remark 9. It is very important to note that a conditional encoding is not necessarily harder to solve than unconditional encodings for QBF solvers. This is not surprising, since the number of quantifier alternations are not the sole deciding factors for hardness of a problem. For instance, in this paper, we presented Ex. 1 which can be solved using unconditional strategy. However, one could easily encode the same problem using a conditional strategy, but this does not increase the intrinsic hardness of the problem.

Indeed, every problem has its intrinsic hardness and it is not trivial to figure out at which layer such a problem belongs to in the polynomial hierarchy.

5 Discussion: What is a better explanation?

Since the unrealizability explanations that we find are the shortest possible (because the bounded search ends as soon

as a satisfiable assignment is found) it is easy to see that a shorter explanation is better than a longer one. However, there are other criteria for measuring quality of explanations: e.g., if we consider *conditionality*, then finding a strategy that is less conditional is better than a more conditional one, because the less conditional the strategy, more of its Skolem functions will be constants. We now formalize this intuition.

Definition 7. We measure the quality of an explanation as a function $q(k, o)$, where k is the length of the explanation and o is the amount of times the environment observes some prefix of the play (note $o = 0$ corresponds to unconditional).

Since the goal is to minimize q , then, given k and o for strategy ρ and k' and o' for explanation ρ' , where $k \leq k'$ and $o \leq o'$, then it is easy to see that $q(k, o) \leq q(k', o')$, which means q is a *better* explanation. However, what if $k \leq k'$ but $o > o'$, or vice-versa? For instance, in Ex. 9, is it better to have a longer unconditional explanation or a shorter conditional one? This is not always an easy choice and its automatization will heavily depend on our definition of q . Moreover, both kinds of explanations can be complementary, allowing us to find different unrealizability sources.

Example 10. Consider a particle scenario where a drone wants to reach a certain goal while avoiding dangerous zones. Also, there is an uncontrollable (environment) wind-turbulence that can affect the position of the drone. Now, the developers (1) train the drone using RL until they reach a desired success rate in satisfying the goal, and afterwards (2) design the specification φ of an infinite-state safety shield that would achieve collision avoidance. However, they find that φ is unrealizable. Moreover, the environment strategy is not easy to interpret (the automata is too big), so they use methods of Sec. 3 and Sec. 4. In particular, they find that there are both conditional and unconditional explanations.

We illustrate this using Fig. 1, where the green dot (A) represents the agent, and the yellow dot (G) represents the goal. Also, the dotted line indicates the agent's intended trajectory, while the solid line represents the actual trajectory executed. Now, in Fig. 1 (Left) the agent collides without external influences. (Middle) The agent attempts to follow a safe trajectory, but unavoidable external perturbations (the wind represented by blue arrows) push it towards the obstacle, resulting in a collision. (Right) The agent initially follows a safe trajectory but is influenced by the wind, causing a deviation from its intended path. Despite this disturbance, the agent successfully recovers and continues towards the goal. However, later, a stronger perturbation pushes the agent in the opposite direction, resulting in a collision. As we can see, both the unconditional and conditional strategies (i.e., **Middle** and **Right** resp.) allow us to discover information about φ to restrict the power of the environment towards a realizable version of φ : **Middle** shows that no matter what the agent does, the turbulence is too strong; whereas **Right** shows that φ is allowing an unrealistic dynamic such as changing the turbulence direction from timestep to timestep.

In summary, we acknowledge there is no unique rigorous formalization of what an explanation is³. In this section, we

proposed that minimal (k, o) pairs seem to correlate with human-judged interpretability.

6 Empirical Evaluation

Experimental setting. The main contribution of this paper is the method for obtaining simple explanations in reactive systems specified in $LTL_{\mathcal{T}}$. In addition, Sec. 5 shows cases in a qualitative manner how the method helps finding such explanations for shielding. Now, we perform a scalability evaluation with a prototype `unrealExplainerT` that takes an $LTL_{\mathcal{T}}$ formula and a bound k , produces the unrolling for both Q-SMT and QBF approaches, and returns the unrealizability explanation (in case it exists).

Concretely, we created two versions for each benchmark in (Rodríguez and Sánchez 2023): (1) if the specification does not have a strategy of length k (e.g., it is realizable), we introduced minimal modifications to have such a strategy; (2) if the specification did contain such strategy, then we made minimal modifications not to have the strategy. We tested both the original and modified versions, both QBF and Q-SMT and both conditional and unconditional cases.

Unconditional encoding. We can see results in Tab. 1, under the block *unconditional*. The first column corresponds to the name of the benchmarks (*nm.*) The two next columns shows variables (*vr.*) and literals (*lt.*) per benchmark, where gray colour indicates that the specification is unrealizable and white colour means the opposite.; The following two groups of columns show results of the execution of QBF and Q-SMT techniques for a different (at most) number of unrollings of the formula: 10, 50 and 100⁴. We show the time needed (in seconds) for each execution, where Preprocessing time (*Pre.*) in the QBF column is the time needed for computing the Boolean abstraction. Results show that, even if QBF has an initial Boolean abstraction cost, it quickly begins to perform better than Q-SMT in time and also reaches higher limits. Note that we encounter false negatives (i.e., the specification is unrealizable but because there is no unconditional strategy), but all the results are sound. Also, note that the time necessary for constructing the unrollings of both QBF and Q-SMT formulae is here negligible.

Conditional encoding. Similar experiments were conducted for conditional versions of the benchmarks, noting that this method offers completeness, but at the cost of an expected dramatical decrease in scalability. The *conditional* block of Tab. 1 confirms these results, where times increased a lot and Q-SMT usually does not go beyond 5 whereas QBF does not go beyond 20 (usually not beyond 10). Therefore, we can argue that, in case conditional explanations are sought, paying the price for the abstraction is absolutely worth it. Moreover, usually Q-SMT could not provide an answer (underlying SMT-solver responded N/A).

tions require cognitive models that are inherently hard to formalize.

⁴Note that an interval of 100 timesteps in industrial benchmarks happens because time of the original specifications is dense: this means that if there is a requirement that must be satisfied in 0.2 seconds and another one in 1 second, this imposes a discrete representation with 5 timestep horizon.

³Papers (Miller 2019) show that human-interpretable explana-

Bn. (nm.)	Cls. (v, l)	Unconditional							Conditional			
		QBF				Q-SMT			QBF		Q-SMT	
		Pre.	10	50	100	10	50	100	10	20	10	20
<i>Li.</i>	(5, 16)	4.41	0.02	0.44	9.85	0.32	0.64	14.08	422	-	611	-
	(5, 16)	3.71	0.02	0.36	8.00	0.26	0.52	11.44	341	-	493	-
<i>Tr.</i>	(19, 36)	5.13	0.01	0.53	11.70	0.38	0.76	16.72	507	-	-	-
	(19, 36)	5.01	0.01	0.57	12.62	0.41	0.82	18.04	544	-	-	-
<i>Con.</i>	(2, 2)	4.37	0.02	0.47	10.52	0.34	0.68	15.04	452	490	654	7029
	(2, 2)	4.34	0.02	0.51	11.39	0.37	0.74	16.28	498	536	741	-
<i>Tn.</i>	(8, 8)	7.04	0.02	0.43	9.54	0.31	0.62	13.64	411	4454	-	-
	(8, 8)	7.38	0.02	0.61	13.55	0.44	0.88	19.36	588	6563	824	-
<i>Air.</i>	(13, 14)	3.29	0.01	0.50	11.14	0.36	0.72	15.92	477	-	6838	-
	(13, 14)	4.51	0.01	0.40	8.93	0.29	0.58	12.76	397	-	5124	-
<i>Coo.</i>	(3, 5)	3.64	0.03	0.46	10.16	0.33	0.66	14.52	443	475	622	681
	(3, 5)	3.56	0.03	0.54	12.01	0.39	0.78	17.16	520	556	754	811
<i>Usb.</i>	(5, 8)	3.93	0.01	0.49	10.78	0.35	0.70	15.40	466	5151	6557	-
	(5, 8)	3.93	0.01	0.56	12.32	0.40	0.80	17.6	5343	5702	7837	-
<i>St.</i>	(11, 14)	6.06	0.02	0.39	8.62	0.28	0.56	12.32	3417	-	-	-
	(11, 14)	2.86	0.02	0.51	11.39	0.37	0.74	16.28	5162	-	-	-

Table 1: Results using both Q-SMT and QBF unrollings to find unconditional and conditional environment strategies of industrial benchmarks.

7 Final Remarks

Related Work. We classify our work in classic incremental SAT/SMT/QBF solving methods like CEGAR-loops (Clarke et al. 2000), which repeatedly check satisfiability while incrementally growing the formula/constraint set and bounded unroll constraints each iteration until the proof is found. Some concrete works are intended to find counter examples and counter traces for SMT solvers (e.g., (Chehida et al. 2021; Reynolds et al. 2015)). Also, QBF solvers have been used for such tasks (e.g., (Balabanov et al. 2015; Janota et al. 2016)) and (Hecking-Harbusch and Tentrop 2018) encodes Petri games in QBF and provides strategies of the environment. QBF has also been used for planning (Shaik and van de Pol 2022; Shaik et al. 2023).

Note that our fully conditional method, is not the same as bounded synthesis (Schewe and Finkbeiner 2007), because the latter bound strategies by size of the controller, not by a temporal size. However, our technique is similar to bounded model checking, BMC, (Biere et al. 1999; Clarke et al. 2001) in the sense that BMC algorithms unroll a finite-state automata for a fixed number of steps k , and check whether a property violation can occur in k or fewer steps. This typically involves encoding the restricted model as an instance of SAT. The process can be repeated with larger and larger values of k . Recently, QBF solving has been used in bounded model checking for hyperproperties (Hsu, Sánchez, and Bonakdarpour 2021), which opens the door for the same study over LTL $_{\mathcal{T}}$ properties (and also with loop conditions (Hsu, César Sánchez, and Bonakdarpour 2023)). Last, we find similar research directions in other temporal logics.; e.g., in STL (Maler and Nickovic 2004), the problem of falsification (i.e., achieving the violation of the proposed requirements) is considered, via different approaches (e.g., (Peltomäki and Porres 2022)).

Limitations and opportunities. In this work, we are restricted to the fragment of LTL $_{\mathcal{T}}$ for which synthesis is decidable. Therefore, future work includes proposing similar methods to temporal logics that allow to transfer richer data across time (e.g., infinite-trace (Geatti, Gianola, and Gigante 2022)). Also, we want to study usability of semi-conditional explanations (see remark 8), because they offer a trade-off between explainability, completeness and performance (e.g., to solve failures of Tab. 1) that seems promising.

Last, we consider that the discussion of Sec. 5 about optimizing different explanation criteria has to be widely made (and note that neurosymbolic approaches seem to be a key direction in order to optimize an eventual explainability function q). Thus, we want to integrate our solution under unified views of explainability. One example is comparing our work with conditional conformant planning (Smith and Weld 1998; Cimatti and Roveri 2000), where the objective is to find a sequence of actions that will guide the system to the desired state, regardless of the nondeterminism.

Conclusion. In this paper, we showed methods to find simple explanations of unrealizable LTL $_{\mathcal{T}}$ safety specifications, mostly designed for shields. We obtain such witnesses via unrollings of the formula up to a certain number k in a semi-complete spirit. We first showed that this method can be designed following an SMT-with-quantifiers (Q-SMT) encoding, but that this method lacks some termination guarantees and might not scale. Then, we showed a second method that uses an QBF encoding preceded by an abstraction process and a posterior deabstraction process. In both cases, we proposed a method to generate unconditional explanations (simpler, but less common) and conditional explanations (more complex, but complete). The paper is the basis for exciting work in other aforementioned directions.

Acknowledgments

This work was funded in part by the DECO Project (PID2022-138072OB-I00) funded by MCIN/AEI/10.13039/501100011033 and by the ESF+.

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