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On the Complexity and Properties of Preferential Propositional Dependence Logic

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Abstract

This paper considers the complexity and properties of KLM-style preferential reasoning in the setting of propositional logic with team semantics and dependence atoms, also known as propositional dependence logic. Preferential team-based reasoning is shown to be cumulative, yet violates System P. We give intuitive conditions that fully characterise those cases where preferential propositional dependence logic satisfies System P. We show that these characterisations do, surprisingly, not carry over to preferential team-based propositional logic. Furthermore, we show how classical entailment and dependence logic entailment can be expressed in terms of non-trivial preferential models. Finally, we present the complexity of preferential team-based reasoning for two natural representations. This includes novel complexity results for classical (non-team-based) preferential reasoning.

1 Introduction

Preferential reasoning in the style of Kraus, Lehmann and Magidor (1990)—henceforth abbreviated by KLM—is one of the main non-monotonic reasoning approaches that is well accepted in knowledge representation and reasoning, with connections to, e.g., belief change (Makinson and Gärdenfors 1991) and human-like reasoning (Ragni et al. 2020); see also Gabbay et al. (1993) and Brewka et al. (1997) for a general placement within non-monotonic reasoning. The semantic core of KLM-style preferential reasoning is its very elegant construction by preferential models. Roughly, a preferential model provides a strict partial order $\prec$ for a set of interpretations of some underlying logic (which is often classical propositional logic). Then, one says a formula ψ *is preferentially entailed from* φ if all $\prec$-minimal models of φ are models of ψ, i.e.,

$$\varphi \mathrel{|\!\sim} \psi \text{ if } \min(\llbracket\varphi\rrbracket, \prec) \subseteq \llbracket\psi\rrbracket.$$

Intuitively, when a non-monotonic inference $\varphi \mathrel{|\!\sim} \psi$ is generically understood as 'when φ holds, then usually ψ holds', the preferential reasoning reading of *'usually'* is *'one expects that'* (Gärdenfors and Makinson 1994). Hence, the intuition is that $\prec$ expresses a degree of exceptionality on the interpretations, i.e., the more preferred interpretations are less exceptional. Another feature of preferential reasoning is that it is exactly characterized by the System P postulates when the underlying logic is classical (KLM, 1990). Because the System P postulates are so widely accepted, preferential reasoning is sometimes considered as the 'conservative core of non-monotonic reasoning' (Pearl 1989; Gabbay 1984).

Team semantics is a logical framework for studying concepts and phenomena that arise in the presence of plurality of objects. These concepts include, e.g., functional dependence ubiquitous in database theory and conditional independence of random variables in statistics. The start of the field of team semantics can be traced back to the introduction of (first-order) dependence logic by Väänänen in (Väänänen 2007). In dependence logic, formulas are interpreted by sets of assignments (teams) instead of single assignments as in the usual classical semantics. Syntactically, dependence logic introduces new atomic formulas called dependence atoms $=\!(\vec{x}, y)$ expressing that the values of the variables $\vec{x}$ functionally determine the value of the variable y. During the past decade, the expressivity and complexity aspects of dependence logic and other team-based logics have been extensively studied and interesting connections have been found to areas such as database theory (Hannula, Kontinen, and Virtema 2020; Hannula and Kontinen 2016), meta-finite model theory (Hannula et al. 2020), inquisitive logic (Ciardelli, Iemhoff, and Yang 2020), and epistemic logic (Galliani 2015). These works focus on logics in the first-order, propositional and modal team semantics, and more recently also in the multiset (Durand et al. 2018a), probabilistic (Durand et al. 2018b) and semiring settings (Barlag et al. 2023).

In this paper, we study preferential propositional dependence logic, i.e., preferential entailment with propositional dependence logic as underlying logic. As far as the authors know, a merger of logics in team semantics and non-monotonic reasoning has not been studied so far except for (Yan 2023), where the former applies a certain (non-monotonic) team-based modal logic to the formal analysis of natural language. In the following, we present the motivation for our study and then present an overview of this paper, including our main contributions.

Motivation. Combining team-based reasoning and preferential reasoning is a promising way to obtain a novel conceptually rich family of reasoning approaches. Consider, for instance, the classical example with *birds* (b), *flies* (f), and penguin (p). First, preferential entailment $=\!(b, f) \mathrel{|\!\sim} \neg p$ reads technically as 'all maximally preferred teams that satisfy $=\!(b, f)$ also satisfy $\neg p$'. There is no obvious way to formulate the latter kind of expression in existing team-based logics, so injecting non-monotonicity is a valuable extension of team logic. Note that '$=(b,f) \mathrel{|\!\sim} \neg p$' does not imply that $=(b,f) \wedge p$ is inconsistent. Then, when employing the typical understanding of preferential reasoning as realising inference by expectation, we obtain the following. The dependence atom $=(b,f)$ expresses that whether it is a *bird* fully determines whether it *flies*. Thus, the (monotonic) entailment $=(b,f) \models \neg p$ states that

'*when whether it is a* bird *(b) determines whether it* fliesflies *(f), then it is not a penguin ($\neg p$)*'

and the preferential entailment $=(b,f) \mathrel{|\!\sim} \neg p$ reads as

'*when whether it is a* bird *(b) determines whether it* flies *(f), then one* ***expects*** *not a penguin ($\neg p$)*'.

This is an expression that preferential reasoning with an underlying classical logic does not permit. But we do not have to stop with this kind of understanding. A team corresponds to a plurality of objects, which permits various understandings of what a team stands for. Moreover, the preferential setting allows us to explore new understandings of the underlying order $\prec$. Dependent on the application context, one reads $=(b,f) \mathrel{|\!\sim} \neg p$, e.g., as follows:

- *Teams as databases:* 'When the value of b determines the value of f in a database, then one expects that the value of p is 0.'
- *Teams as possible worlds.* 'When the agent is convinced that whether f holds in a world always depends on b, then usually the agent expects that p does not hold.'
- *Teams as answers to a question (inquisitive reading).* 'One expects that p does not hold whenever in all answers f depends on b.'
- *Teams as datasets and $\prec$ orders them by reliability.* 'In all most reliable datasets in which b determines the value of f, it does not hold p.'

These are examples of interpretations of preferential dependence logic. We expect that one discovers more potential interpretations and applications of preferential team-based logics, when one considers preferential versions of other team-based logics.

Contributions. In this paper, we consider the complexity and properties of KLM-style preferential logics in the context of team-based logics. Specifically, we will encounter the preferential counterparts of the following logics:

- Propositional logic with classical semantics (CPL)
- Propositional logic with team-based semantics (TPL)
- Propositional dependence logic (PDL)

Our study will focus on preferential propositional dependence logic ($\mathsf{PDL}^{\mathsf{pref}}$). But, we will also discuss preferential entailment of propositional logic with classical semantics ($\mathsf{CPL}^{\mathsf{pref}}$) and team-based semantics ($\mathsf{TPL}^{\mathsf{pref}}$). The following list summarizes the main contributions of this paper:

- [*Relationship of* $\mathsf{PDL}^{\mathsf{pref}}$ *to System P.*] It is shown that $\mathsf{PDL}^{\mathsf{pref}}$ satisfies System C and violates System P. We present two properties, ($\star$) and ($\triangle$) (see p.5), for which each of them precisely characterize those preferential models in which System P is satisfied.
- [*Properties of* $\mathsf{TPL}^{\mathsf{pref}}$.] We observe that characterization of System P via ($\star$) and ($\triangle$) does not carry over to $\mathsf{TPL}^{\mathsf{pref}}$ from $\mathsf{PDL}^{\mathsf{pref}}$. This is surprising, as TPL is a fragment of PDL. It is shown that $\mathsf{TPL}^{\mathsf{pref}}$ still satisfies System C and violates System P.
- [*Complexity of Preferential Reasoning.*] We give a full classification in terms of tractable and intractable cases for the problem of inference from a given preferential model. Note that—unlike the problem of inference from a set of conditional assertions for $\mathsf{CPL}^{\mathsf{pref}}$ (Lehmann and Magidor 1992; Eiter and Gottlob 1992)—the complexity of inference from preferential models for $\mathsf{PDL}^{\mathsf{pref}}$, $\mathsf{CPL}^{\mathsf{pref}}$, and $\mathsf{TPL}^{\mathsf{pref}}$ has not been studied. We prove upper and lower bounds for the complexity of preferential classical propositional logic and preferential propositional dependence logic. Table 1 summarises these complexity results.

Problem	**Tract.**	**Complexity**	**Result**
$\textsc{Ent}(\mathsf{CPL}^{\mathsf{pref}})$	✓	$\in$ P, NC^1-hard	Thm. 28
$\textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})_{>_{\mathrm{lex}}}$	✗	Δ_2^p-complete	Thm. 31
$\textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})$	✗	$\in \Pi_2^p$, Δ_2^p-hard	Thm. 32
$\textsc{Ent}(\mathsf{PDL}^{\mathsf{pref}})$	✗	$\in \Theta_2^p$, NP-hard	Thm. 33
$\textsc{SuccEnt}(\mathsf{PDL}^{\mathsf{pref}})$	✗	$\in \Pi_2^p$, Δ_2^p-hard	Thm. 34
$\textsc{Ent}(\mathsf{TPL}^{\mathsf{pref}})$	✓	$\in$ P, NC^1-hard	Col. 35
$\textsc{SuccEnt}(\mathsf{TPL}^{\mathsf{pref}})$	✗	$\in \Pi_2^p$, Δ_2^p-hard	Col. 35

Table 1: Overview of novel complexity results for entailment based on preferential models. **Tract.** stands for tractability. $\textsc{Ent}/\textsc{SuccEnt}$ are the (succinct) entailment problem for preferential propositional logic (see Section 7 for the definition).

In the next section, we present the preliminaries on logic and computational complexity. Section 3 presents the background on preferential reasoning. In Section 4 we study the relationship of preferential propositional dependence logic to System P. A non-trivial preferential representation of standard entailment is presented in Section 5. In Section 6, we discuss implications for preferential proposition logic with team semantics. Section 7 is dedicated to presenting upper and lower bounds for the complexity of preferential entailment. Finally, Section 8 concludes the paper.

2 Preliminaries

We present the background on propositional logics with classical semantics, team semantics and propositional dependence logic (a survey on team-based logics can be found by Durand, Kontinen, and Vollmer, 2016). Furthermore, we present the background on computational complexity.

Language of Propositional Logic. We denote by $\mathsf{Prop} = \{ p_i \mid i \in \mathbb{N} \}$ the countably infinite set of propositional variables. We will use letters $p, q, r, \ldots$ (with or without subscripts) to stand for elements of Prop. In this paper, we consider propositional formulas in negation normal form, i.e., well-formed PL-formulas φ are formed by the grammar:

$$\varphi ::= p \mid \neg p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi,$$

where $p \in \mathsf{Prop}$, and $\top, \bot$ are the usual syntactic sugars for true and false. We write $\mathsf{Prop}(\varphi)$ for the set of propositional variables occurring in φ.

Classical Propositional Logic (CPL). We consider the *classical semantics* for PL-formulas. If one considers a non-empty finite subset $N \subseteq \mathsf{Prop}$ of propositional variables, then define for valuations $v\colon N \to \{0,1\}$ over N and PL-formulas φ:

$$[\![\varphi]\!]^c = \{\, v\colon N \to \{0,1\} \mid v \models \varphi\}.$$

We also write $v \models p$ in case $v(p) = 1$, and $v \not\models p$ otherwise. The valuation function v is extended to the set of all PL-formulas in the usual way. We denote by $\mathbb{A}_N$ the set of all assignments over N. Furthermore, we will write $\mathrm{PL}(N)$ for all propositional formulas using variables only from N. We write $\varphi \models^c \psi$ for $[\![\varphi]\!]^c \subseteq [\![\psi]\!]^c$ and $\varphi \equiv^c \psi$ if both $\varphi \models^c \psi$ and $\psi \models^c \varphi$. When we talk in this paper about CPL, we refer to the logic with PL-formulas and classical semantics.

Propositional Logic with Team Semantics (TPL). Next, we define *team semantics* for PL-formulas (cf. (Hannula et al. 2018; Yang and Väänänen 2016)). A team X is a set of valuations for some finite set $N \subseteq \mathsf{Prop}$. We write $\mathsf{dom}(X)$ for the domain N of X and $\mathbb{T}_N$ for the set of all such teams.

Definition 1 (Team semantics of PL)**.** *Let X be a team. For any PL-formula φ with $\mathsf{dom}(X) \supseteq \mathsf{Prop}(\varphi)$, the satisfaction relation $X \models \varphi$ is defined inductively as:*

$$\begin{array}{ll}
X \models p & \text{if for all } v \in X : v \models p\\
X \models \neg p & \text{if for all } v \in X : v \not\models p\\
X \models \bot & \text{if } X = \emptyset\\
X \models \top & \text{is always the case}\\
X \models \varphi \wedge \psi & \text{if } X \models \varphi \text{ and } X \models \psi\\
X \models \varphi \vee \psi & \text{if there exist } Y, Z \subseteq X\\
 & \text{ s.t. } X = Y \cup Z, Y \models \varphi, \text{ and } Z \models \psi.
\end{array}$$

The set of all teams X with $X \models \varphi$ is denoted by $[\![\varphi]\!]$. For any two PL-formulas φ, ψ, we write $\varphi \models^t \psi$ if $[\![\varphi]\!] \subseteq [\![\psi]\!]$. Write $\varphi \equiv^t \psi$ if both $\varphi \models^t \psi$ and $\psi \models^t \varphi$. With TPL we refer to the logical setting of PL-formulas with team semantics. We define the following properties for a formula φ:

- $X \models \varphi \iff$ for all $v \in X,\ \{v\} \models \varphi$. **(Flatness)**
- $\emptyset \models \varphi$. **(Empty team)**
- If $X \models \varphi$ and $Y \subseteq X$, then $Y \models \varphi$. **(Downwards closure)**

Proposition 2. TPL *has the flatness property, the empty team property, and the downward closure property.*

Due to the flatness property, logical entailment of propositional logic with team-based semantics $\models^t$ and logical entailment of propositional logic with classical semantics $\models^c$ coincide. However, we will see later that these different semantic approaches will lead to different preferential entailment relations.

Propositional Dependence Logic (PDL). A *(propositional) dependence atom* is a string $=\!(a_1 \dots a_k, b)$, in which $a_1, \dots, a_k, b$ are propositional variables from Prop. The team semantics of dependence atoms is defined as follows, whereby $\vec{a}$ stands for $a_1, \dots, a_k$:

$$X \models =\!(\vec{a}, b) \quad \text{if for all } v, v' \in X, \\ v(\vec{a}) = v'(\vec{a}) \text{ implies } v(p) = v'(p).$$

A dependence atom with the empty sequence in the first component will be abbreviated as $=\!(p)$ and called *constancy atoms*. The team semantics of the constancy atoms is reduced to

$$X \models =\!(p) \ \text{ if for all } v, v' \in X, v(p) = v'(p).$$

We define the language of *propositional dependence logic* (denoted as $\mathrm{PL}(=\!(,))$) as the extension of PL-formulas with dependence atoms. With PDL we refer to the whole logical approach, including the language and the above-mentioned semantics. We consider an example for PDL.

Example 3. *Consider the team X over $\{p, q, r\}$ defined by:*

	p	q	r
v_1	1	0	0
v_2	0	1	0
v_3	0	1	0

We have $X \models =\!(p, q)$ and $X \models =\!(r)$. Moreover, $X \models =\!(p) \vee =\!(p)$ but $X \not\models =\!(p)$.

The following proposition describes properties of PDL.

Proposition 4. PDL *has the empty team property and the downward closure property.*

Note that PDL does not satisfy the flatness property. We can define the flattening ϕ^f of a $\mathrm{PL}(=\!(,))$-formula by replacing all dependence atoms in ϕ by $\top$. Clearly, ϕ^f is a PL formula. Furthermore, one checks easily that $\phi \models^t \phi^f$ and that $\{v\} \models \phi \Leftrightarrow v \models \phi^f$ for all assignments v.

Generic View on Logics. Some parts of this paper will require a generic view on logics. The following provides a view that offers the right abstraction to capture the necessary aspects of logics in this paper in a generic way. A satisfaction system is a triple $\mathbb{S} = \langle \mathcal{L}, \Omega, \models \rangle$, where $\mathcal{L}$ is the set of formulas, Ω is the set of interpretations, and $\models\, \subseteq \Omega \times \mathcal{L}$ is the model-relation. An entailment relation for a satisfaction system is a relation $\mid\!\sim\, \subseteq \mathcal{L} \times \mathcal{L}$. A satisfaction system $\mathbb{S}$ together with an entailment relation $\mid\!\sim$ is denoted as logic $\mathscr{L} = \langle \mathcal{L}, \Omega, \models, \mid\!\sim \rangle$. We say a logic $\mathscr{L}$ is *standard*, if $\mid\!\sim$ is the canonical logical entailment given by $\varphi \mid\!\sim \psi$ if $[\![\varphi]\!] \subseteq [\![\psi]\!]$, whereby $[\![\varphi]\!] = \{v \in \Omega \mid v \models \varphi\}$. The canonical logical entailment is often written with the same symbol $\models$. The propositional logics described in this section provide the following instances of the generic approach for each $N \subseteq \mathsf{Prop}$, which are all standard logics:

$$\begin{aligned}
\mathsf{CPL}_N &= \langle \mathcal{L}_N^{\mathsf{CPL}}, \Omega_N^{\mathsf{CPL}}, \models, \models^{\mathsf{CPL}} \rangle = \langle \mathrm{PL}(N), \mathbb{A}_N, \models, \models^c \rangle\\
\mathsf{TPL}_N &= \langle \mathcal{L}_N^{\mathsf{TPL}}, \Omega_N^{\mathsf{TPL}}, \models, \models^{\mathsf{TPL}} \rangle = \langle \mathrm{PL}(N), \mathbb{T}_N, \models, \models^t \rangle\\
\mathsf{PDL}_N &= \langle \mathcal{L}_N^{\mathsf{PDL}}, \Omega_N^{\mathsf{PDL}}, \models, \models^{\mathsf{PDL}} \rangle = \langle \mathrm{PL}(=\!(,)), \mathbb{T}_N, \models, \models^t \rangle
\end{aligned}$$

Often, we will not mention N explicitly and assume that there is some N of appropriate size. Moreover, we will write $\models$ instead of $\models^t$ when there is no ambiguity.

Computational Complexity. We assume basic familiarity with computational complexity theory (Papadimitriou 1994). We will make use of the complexity classes P and $\Delta_2^p = \mathsf{P}^{\mathsf{NP}}$, as well as standard reducibility notions, e.g., logspace-many-to-one reductions $\leq_{\mathrm{m}}^{\log}$.

Before we define a natural complete problem for Δ_2^p, we need to formally introduce the lexicographic order on assignments. Let $P \subsetneq \mathsf{Prop}$ with $|P| = n$. An assignment function $v\colon P \to \{0,1\}$ is interpreted as a string v_s from $\{0,1\}^n$ in the natural way, i.e., $v_s = v(x_1)\cdots v(x_n)$ if $P = \{x_1,\ldots,x_n\}$. We denote by $v_s[i]$ for $1 \leq i \leq n$ the ith bit of v_s.

Definition 5 (lexicographic order)**.** *For two strings $s, s' \in \{0,1\}^n$ with $s \neq s'$ we say that $s <_{\mathrm{lex}} s'$ if there exists a (possibly empty) prefix $0 \leq j \leq n$ such that $s[k] = s'[k]$ for all $0 \leq k \leq j$ and $s[k+1] < s'[k+1]$.*

Next, we define the decision problem ODDLEXMAXSAT studied by Krentel (1988), which we abbreviate by OLMS.

Problem:	OLMS
Input:	A propositional formula φ over variables $\{x_1,\ldots,x_n\}$.
Question:	φ is satisfiable, and for the largest satisfying assignment θ with respect to $<_{\mathrm{lex}}$ do we have that $\theta(x_n) = 1$?

Proposition 6 (Krentel 1988)**.** OLMS *is Δ_2^p-complete under $\leq_{\mathrm{m}}^{\log}$-reductions.*

The complementary problem to OLMS is denoted here as $\overline{\text{OLMS}}$, where $\theta(x_n) \neq 1$. As Δ_2^p is a deterministic complexity class, $\overline{\text{OLMS}}$ of OLMS is also Δ_2^p-complete.

Now we will present the standard notion of a circuit and the notion of a circuit family. For a comprehensive overview of the topic of circuit complexity, refer to the textbook by Vollmer (1999).

Definition 7. *Let $B = \{\vee,\wedge,\neg,0,1\}$. A* Boolean circuit *over B with n inputs and one output gate is a tuple $C = (V, E, \alpha, \beta, \omega)$, where (V, E) is a finite directed acyclic graph, $\alpha\colon E \to \mathbb{N}$ is an injective function, $\beta\colon V \to B \cup \{x_1,\ldots,x_n\}$, and $\omega\colon V \to \{y_1,\ldots,y_m\} \cup \{*\}$ such that the following is true:*

1. *If $v \in V$ has in-degree 0, then $\beta(v) \in \{x_1,\ldots,x_n\}$ is an input gate, or $\beta(v)$ is a 0-ary Boolean constant from B.*
2. *If $v \in V$ has in-degree $k > 0$, then $\beta(v)$ is a k-ary Boolean function from B.*
3. *For every $1 \leq i \leq n$, there is at most one node $v \in V$ such that $\beta(v) = x_i$.*
4. *For every $1 \leq i \leq m$, there is exactly one node $v \in V$ such that $w(v) = y_i$.*

If $\beta(v) = x_i$ for some i then v is an input node*. If $\omega(v) \neq *$ then v is an* output node.

Definition 8. *Let $B = \{\vee,\wedge,\neg,0,1\}$. A* circuit family *over B is a sequence $\mathcal{C} = (C_0, C_1, \ldots)$, where for every $n \in \mathbb{N}$, C_n is a circuit over B with n inputs. Let f^n be the function computed by C_n. Then we say that $\mathcal{C}$ computes the function $f\colon \{0,1\}^* \to \{0,1\}^*$, defined for every $w \in \{0,1\}^*$ by $f(w) := f^{|w|}(w)$. We write $f = (f^n)_{n\in\mathbb{N}}$ and $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$.*

As usual, AC^0 denotes the class of all polynomial-sized circuit families of constant depth using gates with unbounded fan-in, i.e., the number of inputs a logic gate can handle. NC^1 denotes the class of all polynomial-sized circuit families of logarithmic depth using gates with fan-in two.

Example 9. *Given two binary strings $\bar{a} = a_{n-1}\cdots a_0, \bar{b} = b_{n-1}\cdots b_0$ (so the least significant bit is the rightmost bit), the lexicographic order $<_{\mathrm{lex}}$ can be defined via an AC^0-circuit, where the overline $\overline{\cdot}$ means an outermost negation*

$$\overline{\bigvee_{i=0}^{n-1}\big(a_i > b_i \wedge \bigwedge_{j=i+1}^{n-1}(a_j = b_j)\big)},$$

and conjunctions (disjunctions) over the empty set are defined as true (false). Here $a > b$ is then merely encoded via $a \wedge \neg b$, and $a = b$ with $(a \wedge b) \vee (\neg a \wedge \neg b)$.

3 Background on Preferential Logics

In this section, we present background on preferential logics in the style of Kraus, Lehmann and Magidor (1990) (KLM). In preferential logic, an entailment $\varphi \mathrel{|\!\sim} \psi$ holds, when minimal models of φ are models of ψ. This is formalized via preferential models, which we introduce in the following.

For a strict partial order $\prec \subseteq \mathcal{S} \times \mathcal{S}$ on a set $\mathcal{S}$ and a subset $S \subseteq \mathcal{S}$, an element $s \in S$ is called *minimal in S with respect to* $\prec$ if for each $s' \in S$ holds $s' \not\prec s$. Then, $\min(S, \prec)$ is the set of all $s \in S$ that are minimal in S with respect to $\prec$.

Definition 10 (KLM, 1990)**.** *Let $\mathscr{L} = \langle \mathcal{L}, \Omega, \models, \models^{\mathscr{L}} \rangle$ be a logic. A* preferential model *for $\mathscr{L}$ is a triple $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ where $\mathcal{S}$ is a set, $\ell\colon \mathcal{S} \to \Omega$, $\prec$ is a strict partial order on $\mathcal{S}$, and the following condition is satisfied:*

> [Smoothness] $\mathcal{S}(\varphi) = \{s \in \mathcal{S} \mid \ell(s) \models \varphi\}$ *is smooth with respect to $\prec$ for every formula $\varphi \in \mathcal{L}$, i.e, for each $s \in \mathcal{S}(\varphi)$ we have either $s \in \min(\mathcal{S}(\varphi), \prec)$ or there exists a state $s' \in \min(\mathcal{S}(\varphi), \prec)$ with $s' \prec s$.*

We say a preferential model is finite if $\mathcal{S}$ is finite.

Smoothness guarantees the existence of minimal elements. For convenience, we make use of the following abbreviations: $\min([\![\varphi]\!], \prec) = \{\ell(s) \mid s \in \min(\mathcal{S}(\varphi), \prec)\}$ and we write $s \models \varphi$ for $\ell(s) \models \varphi$.

Definition 11 (KLM, 1990)**.** *Let $\mathscr{L} = \langle \mathcal{L}, \Omega, \models, \models^{\mathscr{L}} \rangle$ be a standard logic. The entailment relation $\mathrel{|\!\sim}_{\mathbb{W}} \subseteq \mathcal{L} \times \mathcal{L}$ for a preferential model $\mathbb{W}$ for $\mathscr{L}$ is given by*

$$\varphi \mathrel{|\!\sim}_{\mathbb{W}} \psi \text{ if } \min([\![\varphi]\!], \prec) \subseteq [\![\psi]\!] \ .$$

An entailment relation $\mathrel{|\!\sim} \subseteq \mathcal{L} \times \mathcal{L}$ is called preferential *if there is a preferential model $\mathbb{W}$ for $\mathscr{L}$ such that $\mathrel{|\!\sim} = \mathrel{|\!\sim}_{\mathbb{W}}$.*

Every preferential model $\mathbb{W}$ for a standard logic $\mathscr{L} = \langle \mathcal{L}, \Omega, \models, \models^{\mathscr{L}} \rangle$ gives rise to a logic $\mathscr{L}_{\mathbb{W}} = \langle \mathcal{L}, \Omega, \models, \mathrel{|\!\sim}_{\mathbb{W}} \rangle$. We say that each such logic $\mathscr{L}_{\mathbb{W}}$ is a preferential logic for $\mathscr{L}$. *Preferential propositional dependence logic* ($\mathsf{PDL}^{\mathsf{pref}}$) refers to the preferential logics for PDL; and analogously for *preferential classical propositional logic* ($\mathsf{CPL}^{\mathsf{pref}}$) and *preferential propositional logic with team semantics* ($\mathsf{TPL}^{\mathsf{pref}}$).

4 On the Relationship of $\mathsf{PDL}^{\mathsf{pref}}$ to System P

In this section, we consider the satisfaction of properties by preferential propositional dependence logic. We make use of the following postulates for non-monotonic entailment $\mid\!\sim$:

$$\frac{}{\varphi \mid\!\sim \varphi}\ \text{(Ref)} \qquad \frac{\varphi \models \psi \quad \gamma \mid\!\sim \varphi}{\gamma \mid\!\sim \psi}\ \text{(RW)}$$

$$\frac{\varphi \equiv \psi \quad \varphi \mid\!\sim \gamma}{\psi \mid\!\sim \gamma}\ \text{(LLE)} \qquad \frac{\varphi \mid\!\sim \psi \quad \varphi \mid\!\sim \gamma}{\varphi \wedge \psi \mid\!\sim \gamma}\ \text{(CM)}$$

$$\frac{\varphi \wedge \psi \mid\!\sim \gamma \quad \varphi \mid\!\sim \psi}{\varphi \mid\!\sim \gamma}\ \text{(Cut)} \qquad \frac{\varphi \mid\!\sim \gamma \quad \psi \mid\!\sim \gamma}{\varphi \vee \psi \mid\!\sim \gamma}\ \text{(Or)}$$

Note that $\models$ is the entailment relation of the underlying monotonic logic, and $\equiv$ the respective semantic equivalence. The rules (Ref), (RW), (LLE), (CM) and (Cut) form *System C*. Notably, the rule (CM) goes back to the foundational paper on non-monotonic reasoning systems by Gabbay (1984) and is a basic weakening of monotonicity. *System P* consists of all rules of System C and the rule (Or). The rule of (Or) is motivated by reasoning by case (Pearl 1989).

A seminal result for CPL by KLM (1990) is the direct correspondence between preferential entailment and System P.

Proposition 12 (KLM, 1990)**.** *An entailment relation $\mid\!\sim$ for* CPL *satisfies System P if and only if $\mid\!\sim$ is preferential.*

System C and System P are of utmost importance for non-monotonic reasoning. System C is considered to be the basic properties of good non-monotonic reasoning, and System P is considered the 'conservative core' of non-monotonic reasoning (KLM, 1990).

Satisfaction of $\mathsf{PDL}^{\mathsf{pref}}$ of System C. Our first observation is that preferential entailment for PDL satisfies System C. This is a novel result as the proof of System C satisfaction for $\mathsf{CPL}^{\mathsf{pref}}$ given by KLM does not carry over to preferential propositional dependence logic.

Proposition 13. $\mathsf{PDL}^{\mathsf{pref}}$ *satisfies System C.*

Proof. We show that $\mid\!\sim_{\mathbb{W}}$ satisfies all rules of System C:
[*Ref.*] Considering the definition of $\mid\!\sim_{\mathbb{W}}$ yields that $\varphi \mid\!\sim_{\mathbb{W}} \varphi$ if for all minimal $s \in \mathcal{S}(\varphi)$ holds $\ell(s) \models \varphi$. By the definition of $\mathcal{S}(\varphi)$, we have $s \in \mathcal{S}(\varphi)$ if $\ell(s) \models \varphi$. Consequently, we have $\varphi \mid\!\sim_{\mathbb{W}} \varphi$.
[*LLE.*] From $\varphi \equiv \psi$, we obtain that $\mathcal{S}(\varphi) = \mathcal{S}(\psi)$ holds. By using this last observation and the definition of $\mid\!\sim_{\mathbb{W}}$, we obtain $\psi \mid\!\sim_{\mathbb{W}} \gamma$ from $\varphi \mid\!\sim_{\mathbb{W}} \gamma$.
[*RW.*] Clearly, by definition of $\varphi \models \psi$ we have $[\![\varphi]\!] \subseteq [\![\psi]\!]$. From the definition of $\gamma \mid\!\sim_{\mathbb{W}} \varphi$, we obtain that $\ell(s) \models \varphi$ holds for each minimal $s \in \mathcal{S}(\gamma)$. The condition $\ell(s) \models \varphi$ in the last statement is equivalent to stating $\ell(s) \in [\![\varphi]\!]$. Because of $[\![\varphi]\!] \subseteq [\![\psi]\!]$, we also have $\ell(s) \in [\![\psi]\!]$; and hence, $\ell(s) \models \psi$ for each minimal $s \in \mathcal{S}(\gamma)$. This shows that $\gamma \mid\!\sim_{\mathbb{W}} \psi$ holds.
[*Cut.*] By unfolding the definition of $\mid\!\sim_{\mathbb{W}}$, we obtain $\min(\mathcal{S}(\varphi \wedge \psi), \prec) \subseteq \mathcal{S}(\gamma)$ from $\varphi \wedge \psi \mid\!\sim_{\mathbb{W}} \gamma$. Analogously, $\varphi \mid\!\sim_{\mathbb{W}} \psi$ unfolds to $\min(\mathcal{S}(\varphi), \prec) \subseteq \mathcal{S}(\psi)$. Moreover, employing basic set theory yields that $\mathcal{S}(\varphi \wedge \psi) = \mathcal{S}(\varphi) \cap \mathcal{S}(\psi) \subseteq \mathcal{S}(\varphi)$ holds. From $\mathcal{S}(\varphi \wedge \psi) \subseteq \mathcal{S}(\varphi)$ and $\min(\mathcal{S}(\varphi), \prec) \subseteq \mathcal{S}(\psi)$, we obtain $\min(\mathcal{S}(\varphi), \prec) \subseteq \mathcal{S}(\varphi \wedge \psi)$. Consequently, we also have that $\min(\mathcal{S}(\varphi), \prec) = \min(\mathcal{S}(\varphi \wedge \psi), \prec)$ holds. Using the last observation and $\min(\mathcal{S}(\varphi \wedge \psi), \prec) \subseteq \mathcal{S}(\gamma)$, we obtain $\min(\mathcal{S}(\varphi), \prec) \subseteq \mathcal{S}(\gamma)$. Hence also $\varphi \mid\!\sim_{\mathbb{W}} \gamma$ holds.
[*CM.*] By unfolding the definition of $\mid\!\sim_{\mathbb{W}}$, we obtain $\min(\mathcal{S}(\varphi), \prec) \subseteq \mathcal{S}(\psi)$ and $\min(\mathcal{S}(\varphi), \prec) \subseteq \mathcal{S}(\gamma)$. We have to show that $\min(\mathcal{S}(\varphi \wedge \psi), \prec) \subseteq \mathcal{S}(\gamma)$ holds. Let s be element of $\min(\mathcal{S}(\varphi \wedge \psi), \prec)$. Clearly, we have that $s \in \mathcal{S}(\varphi)$ holds. We show by contradiction that s is minimal in $\mathcal{S}(\varphi)$. Assume that s is not minimal in $\mathcal{S}(\varphi)$. From the smoothness condition, we obtain that there is an $s' \in \mathcal{S}(\varphi)$ such that $s' \prec s$ and s' is minimal in $\mathcal{S}(\varphi)$ with respect to $\prec$. Because s' is minimal and because we have $\min(\mathcal{S}(\varphi), \prec) \subseteq \mathcal{S}(\psi)$, we also have that $s' \in \mathcal{S}(\psi)$ holds and hence that $s' \in \mathcal{S}(\varphi \wedge \psi)$ holds. The latter contradicts the minimality of s in $\mathcal{S}(\varphi \wedge \psi)$. Consequently, we have that $s \in \min(\mathcal{S}(\varphi), \prec)$ holds. Because we have $\min(\mathcal{S}(\varphi), \prec) \subseteq \mathcal{S}(\gamma)$, we obtain $\varphi \wedge \psi \mid\!\sim_{\mathbb{W}} \gamma$. ☐

Relationship of $\mathsf{PDL}^{\mathsf{pref}}$ to System P. The following example shows that Proposition 12 does not carry over to PDL, i.e., there are preferential entailments for PDL that witness a violation of (Or).

Example 14. *Assume that $N = \{p, q\} \subseteq \mathsf{Prop}$ holds. The following valuations v_1, v_2, v_3 will be important:*

$$v_1(p) = v_1(q) = v_2(q) = 1 \quad v_2(p) = v_3(p) = v_3(q) = 0$$

We consider the teams $X_{pq} = \{v_1\}$, $X_{\overline{p}q} = \{v_2\}$, and $X_{p\leftrightarrow q} = \{v_1, v_3\}$. Let $\mathbb{W}_{pq} = \langle \mathcal{S}_{pq}, \ell_{pq}, \prec_{pq} \rangle$ be the preferential model such that

$$\mathcal{S}_{pq} = \{s_X \mid X \text{ is a non-empty team}\} \quad \ell_{pq}(s_X) = X$$

holds, and such that $\prec_{pq}$ is the strict partial order given by (for the sake of readability, we identify s_X with X)

$$X_{p\leftrightarrow q} \prec_{pq} X_{pq} \qquad X_{pq} \prec_{pq} X$$
$$X_{p\leftrightarrow q} \prec_{pq} X_{\overline{p}q} \qquad X_{\overline{p}q} \prec_{pq} X$$

where X stands for every team different from $X_{\overline{p}q}$ and $X_{p\leftrightarrow q}$. We obtain the following preferential entailments:

$$p \mid\!\sim_{\mathbb{W}_{pq}} q \qquad \neg p \mid\!\sim_{\mathbb{W}_{pq}} q \qquad p \vee \neg p \not\mid\!\sim_{\mathbb{W}_{pq}} q$$

This shows that $\mid\!\sim_{\mathbb{W}_{pq}}$ violates (Or)*, and thus, System P.*

Proposition 15. $\mathsf{PDL}^{\mathsf{pref}}$ *violates System P.*

We will now identify two properties that will fully capture those preferential entailment relations that satisfy all rules of System P within PDL. First, we can observe that one obtains System P, when the underlying preferential model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ satisfies the following property:

$$\min([\![\varphi \vee \psi]\!], \prec) \subseteq \min([\![\varphi]\!], \prec) \cup \min([\![\psi]\!], \prec) \quad (\star)$$

One easily checks that $(\star)$ guarantees satisfaction of (Or).

Proposition 16. *Let $\mathbb{W}$ be a preferential model for* PDL*. If $(\star)$ is satisfied for all formulas φ, ψ, then $\mid\!\sim_{\mathbb{W}}$ satisfies* (Or)*.*

Second, we say that $\mathbb{W}$ satisfies the $(\triangle)$-property if for all state $s \in \mathcal{S}$ hold:

$$\text{If } |\ell(s)| > 1, \text{then } \ell(s') \subsetneq \ell(s) \text{ and } s' \prec s \text{ for some } s' \in \mathcal{S}. \quad (\triangle)$$

The ($\triangle$)-property demands (when understanding states as teams) that for each non-singleton team X exists a proper subteam Y of X that is preferred in $\mathbb{W}$ over X.

In the following theorem, we show that the $\triangle$-property and ($\star$)-property guarantee satisfaction of System P.

Theorem 17. *Let $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ be a preferential model for* PDL. *The following statements are equivalent:*

(a) $\mathrel{|}\!\sim_{\mathbb{W}}$ satisfies System P.

(b) $\mathbb{W}$ satisfies the $\triangle$-property.

(c) The ($\star$)-property holds for all $\varphi, \psi \in$ PDL.

We prepare the proof of Theorem 17 via the following lemmata. For the first lemma, assume that $N = \{p_1, \ldots, p_n\}$, and let X be an N-team. We define the following formula:

$$\Theta_X := \bigvee_{v \in X} (p_1^{v(1)} \wedge \cdots \wedge p_n^{v(n)}) \, ,$$

whereby $p_i^{v(i)}$ is p_i if $v(p_i) = 1$ and $\neg p_i$ otherwise. Note that for a team X, the powerset $\mathcal{P}(X)$ is the set of all subteams of X. It is straightforward to check the following.

Lemma 18. *Θ_X defines the family of subteams of X, i.e., we have that $[\![\Theta_X]\!] = \mathcal{P}(X)$.*

The next lemma guarantees that for a sufficiently large team X, there exist formulas φ, ψ such that X is a model of the disjunction $\varphi \vee \psi$, but X is not a model of φ and ψ.

Lemma 19. *For each team X with $|X| > 1$, there exist formulas φ and ψ such that*

$$X \models \varphi \vee \psi \, , \; X \not\models \varphi \, , \text{ and } X \not\models \psi .$$

Proof. Since $|X| > 1$ there exists non-empty $Y, Z \subseteq X$ such that $X = Y \cup Z$ and $Y \neq X$ and $Z \neq X$. There are formulas φ and ψ such that $[\![\varphi]\!] = \mathcal{P}(Y)$ and $[\![\psi]\!] = \mathcal{P}(Z)$, namely $\varphi = \Theta_Y$ and $\psi = \Theta_Z$ from Lemma 18. ☐

We will now show that the ($\triangle$)-property implies the ($\star$)-property, i.e., if a preferential model satisfies the ($\triangle$)-property, then the model also satisfies the ($\star$)-property.

Lemma 20. *If a preferential model $\mathbb{W}$ for* PDL *satisfies the ($\triangle$)-property, then it also satisfies the ($\star$)-property.*

Proof. Assume ($\triangle$) holds. Then it is easy to see that the minimal elements of the order $\prec$ are states that are mapped, via ℓ, to singleton teams. Furthermore, by the downward closure property, for any $\varphi \vee \psi$ the minimal teams satisfying the formula are all singletons. Since for singleton teams the interpretation of $\vee$ is equivalent to that of the Boolean disjunction, the ($\star$)-property follows. ☐

Next, we will show that satisfaction of (Or) guarantees, that the ($\triangle$)-property is always satisfied.

Lemma 21. *Let $\mathbb{W}$ be a preferential model for* PDL. *If $\mathrel{|}\!\sim_{\mathbb{W}}$ satisfies* (Or)*, then $\mathbb{W}$ satisfies the ($\triangle$)-property.*

Proof. The proof is by contraposition. For that, assume that ($\triangle$) fails. Then there exists a team X of size $j \geq 2$ such that for all $Y \subseteq X$, $Y \not\prec X$. Let $j = l + k$ ($l, k \geq 1$ and $l \leq k$) and define

$$\varphi := \Theta_X \wedge (\theta \vee \cdots \vee \theta),$$

where $\theta := \bigwedge_{1 \leq i \leq n} {=}(p_i)$ and φ has l many copies of θ. It is easy to check that φ is satisfied by subteams of X of cardinality at most l. The formula ψ is defined similarly with k copies of θ in the disjuncts. Now it holds that $\psi \models \psi$, $\varphi \models \psi$ but $X \not\models \varphi, \psi$. Using reflexivity and right weakening, it follows that $\psi \mathrel{|}\!\sim_{\mathbb{W}} \psi$ and $\varphi \mathrel{|}\!\sim_{\mathbb{W}} \psi$. On the other hand, since X is now a minimal model of $\varphi \vee \psi$ that does not satisfy ψ we have shown $\varphi \vee \psi \mathrel{|}\!\not\sim_{\mathbb{W}} \psi$ and that (Or) fails for $\mathrel{|}\!\sim_{\mathbb{W}}$. ☐

Now we are ready to give the proof of Theorem 17.

Proof of Theorem 17. From Lemma 20, we obtain ($\triangle$)⇒($\star$). From Proposition 16 we obtain ($\star$)⇒(Or). Moreover, from Lemma 21 we obtain (Or)⇒($\triangle$). Hence, we have a closed ring of implications and conclude (Or) ⇔ ($\triangle$) ⇔ ($\star$). ☐

In conformance with Theorem 17, the model $\mathbb{W}_{\mathsf{pq}}$ from Example 14 violates the ($\triangle$)-property and ($\star$)-property.

On the Expressivity on PDL$^{\mathsf{pref}}$ with System P. Next, we show that preferential models for System P reasoning are quintessentially the same as their flat (see Section 2) counterpart in CPL$^{\mathsf{pref}}$.

Theorem 22. *Let $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ be a preferential model for* PDL *such that $\mathrel{|}\!\sim_{\mathbb{W}}$ satisfies System P. Then, $\varphi \mathrel{|}\!\sim_{\mathbb{W}} \psi$ if and only if $\varphi^f \mathrel{|}\!\sim_{\mathbb{W}'} \psi^f$, where $\mathbb{W}' = \langle \mathcal{S}', \ell', \prec' \rangle$ denotes the preferential model for* CPL *induced by $\mathbb{W}$, i.e., one obtains $\mathbb{W}'$ from $\mathbb{W}$ by first removing all states labelled by non-singleton teams, then replacing labels of the singleton teams by their content.*

Proof. By construction, for all valuations s, s' it holds that $s \prec' s'$ if and only if $\{s\} \prec \{s'\}$. By Theorem 17, $\mathbb{W}$ satisfies the ($\triangle$)-property and hence the minimal elements of $\prec$ are singleton teams. Hence $\varphi \mathrel{|}\!\sim \psi$, if and only if, for all minimal $\{s\} \in [\![\varphi]\!]$ with $\{s\} \models \psi$, if and only if, for all $\prec'$-minimal $s \in [\![\varphi^f]\!] : s \models \psi^f$. The last equivalence holds due to the remark on flattening after Proposition 4. ☐

Theorem 22 demonstrates that System P reasoning in PDL$^{\mathsf{pref}}$ does not fully employ the underlying team semantics.

5 Preferential Reconstruction of PDL and CPL

In this section, we characterize the logical entailment $\models^t$ for PDL, as well as the logical entailment $\models^c$ for propositional logic with classical semantics in a non-trivial canonical way. Let $\mathbb{W}_{\mathsf{sub}} = \langle \mathcal{S}_{\mathsf{sub}}, \ell_{\mathsf{sub}}, \prec_{\mathsf{sub}} \rangle$ and $\mathbb{W}_{\mathsf{sup}} = \langle \mathcal{S}_{\mathsf{sup}}, \ell_{\mathsf{sup}}, \prec_{\mathsf{sup}} \rangle$ be the preferential models such that the following holds:

$$\begin{array}{l} \mathcal{S}_{\mathsf{sub}} = \mathcal{S}_{\mathsf{sup}} = \{ s_X \mid X \text{ is a non-empty team} \} \\ \ell_{\mathsf{sub}}(s_X) = \ell_{\mathsf{sup}}(s_X) = X \\ Y \prec_{\mathsf{sub}} X \text{ if } Y \subsetneq X \qquad Y \prec_{\mathsf{sup}} X \text{ if } X \subsetneq Y \end{array}$$

In $\mathbb{W}_{\mathsf{sub}}$ and $\mathbb{W}_{\mathsf{sup}}$, for each team X there is exactly one state s_X that is labelled by X. In $\prec_{\mathsf{sub}}$, subsets of a team are preferred, whereas in $\prec_{\mathsf{sup}}$, superset teams are preferred.

The preferential model $\mathbb{W}_{\mathsf{sup}}$ gives rise to the PDL entailment relation $\models$, and the preferential model $\mathbb{W}_{\mathsf{sup}}$ gives rise to CPL entailment of the flattening $\models^c$.

Proposition 23. *For all* $\mathrm{PL}(=(,))$*-formulas* φ, ψ *we have:*

(a) $\varphi \mathrel{|\!\sim}_{\mathbb{W}_{sub}} \psi$ *if and only if* $\varphi^f \models^c \psi^f$

(b) $\varphi \mathrel{|\!\sim}_{\mathbb{W}_{sup}} \psi$ *if and only if* $\varphi \models \psi$

Proof. We show statements (a) and (b) separately.
(a) First, observe at first that we have $\varphi \mathrel{|\!\sim}_{\mathbb{W}_{\text{sub}}} \psi$ exactly when we also have $\min(\llbracket\varphi\rrbracket, \prec_{\text{sub}}) \subseteq \llbracket\psi\rrbracket$. Because $\mathrm{PL}(=(,))$ has the downwards closure property, we also have that stating $\min(\llbracket\varphi\rrbracket, \prec_{\text{sub}}) \subseteq \llbracket\psi\rrbracket$ is equivalent to stating that for all singleton teams $\{v\}$ it holds that $\{v\} \models \varphi$ implies $\{v\} \models \psi$. The latter statement is equivalent to stating that for the flattening φ^f and ψ^f it holds that for all valuations v it holds that $v \models \varphi^f$ implies $v \models \psi^f$ (see also Section 2). Hence, we have $\varphi \mathrel{|\!\sim}_{\mathbb{W}_{\text{sub}}} \psi$ if and only if $\varphi^f \models^c \psi^f$.
(b) We obtain that holds $\models \subseteq \mathrel{|\!\sim}_{\mathbb{W}_{\text{sup}}}$ immediately by the definition of $\mathrel{|\!\sim}_{\mathbb{W}_{\text{sup}}}$. Next, we show that $\mathrel{|\!\sim}_{\mathbb{W}_{\text{sup}}} \subseteq \models$ holds. The statement $\varphi \models \psi$ is equivalent to $\llbracket\varphi\rrbracket \subseteq \llbracket\psi\rrbracket$. Because $\llbracket\varphi\rrbracket$ is downward-closed, there are (pairwise $\subseteq$-incomparable) teams $X_1, \ldots, X_n$ such that $\llbracket\varphi\rrbracket = \mathcal{P}(X_1) \cup \ldots \cup \mathcal{P}(X_n)$. Because of the last property, we have that $\varphi \models \psi$ holds exactly when $\{X_1, \ldots, X_n\} \subseteq \llbracket\psi\rrbracket$ holds. By construction of $\mathbb{W}_{\text{sup}}$ we have $\min(\llbracket\varphi\rrbracket, \prec_{\text{sup}}) = \{X_1, \ldots, X_n\}$ for φ. Thus, we also have that $\varphi \mathrel{|\!\sim}_{\mathbb{W}_{\text{sup}}} \psi$ holds and consequently, we also have $\mathrel{|\!\sim}_{\mathbb{W}_{\text{sup}}} \subseteq \models$. □

6 Implications for $\mathsf{TPL}^{\mathsf{pref}}$

We consider the preferential version of the fragment TPL of PDL. By inspecting Example 14 and the proof of Proposition 13, we observe that they also apply to $\mathsf{TPL}^{\mathsf{pref}}$.

Proposition 24. $\mathsf{TPL}^{\mathsf{pref}}$ *satisfies System C and violates System P.*

Surprisingly, Theorem 17 does not carry over to $\mathsf{TPL}^{\mathsf{pref}}$. In the following, we consider an example that witnesses that $(\star)$ and $(\triangle)$ do not characterize System P in $\mathsf{TPL}^{\mathsf{pref}}$.

Example 25. *Assume that* $N = \{p\} \subseteq \mathsf{Prop}$ *holds. There are exactly two valuations* v_p *and* $v_{\overline{p}}$ *with* $v_p(p) = 1$ *and* $v_{\overline{p}}(p) = 0$*. We consider the teams* $X_p = \{v_p\}$, $X_{\overline{p}} = \{v_{\overline{p}}\}$*, and* $X_{p\overline{p}} = \{v_1, v_2\}$*. Let* $\mathbb{W}_{\circledast} = \langle \mathcal{S}_{\circledast}, \ell_{\circledast}, \prec_{\circledast}\rangle$ *be the preferential model such that*

$$\mathcal{S}_{\circledast} = \{s_{X_p}, s_{X_{\overline{p}}}, s_{X_{p\overline{p}}}\} \qquad \ell_{\circledast}(s_X) = X$$

holds, and such that $\prec_{pq}$ *is the strict partial order given by (for the sake of readability, we identify* s_X *with* X*):*

$$X_{p\overline{p}} \prec_{\circledast} X_p \qquad\qquad X_{p\overline{p}} \prec_{\circledast} X_{\overline{p}}$$

One can check that for all PL*-formulas the postulates* (Or) *is satisfied. System P satisfaction follows then from Proposition 24. Clearly,* $X_{p\overline{p}} \prec_{\circledast} X_p$ *witness a violation of the* $(\triangle)$*-property. For a violation of the* $(\star)$*-property, we make the following observation:*

$$\begin{aligned}\min(\llbracket p \vee \neg p\rrbracket, \prec_{\circledast}) = \{X_{p\overline{p}}\} &\not\subseteq \{X_p\} \cup \{X_{\overline{p}}\} \\ &= \min(\llbracket p\rrbracket, \prec_{\circledast}) \cup \min(\llbracket \neg p\rrbracket, \prec_{\circledast})\end{aligned}$$

In summary, we obtain from Example 25 the following.

Proposition 26. $\mathbb{W}_{\circledast}$ *is a preferential model for* TPL *that violates* $(\star)$ *and* $(\triangle)$*, yet* $\mathrel{|\!\sim}_{\mathbb{W}_{\circledast}}$ *satisfies System P.*

Unfortunately, a corresponding alternative for $(\star)$ and $(\triangle)$ that characterizes System P within TPL eludes us so far.

Remark 27. *One might note that* $\mathbb{W}_{\circledast}$ *from Example 25 is also a preferential model for* PDL*. However, in compliance with Theorem 17, System P is violated by* $\mathrel{|\!\sim}_{\mathbb{W}_{\circledast}}$ *in* PDL*. To see this, we are setting* φ*,* ψ *and* γ *in* (Or) *to* $=(p)$*. When doing so, we obtain that* $=(p) \mathrel{|\!\sim}_{\mathbb{W}_{\circledast}} =(p)$ *and* $=(p) \vee =(p) \mathrel{|\!\not\sim}_{\mathbb{W}_{\circledast}} =(p)$ *hold. The latter is the case, because* $\min(\llbracket =(p) \vee =(p)\rrbracket, \prec_{\circledast}) = \{X_{p\overline{p}}\}$ *and* $X_{p\overline{p}} \not\models =(p)$ *holds. Clearly,* $=(p)$ *is not a* PL*-formula, and thus,* $=(p)$ *is also not a formula of* TPL*.*

7 Complexity of Entailment

In this section, we study the computational complexity of entailment-related problems. Informally this means, that we are given a preferential model $\mathbb{W}$ and two formulas φ, ψ as inputs. Then we ask whether $\varphi \mathrel{|\!\sim}_{\mathbb{W}} \psi$ is true.

Preferential Propositional Logic. Preferential models $\mathbb{W}$ encompass three components: a set $\mathcal{S}$, a labelling function ℓ, and an order $\prec$. Let us first define the problem of interest.

Problem:	$\textsc{Ent}(\mathsf{CPL}^{\mathsf{pref}})$ — entailment problem for preferential propositional logic
Input:	A finite preferential model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec\rangle$ for CPL and $\varphi, \psi \in \mathrm{PL}$.
Question:	Is it true that $\varphi \mathrel{|\!\sim}_{\mathbb{W}} \psi$?

First, we see how this problem can be solved in polynomial time in a brute-force approach.

Theorem 28. $\textsc{Ent}(\mathsf{CPL}^{\mathsf{pref}})$ *is in* P *and* NC^1*-hard under* $\leq_m^{\mathsf{AC}^0}$*-reductions.*

Proof. We show membership and hardness separately.
[*Membership*.] Consider an input $\mathbb{W}, \varphi, \psi$ with $\mathbb{W} = \langle \mathcal{S}, \ell, \prec\rangle$ as defined above. We construct a polynomial-time algorithm in the following:

1. Check for every $s \in \mathcal{S}$ whether $s \models \varphi$ and place a mark in $\mathcal{S}$ at this element if yes.
2. In the corresponding graph $(\mathcal{S}, \prec)$ search for all marked minimal elements s and check if $s \models \psi$. If not, reject.
3. Accept.

The $\models$ checks are in $\mathsf{NC}^1 \subseteq \mathsf{P}$ (Buss et al. 1992). The minimum search is a simple graph search for minimal elements in DAGs, which can be done in time linear in the size of the graph $(\mathcal{S}, \prec)$.
[*Hardness*.] The model checking problem for CPL is NC^1-complete (Buss et al. 1992). Given a propositional assignment θ and a propositional formula φ, reduce it quite directly as follows showing NC^1-hardness:

$$(\theta, \varphi) \mapsto ((\{\theta\}, \mathrm{id}_{\mathcal{S}}, \emptyset), \top, \varphi).$$

As $\mathcal{S}$ contains only one element which also is satisfied by the first formula $\top$, we require it to satisfy also φ. This is equivalent to the model checking problem for CPL. The reduction is a sheer mapping of values or constant parts, so computable by an AC^0 circuit family. $\square$

Clearly, there are exponentially many assignments in the number of variables of a considered formula. This can easily result in an exponentially large set $\mathcal{S}$ and hides the "real" complexity of the problem. Accordingly, we want to consider a more succinct version of the problem. We approach this observation with the following definition.

Definition 29. *Let $N \subseteq \mathsf{Prop}$ be a set of propositions with $|N| = n$, $\mathcal{S} = \{0,1\}^m$ be a set for $m \in n^{O(1)}$, and $\prec \subseteq \mathcal{S} \times \mathcal{S}$ be a strict partial order. Now let $\mathbb{W}$ be a preferential model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ such that $\ell\colon \mathcal{S} \rightharpoonup \mathbb{A}_N$ is a partial labelling. Let there be two $n^{O(1)}$-sized circuit families $\mathcal{L}, \mathcal{O}$ (labelling, ordering) such that the following is true:*

1. *ℓ is computed by $\mathcal{L}$,*
2. *$\mathcal{O}\colon \mathcal{S} \times \mathcal{S} \rightharpoonup \{0,1\}$ is a partial function such that for $s, s' \in \mathcal{S}$, the circuit outputs 1 if and only if $s \prec s'$ is true.*

We call $(\mathcal{L}, \mathcal{O})$ an $n^{O(1)}$-sized circuit representation *of* $\mathbb{W}$.

Remark 30. *Notice that the size of $\mathcal{S}$ is always 2^m. However, the two circuit families $\mathcal{L}$ and $\mathcal{O}$ need to deal with so-to-speak irrelevant input strings in a reasonable way. In this light, the preimage of the partial function ℓ induces what part of $\mathcal{S}$ is relevant.*

Now, let $\textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})$ be the problem considering only instances that have a $n^{O(1)}$-sized circuit representation of the preferential model, i.e., the input then is of the form $\langle\langle \mathcal{O}, \mathcal{L}\rangle, \varphi, \psi\rangle$.

For some $\prec$ order, we write $\textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})_\prec$ for the problem $\textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})$ where the order for given instances is fixed to $\prec$.

Theorem 31. $\textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})_{>_{\mathrm{lex}}}$ *is Δ_2^p-complete under $\leq_{\mathrm{m}}^{\log}$-reductions.*

Proof. We show hardness and membership separately.
[*Hardness*.] We state a reduction from $\overline{\mathrm{OLMS}}$ to $\textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})$. By virtue of Example 9, there exists a polynomial-sized (in the number of variables) circuit family (even in AC^0) that defines the lexicographic order on binary strings $<_{\mathrm{lex}}$. Now, just swap the inputs of this circuit and thereby define the lexicographic order $>_{\mathrm{lex}}$. Call this circuit $\mathcal{O}$. Let $\varphi(x_1, \ldots, x_n)$ be the input of $\overline{\mathrm{OLMS}}$. Regarding the circuit representation of the preferential model, we let $m = n$, so $\mathcal{S} = \mathbb{A}_N$ where $N = \{x_1, \ldots, x_n\}$. As a result, $\mathcal{L} = \mathrm{id}_{\mathcal{S}}$, where $\mathrm{id}_{\mathcal{S}}$ is the identity function on $\mathcal{S}$. Then, we define the reduction

$$\varphi \overset{f}{\mapsto} ((\mathcal{L}, \mathcal{O}), \varphi, \neg x_n)$$

Example 9 shows that $\mathcal{O}$ is logspace-constructible and thereby f is logspace-constructible.

We claim that the reduction is correct, i.e., $\varphi \in \overline{\mathrm{OLMS}}$ if and only if $f(\varphi) \in \textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})$.

"$\Rightarrow$": Let φ be a positive instance of $\overline{\mathrm{OLMS}}$ and θ be the $>_{\mathrm{lex}}$-maximal satisfying assignment. Then there are two possibilities.

1. φ is unsatisfiable. Then, because $\mathrel{|\!\!\sim}_{\mathbb{W}}$ suffers logical explosion all formulas are implied and $\varphi \mathrel{|\!\!\sim}_{\mathbb{W}} \neg x_n$ is true.
2. φ is satisfiable, but for the $>_{\mathrm{lex}}$-minimal assignment θ (notice that this is the $<_{\mathrm{lex}}$-largest assignment) we have that $\theta(x_n) = 0$. Hence, θ is not a model for x_n and thereby $\varphi \mathrel{|\!\!\sim}_{\mathbb{W}} \neg x_n$ holds.

"$\Leftarrow$": Let $\varphi \notin \overline{\mathrm{OLMS}}$. Clearly, $\theta(x_n) = 1$ by requirement of $\overline{\mathrm{OLMS}}$. Then, the string representation of θ is minimal w.r.t. $>_{\mathrm{lex}}$. Furthermore, for any model that satisfies x_n we have that x_n is assigned 1, hence, also θ is in the set of models of x_n. As a result $\varphi \mathrel{|\!\!\not\sim}_{\mathbb{W}} \neg x_n$, and $f(\varphi) \notin \textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})$.
[*Membership*.] We sketch a polynomial time algorithm that uses an oracle for proposition satisfiability. Let φ, ψ be the input formulas and $(\mathcal{L}, \mathcal{O})$ be the circuit-representation of the preferential model. Then, we use the SAT oracle as follows.

For $c \in \{0,1\}$, we let $\varphi(x_i = c)$ be the formula φ where every occurrence of x_i is substituted by the value of c. Now the Δ_2^p-algorithm works as follows. Ask the SAT oracle if $\varphi(x_1 = 0)$ is satisfiable. If yes, then proceed similarly with x_2 for $\varphi(x_1 = 0)$. If no, then proceed similarly with x_2 for $\varphi(x_1 = 1)$. In the end, we know the lexicographic maximal assignment and need to merely check if it satisfies ψ. $\square$

Because of the previous result, we now have of a complete problem regarding the specific order $>_{\mathrm{lex}}$.

Theorem 32. $\textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})$ *is in Π_2^p and Δ_2^p-hard under $\leq_{\mathrm{m}}^{\log}$-reductions.*

Proof. Hardness is given by Theorem 31. Hence, it suffices to show that $\textsc{SuccEnt}(\mathsf{CPL}^{\mathsf{pref}})$ is in Π_2^p. Let $\langle\langle \mathcal{O}, \mathcal{L}\rangle, \varphi, \psi\rangle$ be the input. Without loss of generality, assume that the set of propositions is $N = \{x_1, \ldots, x_n\}$. We describe the behaviour of the Π_2^p-machine that decides the problem.

1. Univerisally nondeterministically branch on all elements $j \in \mathcal{S}$ specified by inputs to $\mathcal{O}$, and all assignments $s\colon \{x_1, \ldots, x_n\} \to \{0,1\}$.
2. Existentially nondeterministically branch on all assignments $s'\colon \{x_1, \ldots, x_n\} \to \{0,1\}$.
3. If $j \neq \mathcal{L}(s)$ then accept.
4. If $j \models \psi$ then accept.
5. If $\mathcal{L}(s') \models \varphi$ and $\mathcal{O}(s', s)$ then accept.
6. Reject.

The nondeterminism induced by 1./2. is $\forall\exists$, hence Π_2^p. Steps 3. and 5. make use of the circuit family $\mathcal{L}$ resulting in a P computation. Again, the $\models$-checks in 4./5. are in NC^1 (Buss et al. 1992). The computation of the circuit value $\mathcal{O}(s, s')$ in 5. is in P. Reaching 6. means that

$$(j = \ell(s)) \wedge (j \not\models \psi) \wedge (\ell(s') \not\models \varphi \vee s' \not\prec s)$$

resulting in a negative answer to the input. $\square$

Preferential Propositional Dependence Logic. The following version of $\textsc{Ent}(\mathsf{CPL}^{\mathsf{pref}})$ lifts the problem to team semantics and the logic PDL by similar definitions.

Problem:	$\textsc{Ent}(\mathsf{PDL}^{\mathsf{pref}})$ — entailment problem for preferential propositional dependence logic
Input:	A finite preferential model $\mathbb{W} = \langle \mathcal{S}, \ell, \prec \rangle$ for PDL and $\varphi, \psi \in \mathrm{PL}(=(,))$.
Question:	Is it true that $\varphi \mathrel{|\!\sim_{\mathbb{W}}} \psi$?

In the following, we will state a result regarding the less known complexity class Θ_2^p. This class is defined as $\mathsf{P}^{\mathsf{NP}[\log]}$ meaning a restriction to logarithmic many calls to the NP oracle. By definition, we then have the containment $\Theta_2^p \subseteq \Delta_2^p$. Also it can be characterised by $\mathsf{P}^{||\mathsf{NP}}$ which is having non-adaptive but unrestricted many parallel NP oracle calls (Buss and Hay 1991; Hemachandra 1989).

Theorem 33. $\textsc{Ent}(\mathsf{PDL}^{\mathsf{pref}})$ *is in* Θ_2^p *and* NP*-hard under* $\leq_{\mathrm{m}}^{\log}$*-reductions.*

Proof. We show membership and hardness separately.
[*Membership*.] We present a Θ_2^p algorithm deciding the problem. The model checking problem for PDL is NP-complete (Ebbing and Lohmann 2012, Thm. 1). We use it as an oracle here.

1. In parallel, ask the NP-oracle for each team $T \in \mathcal{S}$ whether $T \models \varphi$ and $T \models \psi$.
2. For every minimal element in the order induced graph $(\mathcal{S}, \prec)$, if the oracle answers were of the form $(1, 0)$ (that is, φ was satisfied but ψ not) then reject.
3. Accept.

An oracle answer does not imply a different call afterwards. As a result, the oracle calls are non-adaptive and can be asked in parallel. As the input consists of $\mathcal{S}$, we have enough time to browse through all elements which also allows of identifying the minimal elements in the graph $(\mathcal{S}, \prec)$. The algorithm is correct as Step 3. is executed if no contradiction of the form that a minimal assignment satisfies φ but falsifies ψ occurs.
[*Hardness*.] The model checking problem for PDL is NP-complete (Ebbing and Lohmann 2012, Thm. 1). Now reduce it quite directly as follows showing NP-hardness:

$$(T, \varphi) \mapsto ((\{T\}, \mathrm{id}_{\mathcal{S}}, \emptyset), \top, \varphi).$$

As $\mathcal{S}$ contains only one element which also is satisfied by the first formula $\top$, we require it to satisfy also φ. This is equivalent to the model checking problem for PDL. $\square$

Analogously as before, we assume for the succinct version $\textsc{SuccEnt}(\mathsf{PDL}^{\mathsf{pref}})$, that the circuit families now are of size $(2^n)^{O(1)}$ (saving one exponential step via succinct representations), where n is the number of variables in φ and ψ. Notice that, while the inputs can be still of exponential size in n (a single team can have this size), it is still meaningful to have smaller inputs (avoiding doubly exponentially many such teams as trivial bound for $|\mathcal{S}|$).

It might be a bit surprising at first sight that having a harder model checking problem does not increase the complexity, but as stated in the proof above, this is compensated by the $\forall\exists$ structure of Π_2^p.

Theorem 34. $\textsc{SuccEnt}(\mathsf{PDL}^{\mathsf{pref}})$ *is in* Π_2^p *and* Δ_2^p*-hard under* $\leq_{\mathrm{m}}^{\log}$*-reductions.*

Proof. [*Membership*.] The algorithm stated in the proof of Theorem 32 works if it is modified as follows. While Steps 4. and 5. have a higher model checking complexity for PDL, that is, NP compared to NC^1, we can guess the required certificates (to obtain membership in P) also in Step 2. This does not increase complexity, as Step 2. is already existential. That is why this still yields a Π_2^p algorithm.
[*Hardness*.] Use a circuit family that encodes $\mathcal{S}$ as a set of singleton teams: $\mathcal{S} \coloneqq \{\ \{v\} \mid v \colon \{p_1, \ldots, p_n\} \to \{0, 1\}\ \}$. Flatness and the hardness proof of Theorem 31 then yields the result. $\square$

Preferential Propositional Logic over Teams. From Theorem 33 and 34, we can deduce similar ones for the team logic without dependence atoms TPL. Note, that the model checking problem for TPL is (potentially) easier than for PDL, as has been shown in P (Ebbing and Lohmann 2012, Tab. 1). As a result, the influence of the NP completeness of model checking for PDL, needs to be reconsidered.

Corollary 35. *The following holds:*

1. $\textsc{Ent}(\mathsf{TPL}^{\mathsf{pref}})$ *is in* P *and* NC^1*-hard under* $\leq_m^{\mathsf{AC}^0}$*-reductions.*
2. $\textsc{SuccEnt}(\mathsf{TPL}^{\mathsf{pref}})$ *is in* Π_2^p *and* Δ_2^p*-hard under* $\leq_{\mathrm{m}}^{\log}$*-reductions.*

8 Conclusion

We have established a foundation for preferential non-monotonic reasoning within the framework of team semantics. Our results also provide new insights into the algorithmic properties of preferential reasoning for classical propositional logic.

Team-based logics have a wide range of applications, e.g., in the formal semantics of natural language, in the semantics of statements and questions (as in inquisitive logic), and in modelling free choice inferences (as addressed by BSML modal logics (Aloni 2022)). For the combination of team semantics and non-monotonic reasoning—as studied here—we expect a large variety of applications in these and other domains. As one of many areas of application, we want to highlight reasoning over datasets of different qualities. In this scenario, datasets are modelled as teams. The highly flexible approach of putting preferences on teams allows for imposing a metric on datasets. Within this setting, preferential team-based reasoning becomes a framework for studying selective reasoning of preferred datasets. We think this is a highly relevant matter, for which our work provides a foundation.

In future work, we plan to extend our axiomatics studies of preferential reasoning over teams to other team-based logics, e.g., inclusion logic. For that, we will build upon recent results on the axiomatics of choice in restricted settings (Sauerwald et al. 2025). In the realm of complexity, we will study the query and data complexity of the problems discussed in Section 7, and find tight complexity bounds for the problems studied. This includes investigating the complexity of a team variant of OLMS (see Section 2).

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