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Compactness and Preservation in Logical Argumentation Frameworks

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Abstract

Logic-based argumentation is a formal method for constructing, evaluating and comparing arguments. In this paper we address two (related) key issues concerning the representation of logical argumentation frameworks: how to describe them in a compact way, and how to move from one framework to another while preserving their basic logical characteristics. The results are applied to various forms of attack rules and different kinds of argumentative semantics, and are demonstrated for transitions between several 3-valued logics and classical logic. As a byproduct, our results are also used for converting logic-based argumentation frameworks to assumption-based argumentation frameworks.

1 Introduction

Logic-based argumentation (Besnard et al. 2014; Arieli et al. 2021) is a formal method for defining, evaluating, and deriving logical arguments from a fixed knowledge-base. It has been shown useful for a variety of purposes, such as conflict resolution (especially in complex domains like medicine (Hunter and Williams 2012), law (Prakken 2017), ethical reasoning (Straßer and Arieli 2019), etc), modeling of defeasible reasoning (Pollock 1992), logic programs (García and Simari 2004), decision making (Tamani and Croitoru 2014), database systems (Deagustini et al. 2017), and bridging between Philosophy and AI (van Berkel 2023).

In this paper we consider two important aspects in the description of logical argumentation frameworks:

- *Compact Representation:* By nature, logic-based argumentation frameworks should be deductively closed, or at least capable of supporting sound (and frequently also complete) logical inferences. As such, these frameworks are demanding in terms of space and computational complexity. In this paper, we consider compact representation forms for logical argumentation frameworks and show that as long as the set of the premises is kept finite and the attacks between arguments depend only on the support sets of the attacked arguments, a logical framework can be translated to a finite equivalent framework.
- *Preservation of Logical Properties:* In relation to the previous item, for reducing the number of arguments to a minimum and/or having more adequate settings, it is sometimes convenient to switch from one base logic to another. The second subject of this paper thus involves various considerations regarding such transitions and the possibility to keep the characteristics of the original frameworks. Most importantly: preserving their logical conclusions and the selections of arguments obtained by standard argumentation semantics (Dung 1995).

To address the two issues above we first recall, in the next section, the basic notions that are related to the definition of logical argumentation frameworks and the entailment relations that are induced by them. Then, in Sections 3 and 4 we respectively consider compactness and logical preservations in such frameworks. The results are applied to different forms of attack rules and with respect to a variety of Dung-style semantics, and are demonstrated for the transition to classical logic from three base logics, all of them are 3-valued: Bochvar $\mathsf{B3}$, Kleene $\mathsf{K3}$ and Priest LP. Finally, in Section 5 we use the results in the paper for relating logic-based argumentation frameworks and assumption-based argumentation frameworks, and in Section 6 we conclude.

2 Logic-Based Argumentation Frameworks

For defining logical argumentation frameworks, and arguments in particular, we have to specify an underlying logic. We start with a general definition of a logic (Tarski 1941).

Definition 1. A (propositional) *logic* is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a *consequence relation* for $\mathcal{L}$, that is: $\vdash$ is a binary relation between sets of formulas and formulas in $\mathcal{L}$, satisfying the following conditions:

Reflexivity: if $\psi \in \mathcal{S}$ then $\mathcal{S} \vdash \psi$,
Monotonicity: if $\mathcal{S} \vdash \psi$ and $\mathcal{S} \subseteq \mathcal{S}'$ then $\mathcal{S}' \vdash \psi$,
Transitivity: if $\mathcal{S} \vdash \psi$ and $\mathcal{S}', \psi \vdash \phi$ then $\mathcal{S}, \mathcal{S}' \vdash \phi$.

In what follows we shall denote by $\mathsf{Cn}_\vdash(\mathcal{S})$ the $\vdash$-transitive closure of $\mathcal{S}$, that is: $\mathsf{Cn}_\vdash(\mathcal{S}) = \{\psi \mid \mathcal{S} \vdash \psi\}$.

In addition, it is usual to assume that $\vdash$ is *structural* (i.e., closed under substitutions: for any substitution θ, if $\mathcal{S} \vdash \psi$ then $\theta(\mathcal{S}) \vdash \theta(\psi)$), *non-trivial* (i.e., $p \not\vdash q$ for every two atomic formulas p, q), and *finitary* (if $\mathcal{S} \vdash \psi$, there is a *finite* set $\mathcal{S}' \subseteq \mathcal{S}$ such that $\mathcal{S}' \vdash \psi$).[1]

[1]Finitariness is often essential for practical reasoning, e.g., expressing arguments by a finite number of assumptions.

Unless referring to a specific language, we assume that $\mathcal{L}$ contains at least the following connectives and constant:

$\vdash$-*negation* $\neg$: $p \not\vdash \neg p$ and $\neg p \not\vdash p$ (for every atomic p),

$\vdash$-*conjunction* $\wedge$: $\mathcal{S} \vdash \psi \wedge \phi$ iff $\mathcal{S} \vdash \psi$ and $\mathcal{S} \vdash \phi$,

$\vdash$-*disjunction* $\vee$: $\mathcal{S}, \phi \vee \psi \vdash \sigma$ iff $\mathcal{S}, \phi \vdash \sigma$ and $\mathcal{S}, \psi \vdash \sigma$.

$\vdash$-*truth constant* T: $\mathcal{S} \vdash \mathsf{T}$ for every $\mathcal{S}$.

The set of the (well-formed) formulas of $\mathcal{L}$ is denoted $\mathsf{WFF}(\mathcal{L})$. For a finite set of formulas $\mathcal{S}$ we denote by $\bigwedge\mathcal{S}$ (respectively, by $\bigvee\mathcal{S}$) the conjunction (respectively, the disjunction) of all the formulas in $\mathcal{S}$. We denote by $\wp(\mathcal{S})$ (by $\wp_{\mathsf{fin}}(\mathcal{S})$) the set of the (finite) subsets of $\mathcal{S}$, and by $\mathsf{Atoms}(\mathcal{S})$ the set of atoms that appear in the formulas of $\mathcal{S}$. We say that a set $\mathcal{S}$ is $\vdash$-*inconsistent*, if $\mathcal{S} \vdash \neg\bigwedge\mathcal{S}'$ for some $\mathcal{S}' \in \wp_{\mathsf{fin}}(\mathcal{S})$.

Arguments for a base logic are defined as follows:

Definition 2. Given a logic $\mathfrak{L} = \langle\mathcal{L}, \vdash\rangle$, an $\mathfrak{L}$-*argument* (an *argument* for short) is a pair $A = \langle\mathcal{S}, \psi\rangle$, where $\mathcal{S}$ (the *support* of A) is a finite set of $\mathcal{L}$-formulas and ψ (the *conclusion* of A) is an $\mathcal{L}$-formula, such that $\mathcal{S} \vdash \psi$ (i.e., $\psi \in \mathsf{Cn}_{\vdash}(\mathcal{S})$). We denote: $\mathsf{Supp}(\langle\mathcal{S}, \psi\rangle) = \mathcal{S}$ and $\mathsf{Conc}(\langle\mathcal{S}, \psi\rangle) = \psi$. Arguments of the form $\langle\emptyset, \psi\rangle$ are called *tautological*.

Example 1. The pairs $\langle\emptyset, p \vee \neg p\rangle$, $\langle\{p\}, p\rangle$ and $\langle\{p, \neg p\}, q\rangle$, are all $\mathfrak{L}$-arguments for $\mathfrak{L} = \mathsf{CL}$ (classical logic). Note that the first argument is tautological since its support is empty, the second argument holds in every logic (by reflexivity), and the last argument is not valid in any paraconsistent base logic (such as LP that will be considered in Section 4.3).

Arguments may attack and counter-attack each other according to pre-defined attack rules. Some of the better known rules are listed in Table 1. Further attack rules are considered, e.g., in (Gorogiannis and Hunter 2011; Straßer and Arieli 2019). Each rule $\mathcal{R}$ in this table is equipped with a set $\mathsf{C}_{\mathcal{R}}$ of conditions (on the rightmost column of the table), the satisfaction of which enables the application of the rule. For instance, according to the rule Def an argument $\langle\mathcal{S}_1, \psi_1\rangle$ attacks an argument $\langle\mathcal{S}_2 \cup \mathcal{S}_2', \psi_2\rangle$, if the conclusion ψ_1 of the attacking argument entails $\neg\bigwedge\mathcal{S}_2$, the negation of (part of) the support of the attacked argument.

Clearly, the rules in Table 1 are not unrelated (see, e.g., (Arieli et al. 2021, Remark 7)).

Argumentation frameworks are now defined follows:

Definition 3. Let $\mathfrak{L} = \langle\mathcal{L}, \vdash\rangle$ be a logic and $\mathcal{A}$ a set of attack rules for $\mathfrak{L}$. Let also $\mathcal{S}$ be a set of $\mathcal{L}$-formulas. The *(logical) argumentation framework* for $\mathcal{S}$, induced by $\mathfrak{L}$ and $\mathcal{A}$, is the pair $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A})\rangle$,[2] where

- $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ is the set of the $\mathfrak{L}$-arguments whose supports are subsets of $\mathcal{S}$, and
- $\mathit{Attack}(\mathcal{A})$ is a relation on $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \times \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, defined by $(A_1, A_2) \in \mathit{Attack}(\mathcal{L})$ iff there is some $\mathcal{R} \in \mathcal{A}$ such that A_1 $\mathcal{R}$-attacks A_2.

[2] In what follows we shall sometimes omit the subscripts and write just $\mathcal{AF}(\mathcal{S})$ for $\langle\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A})\rangle$.

Acronym	Attacking Argument	Attacked Argument	Conditions
Def	$\langle\mathcal{S}_1, \psi_1\rangle$	$\langle\mathcal{S}_2 \cup \mathcal{S}_2', \psi_2\rangle$	$\psi_1 \vdash \neg\bigwedge\mathcal{S}_2$
FullDef	$\langle\mathcal{S}_1, \psi_1\rangle$	$\langle\mathcal{S}_2, \psi_2\rangle$	$\psi_1 \vdash \neg\bigwedge\mathcal{S}_2$
DirDef	$\langle\mathcal{S}_1, \psi_1\rangle$	$\langle\{\varphi\} \cup \mathcal{S}_2', \psi_2\rangle$	$\psi_1 \vdash \neg\varphi$
Ucut	$\langle\mathcal{S}_1, \psi_1\rangle$	$\langle\mathcal{S}_2 \cup \mathcal{S}_2', \psi_2\rangle$	$\psi_1 \dashv\vdash \neg\bigwedge\mathcal{S}_2$
FullUcut	$\langle\mathcal{S}_1, \psi_1\rangle$	$\langle\mathcal{S}_2, \psi_2\rangle$	$\psi_1 \dashv\vdash \neg\bigwedge\mathcal{S}_2$
DirUcut	$\langle\mathcal{S}_1, \psi_1\rangle$	$\langle\{\varphi\} \cup \mathcal{S}_2', \psi_2\rangle$	$\psi_1 \dashv\vdash \neg\varphi$
CmpUcut	$\langle\mathcal{S}_1, \neg\bigwedge\mathcal{S}_2\rangle$	$\langle\mathcal{S}_2 \cup \mathcal{S}_2', \psi_2\rangle$	
CmpFullUcut	$\langle\mathcal{S}_1, \neg\bigwedge\mathcal{S}_2\rangle$	$\langle\mathcal{S}_2, \psi_2\rangle$	
CmpDirUcut	$\langle\mathcal{S}_1, \neg\varphi\rangle$	$\langle\{\varphi\} \cup \mathcal{S}_2', \psi_2\rangle$	
ConUcut	$\langle\emptyset, \neg\bigwedge\mathcal{S}_2\rangle$	$\langle\mathcal{S}_2 \cup \mathcal{S}_2', \psi_2\rangle$	
Reb	$\langle\mathcal{S}_1, \psi_1\rangle$	$\langle\mathcal{S}_2, \psi_2\rangle$	$\psi_1 \dashv\vdash \neg\psi_2$
DefReb	$\langle\mathcal{S}_1, \psi_1\rangle$	$\langle\mathcal{S}_2, \psi_2\rangle$	$\psi_1 \vdash \neg\psi_2$

Table 1: Argumentative attacks. Def, Ucut, and Reb abbreviate, respectively, Defeat, Undercut, and Rebuttal. Each of these attack types has variations: Full, Dir (for direct), Cmp (for compact), and Con (for consistency). For instance, ConUcut stands for consistency undercut, i.e., an undercut attack by a tautological attacker.

A logical argumentation framework $\mathcal{AF}(\mathcal{S})$ may be associated with a directed graph, in which the nodes are arguments in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and the edges represent attacks between arguments in $\mathit{Attack}(\mathcal{A})$ (see Example 2 below). What can be *deduced* from $\mathcal{AF}(\mathcal{S})$ is defined in terms of Dung-style semantics (Dung 1995) and the corresponding entailment relations, as indicated in the next two definitions.

Definition 4. Let $\mathcal{AF}(\mathcal{S}) = \langle\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A})\rangle$ be a logical argumentation framework, and let $\mathcal{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$.

- $\mathcal{E}$ *attacks* argument A if there is an argument $B \in \mathcal{E}$ s.t. $(B, A) \in \mathit{Attack}(\mathcal{A})$. $\mathcal{E}$ *defends* A if $\mathcal{E}$ attacks every argument that attacks A. $\mathcal{E}^+$ is the set of arguments that are attacked by $\mathcal{E}$, and $\mathcal{E} \cup \mathcal{E}^+$ is called the *range* of $\mathcal{E}$.
- $\mathcal{E}$ is called *conflict-free* if it does not attack any of its elements (i.e., $\mathcal{E}^+ \cap \mathcal{E} = \emptyset$). A maximally conflict-free set is called a *naive extension* of $\mathcal{AF}(\mathcal{S})$. A set $\mathcal{E}$ whose range is $\subseteq$-maximal among the conflict-free sets is a *stage extension* of $\mathcal{AF}(\mathcal{S})$. A conflict-free set $\mathcal{E}$ whose range is equal to $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ is a *stable extension* of $\mathcal{AF}(\mathcal{S})$.
- An *admissible set* of $\mathcal{AF}(\mathcal{S})$ is a subset of $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ that is conflict-free and defends all its elements. A $\subseteq$-maximally admissible set is a *preferred extension* of $\mathcal{AF}(\mathcal{S})$. The *ideal extension* of $\mathcal{AF}(\mathcal{S})$ is the $\subseteq$-maximal admissible set that is included in each preferred extension.
- A *complete extension* of $\mathcal{AF}(\mathcal{S})$ is an admissible extension that contains all the arguments that it defends. The $\subseteq$-minimally complete extension of $\mathcal{AF}(\mathcal{S})$ is the *grounded extension* of $\mathcal{AF}(\mathcal{S})$. A *semi-stable extension* of $\mathcal{AF}(\mathcal{S})$ is a complete extension with a $\subseteq$-maximal range, and the *eager extension* of $\mathcal{AF}(\mathcal{S})$ is the $\subseteq$-

maximal admissible set that is included in every semi-stable extension.[3]

We shall denote by $\mathsf{Adm}(\mathcal{AF}(\mathcal{S}))$ [respectively, by $\mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Grd}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Prf}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Stb}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Sstb}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Stg}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Idl}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Egr}(\mathcal{AF}(\mathcal{S}))$] the set of all the admissible [respectively, the complete, grounded, preferred, stable, semi-stable, stage, ideal, eager] extensions of $\mathcal{AF}(\mathcal{S})$. The selections of arguments by argumentation semantics are the basis of the argumentative entailments, defined next.

Definition 5. Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack(\mathcal{A})\rangle$ be a logical argumentation framework, and let $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}, \mathsf{SStb}, \mathsf{Stg}, \mathsf{Idl}, \mathsf{Egr}\}$. We denote:

- $\mathcal{S} \mathrel{|\!\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Sem}} \psi$ if there is $\langle \Gamma, \psi\rangle \in \bigcup \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))$,
- $\mathcal{S} \mathrel{|\!\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Sem}} \psi$ if there is $\langle \Gamma, \psi\rangle \in \bigcap \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))$,
- $\mathcal{S} \mathrel{|\!\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Sem}} \psi$ if for every $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))$ there is some $\Gamma_{\mathcal{E}} \subseteq \mathcal{S}$ such that $\langle \Gamma_{\mathcal{E}}, \psi\rangle \in \mathcal{E}$.

When the framework is clear from the context, we shall write $\mathcal{S} \mathrel{|\!\!\sim}_{\star\mathsf{Sem}} \psi$ instead of $\mathcal{S} \mathrel{|\!\!\sim}^{\mathfrak{L},\mathcal{A}}_{\star\mathsf{Sem}} \psi$ (for $\star \in \{\cup, \cap, \Cap\}$).

By their definitions, clearly $\mathrel{|\!\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Sem}} \subset \mathrel{|\!\!\sim}^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Sem}} \subset \mathrel{|\!\!\sim}^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Sem}}$.

Example 2. Let $\mathfrak{L} = \mathsf{CL}$ and $\mathcal{S} = \{p, \neg p, q\}$. Some of the elements in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ are considered in Example 1. Suppose that $\mathcal{A}$ consists of Ucut (Undercut) and ConUcut (Consistency Undercut). Part of $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ is presented in Figure 1.

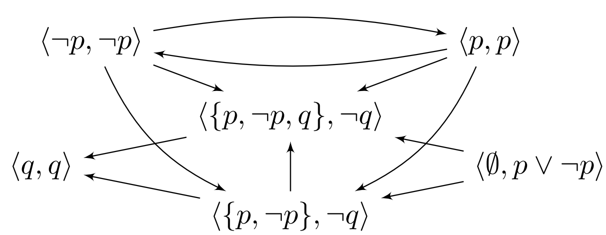

Figure 1: Part of the argumentation framework of Example 2.

Here, the tautological argument $\langle \emptyset, p \vee \neg p\rangle$ defends $\langle q, q\rangle$ from any possible attacker, thus the grounded extension $\mathcal{E}_{\mathsf{grd}}$ in the figure above consists of these two arguments. The preferred (and also the [semi]-stable) extensions in this figure are $\mathcal{E}_{\mathsf{grd}} \cup \{\langle p, p\rangle\}$ and $\mathcal{E}_{\mathsf{grd}} \cup \{\langle \neg p, \neg p\rangle\}$.

When the whole framework $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ is considered, the corresponding grounded extension is $\mathsf{Arg}_{\mathfrak{L}}(\{q\})$ and the preferred, stable, semi-stable and stage extensions are $\mathsf{Arg}_{\mathfrak{L}}(\{q, p\})$ and $\mathsf{Arg}_{\mathfrak{L}}(\{q, \neg p\})$. Since the grounded extension is also the ideal and the eager extension in this case, it follows that q is entailed by $\mathcal{S}$ according to all the entailments in Definition 5 and for every $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{Prf}, \mathsf{SStb}, \mathsf{Stg}, \mathsf{Idl}, \mathsf{Egr}\}$, as expected.

[3] In (Dung 1995) it is shown that the grounded extension of $\mathcal{AF}(\mathcal{S})$ is unique, the preferred extensions are maximally complete, and every stable extension is also preferred. Thus, every stable extension (if exists) is also semi-stable and stage, and the ideal and the eager extensions are complete. For further facts and other extensions, see e.g., (Baroni, Caminada, and Giacomin 2011; Baroni, Caminada, and Giacomin 2018).

3 Compact Representations of Logical Argumentation Frameworks

Logical argumentation frameworks are, by their nature, space demanding. For instance, for every argument $\langle \Gamma, \phi\rangle$ in a given framework $\mathcal{AF}_{\mathsf{CL},\mathcal{A}}(\mathcal{S})$ there are infinitely many arguments based on the same support (such as $\langle \Gamma, \phi \wedge \phi\rangle$, $\langle \Gamma, \phi \vee \psi\rangle$, $\langle \Gamma, \neg\neg\phi\rangle$, etc.). In this section, we consider compact representations of logical frameworks without loosing information. In particular, we show that as long as the set of premises is kept finite and the attacks depend only on the support set of the attacked arguments, any logical argumentation framework can be translated to an equivalent framework with a finite number of arguments.

The next definition identifies attack rules that are triggered only by the content of the supports of the attacked arguments. This includes all the rules in Table 1, except of rebuttal attacks.

Definition 6. An attack rule $\mathcal{R}$ is *support-driven*, if there is a function $\mathsf{C}_{\mathcal{R}} : \wp_{\mathsf{fin}}(\mathsf{WFF}(\mathcal{L})) \times \mathsf{WFF}(\mathcal{L}) \times \wp_{\mathsf{fin}}(\mathsf{WFF}(\mathcal{L})) \to \{\mathsf{true}, \mathsf{false}\}$, such that for every $\mathcal{S}_1, \mathcal{S}_2 \in \wp_{\mathsf{fin}}(\mathsf{WFF}(\mathcal{L}))$ and $\psi_1, \psi_2 \in \mathsf{WFF}(\mathcal{L})$, the following conditions are met:
(1) $\langle \mathcal{S}_1, \psi_1\rangle$ $\mathcal{R}$-attacks $\langle \mathcal{S}_2, \psi_2\rangle$ iff $\mathsf{C}_{\mathcal{R}}(\mathcal{S}_1, \psi_1, \mathcal{S}_2)$ holds,
(2) $\mathsf{C}_{\mathcal{R}}(\mathcal{S}_1, \psi_1, \emptyset)$ is false.

Thus, $\mathcal{R}$ is support-driven if its condition (if any) refers only to the support of the attacked argument (apart of the attacking argument), and the condition is satisfied provided that this support is non-empty.

Note 1. Two remarks are in order here:

(a) The function $\mathsf{C}_{\mathcal{R}}$ in Definition 6 allows to abstractly represent support-driven attacks, which exclusively depend only on the supports of the attacking and the attacked arguments, and the conclusion of the attacking argument. This function reduces questions of attacks between specific arguments to relations between equivalence classes representing supports sets that are logically equivalent (see also Note 2). As we shall show in what follows, this enables finite representations of support-driven attacks.

(b) We concentrate on attack on the supports of arguments, since in logical argumentation conclusion-based attacks (and rebuttal attacks in particular) are rather problematic. To see this, consider again the set $\mathcal{S} = \{p, \neg p, q\}$ in Example 2. As shown in that example, the use of support-driven attacks rules yields the expected conclusions, including tautological arguments (such as $\langle \emptyset, p \vee \neg p\rangle$) and arguments supported by q (such as $\langle q, q\rangle$), which is not involved in the contradiction in $\mathcal{S}$. However, *neither of* $\langle q, q\rangle$ and $\langle \emptyset, p \vee \neg p\rangle$ is in the grounded extension once rebuttals are incorporated. To see this, note that these arguments are rebutted by $\langle \{p, \neg p\}, \neg q\rangle$ and $\langle \{p, \neg p\}, \neg(p \vee \neg p)\rangle$. Another problem is, e.g., that for $\mathcal{S} = \{p, q, \neg(p \wedge q)\}$ with rebuttals there is a complete extension containing $\langle p, p\rangle$, $\langle q, q\rangle$, and $\langle \neg(p \wedge q), \neg(p \wedge q)\rangle$. Such an extension is not consistent.[4]

[4] An extension $\mathcal{E}$ is inconsistent if so is the set of its arguments' conclusions: $\{\mathsf{Conc}(s) \mid s \in \mathcal{E}\}$.

A compact representation of support-induced frameworks is described by the following structures:

Definition 7. Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A}) \rangle$ be an argumentation framework in which all the rules in $\mathcal{A}$ are support-driven. The *support-induced argumentation framework* (SAF), based on the logic $\mathfrak{L}$, the attack rules in $\mathcal{A}$, and the set of premises $\mathcal{S}$ is the framework $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \wp_{\mathsf{fin}}(\mathcal{S}), S\text{-}\mathit{Attack}(\mathcal{A}) \rangle$, where $(\mathcal{S}_1, \mathcal{S}_2) \in S\text{-}\mathit{Attack}(\mathcal{A})$ iff there is an attack rule $\mathcal{R} \in \mathcal{A}$ such that $\mathsf{C}_{\mathcal{R}}(\mathcal{S}_1, \psi_1, \mathcal{S}_2)$ holds for some ψ_1 such that $\langle \mathcal{S}_1, \psi_1 \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and $\mathcal{S}_2 \subseteq \mathcal{S}$.

Example 3. The support-induced argumentation framework that corresponds to the argumentation framework $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ in Example 2 is represented in Figure 1

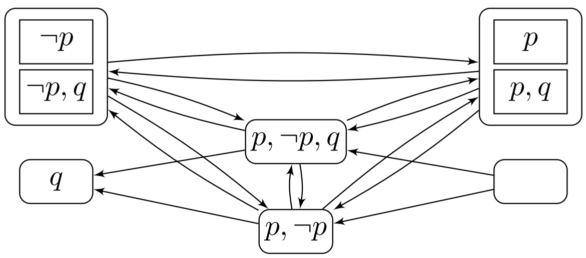

Figure 2: A support-induced framework for Example 2. The nodes $\{\neg p, q\}$, $\{\neg p\}$ and $\{p, q\}$, $\{p\}$ are grouped to two outer nodes, as the inner nodes within each group share the same incoming and outgoing edges. The unlabeled node represents $\emptyset \in \wp_{\mathsf{fin}}(\mathcal{S})$.

Note that while the graph of $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ is not finite, the graph of $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ contains only eight nodes (the size of the power-set of $\mathcal{S}$). Thus, for instance, all the arguments of the form $\langle p \wedge \neg p, \psi \rangle$ for some formula ψ are reduced to one node (that of $\{p \wedge \neg p\}$) in the graph of $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$.

Note 2. The support-induced argumentation framework $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ gives rise to a quotient structure for $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ by a simple translation. Indeed, let $\sim\, \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \times \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ be defined by $A \sim A'$ iff $\mathsf{Supp}(A) = \mathsf{Supp}(A')$. It is easy to see that $\sim$ is an equivalence relation on $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Let $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})_{\sim}$ be the set of equivalence classes induced by $\sim$, and define $\pi : \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \to \wp(\mathcal{S})$ by $[A] \mapsto \mathsf{Supp}(A)$. Due to the reflexivity of $\vdash_{\mathfrak{L}}$, π is a bijection, if $\mathfrak{L}$ has theorems (that is, if $\mathsf{Cn}_{\vdash_{\mathfrak{L}}}(\emptyset) \neq \emptyset$). Let $([A]_{\sim}, [A']_{\sim}) \in \mathsf{Attack}(\mathcal{S})_{\sim}$ iff $(A, A') \in \mathsf{Attack}(\mathcal{S})$. Then, $\langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})_{\sim}, \mathsf{Attack}_{\sim} \rangle$ is a quotient structure for $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$. The latter is isomorphic to $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$, in case that the base logic $\mathfrak{L}$ has theorems.[5]

Note that, given a finite set $\mathcal{S}$ of premises, and assuming that the rules in $\mathcal{A}$ are support-driven, the support-induced argumentation framework $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ is *finite*. It is therefore interesting to check whether $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ and $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ give rise to the same extensions (under the translation which associates arguments of the form $\langle \mathcal{S}', \psi \rangle \in \mathsf{Arg}(\mathcal{S})$ with their support $\mathcal{S}'$). This is confirmed by the following proposition.

[5]If $\mathfrak{L}$ has no theorems, then π is injective with co-domain $\wp_{\mathsf{fin}}(\mathcal{S}) \setminus \{\emptyset\}$. By the reflexivity of $\vdash_{\mathfrak{L}}$, $\langle \mathcal{S}', \psi \rangle$ is an argument for every $\emptyset \neq \mathcal{S}' \subseteq \mathcal{S}$ and $\psi \in \mathcal{S}'$. In this case $\langle \mathsf{Arg}_{\mathcal{L}}(\mathcal{S})_{\sim}, \mathsf{Attack}_{\sim} \rangle$ is isomorphic to $\langle \wp_{\mathsf{fin}}(\mathcal{S}) \setminus \{\emptyset\}, S\text{-}\mathit{Attack}(\mathcal{A}) \cap (\wp_{\mathsf{fin}}(\mathcal{S}) \setminus \{\emptyset\})^2 \rangle$.

Proposition 1. *Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A}) \rangle$ be an argumentation framework with only support-driven rules, and let $\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \wp_{\mathsf{fin}}(\mathcal{S}), S\text{-}\mathit{Attack}(\mathcal{A}) \rangle$ be the corresponding support-induced argumentation framework. For every* $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$,

1. *if $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$ then* $\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\} \in \mathsf{Sem}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$*, and*
2. *if $\Xi \in \mathsf{Sem}(\mathcal{SAF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$ then* $\{A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \Xi\} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$.

The proof of Proposition 1 is rather long and thus it is postponed to the full version of the paper (like the proofs of some of the other results in this paper). Here we only show a basic property that is required for the proof: If a (complete or stage) extension of an argumentation-framework based on support-driven rules includes an argument $\langle \Delta, \delta \rangle$, then it includes all the other arguments that are based on the same support set Δ.

Lemma 1. *Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}(\mathcal{A}) \rangle$ be an argumentation framework with only support-driven rules, and let $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})) \cup \mathsf{Stg}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}})(\mathcal{S})$.[6] Then, if $\langle \Delta, \delta \rangle, \langle \Delta, \delta' \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and $\langle \Delta, \delta \rangle \in \mathcal{E}$, so $\langle \Delta, \delta' \rangle \in \mathcal{E}$.*

Proof. Suppose that $\langle \Theta, \theta \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ $\mathcal{R}$-attacks the argument $\langle \Delta, \delta' \rangle$ for some $\mathcal{R} \in \mathcal{A}$. Then, $\mathsf{C}_{\mathcal{R}}(\Theta, \theta, \Delta)$ holds, and so $\langle \Theta, \theta \rangle$ also $\mathcal{R}$-attacks $\langle \Delta, \delta \rangle \in \mathcal{E}$.

Suppose first that $\mathcal{E} \in \mathsf{Cmp}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. Since $\mathcal{E}$ is admissible, there is an argument $\langle \Lambda, \lambda \rangle \in \mathcal{E}$ that $\mathcal{R}'$-attacks $\langle \Theta, \theta \rangle$ for some $\mathcal{R}' \in \mathcal{A}$. Thus, $\mathcal{E}$ defends $\langle \Delta, \delta' \rangle$ and by the completeness of $\mathcal{E}$, $\langle \Delta, \delta' \rangle \in \mathcal{E}$.

Suppose now that $\mathcal{E} \in \mathsf{Stg}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. So, $\langle \Theta, \theta \rangle \notin \mathcal{E}$ by the conflict-freeness of $\mathcal{E}$. Thus, $\mathcal{E} \cup \{\langle \Delta, \delta' \rangle\}$ is conflict-free. By the $\subseteq$-maximality of $\mathcal{E}$, $\langle \Delta, \delta' \rangle \in \mathcal{E}$. $\square$

Example 4. Consider again the support induced framework of Example 3. By Proposition 1 and Example 2 we get that the grounded, ideal and eager extension in this case is $\{\emptyset, \{q\}\}$, while the preferred, stable, semi-stable and stage extensions of the framework are $\{\emptyset, \{q\}, \{p\}, \{q, p\}\}$ and $\{\emptyset, \{q\}, \{\neg p\}, \{q, \neg p\}\}$.

4 Argumentative Preservation of Logical Inclusion

The ability to compactly represent logic-based argumentation frameworks raises questions about transitions of representations in more general cases: Given two logics with the same language $\mathfrak{L}_1 = \langle \mathcal{L}, \vdash_{\mathfrak{L}_1} \rangle$ and $\mathfrak{L}_2 = \langle \mathcal{L}, \vdash_{\mathfrak{L}_2} \rangle$, where $\mathfrak{L}_1$ is included in $\mathfrak{L}_2$ (i.e., $\vdash_{\mathfrak{L}_1} \subseteq \vdash_{\mathfrak{L}_2}$), under what conditions is the inclusion of the logics preserved when reasoning argumentatively with these logics? As we will show in what follows, in some cases such preservations allow for compact representations of the argumentative reasoning based on a given logic (such as classical logic) by means of another logic (e.g., a 3-valued logic).

Formally, then, we would like to keep the following argumentative inclusion:

[6]By Footnote 3, this covers all the semantics in Proposition 1.

Definition 8. Given a semantics Sem, a logical framework $\mathcal{AF}_1 = \mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S})$ is Sem-*argumentatively included* in a logical framework $\mathcal{AF}_2 = \mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})$, if $\mathfrak{L}_1$ is included in $\mathfrak{L}_2$ and the following conditions hold:

- **Inc1:** If $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_1)$ then $\mathcal{E}^{\uparrow} \in \mathsf{Sem}(\mathcal{AF}_2)$ and
- **Inc2:** If $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_2)$ then $\mathcal{E}^{\downarrow} \in \mathsf{Sem}(\mathcal{AF}_1)$, where:

$\mathcal{E}^{\uparrow} = \{A \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S}) \mid \exists B \in \mathcal{E} \text{ s.t. } \mathsf{Supp}(A) = \mathsf{Supp}(B)\}$,
$\mathcal{E}^{\downarrow} = \{A \in \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S}) \mid \exists B \in \mathcal{E} \text{ s.t. } \mathsf{Supp}(A) = \mathsf{Supp}(B)\}$.

A primary benefit of argumentative inclusion is that it allows for a preservation of logical entailments: If for every $\mathcal{S}$ it holds that $\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S})$ is Sem-argumentatively included in $\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})$, then $\mathrel{|\!\sim}^{\mathfrak{L}_1,\mathcal{A}}_{\circ\mathsf{Sem}} \subseteq \mathrel{|\!\sim}^{\mathfrak{L}_2,\mathcal{A}}_{\circ\mathsf{Sem}}$ for every $\circ \in \{\cup, \cap, \Cap\}$ (recall Definition 5).

As we show below, argumentation inclusion is guaranteed by the following property:

Definition 9. Two support-driven attack relations $\mathcal{R}_1$ and $\mathcal{R}_2$ are *corresponding* relative to two base logics $\mathfrak{L}_1$ and $\mathfrak{L}_2$, if for every set of formulas $\mathcal{S}$ the following two conditions are satisfied:

- **Att1:** If A $\mathcal{R}_1$-attacks B for some $A, B \in \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$, then there is $A' \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S})$ with $\mathsf{Supp}(A) = \mathsf{Supp}(A')$ and A' $\mathcal{R}_2$-attacks B.
- **Att2:** If A $\mathcal{R}_2$-attacks B for some $A, B \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S})$, then there is $A' \in \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$ with $\mathsf{Supp}(A) = \mathsf{Supp}(A')$ and A' $\mathcal{R}_1^{\downarrow}$-attacks B.[7]

The pairs $\langle \mathfrak{L}_1, \mathcal{A}_1 \rangle$ and $\langle \mathfrak{L}_2, \mathcal{A}_2 \rangle$ have *corresponding attacks*, if $\mathcal{A}_1$ and $\mathcal{A}_2$ are sets of support-driven attacks, and for each $\mathcal{R}_1 \in \mathcal{A}_1$ there is a corresponding attack $\mathcal{R}_2 \in \mathcal{A}_2$ (relative to $\mathfrak{L}_1$ and $\mathfrak{L}_2$), and vice versa.

We now show that having corresponding attacks is a sufficient condition for argumentative inclusion with respect to all standard semantics. For this, we show that the support induced argumentation frameworks of $\mathcal{AF}_1$ and $\mathcal{AF}_2$ coincide.

Proposition 2. *Suppose that $\mathfrak{L}_1$ is included in $\mathfrak{L}_2$ and that $\langle \mathfrak{L}_1, \mathcal{A}_1 \rangle$ and $\langle \mathfrak{L}_2, \mathcal{A}_2 \rangle$ have corresponding attacks. Then, for every set $\mathcal{S}$, we have: $\mathcal{SAF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}) = \mathcal{SAF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})$.*

Proposition 2 follows directly from the next lemma.

Lemma 2. *In the notations and assumptions of Proposition 2, let $\mathcal{R}_1 \in \mathcal{A}_1$ and $\mathcal{R}_2 \in \mathcal{A}_2$ be corresponding attacks. For all $\Gamma, \Theta \in \wp_{\mathsf{fin}}(\mathcal{S})$, Γ $\mathcal{R}_1$-attacks Θ iff Γ $\mathcal{R}_2$-attacks Θ.*

Proof. Suppose that Γ $\mathcal{R}_1$-attacks Θ. Then there are $A, B \in \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$ such that $\mathsf{Supp}(A) = \Gamma$, $\mathsf{Supp}(B) = \Theta$, and $\mathsf{C}_{\mathcal{R}_1}(\Gamma, \mathsf{Conc}(A), \Theta)$ holds. Thus, A $\mathcal{R}_1$-attacks B. By **Att1**, there is an argument $A' \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S})$ with $\mathsf{Supp}(A') = \mathsf{Supp}(A)$, and A' $\mathcal{R}_2$-attacks B. So, $\mathsf{C}_{\mathcal{R}_2}(\Gamma, \mathsf{Conc}(A'), \Theta)$ also holds, and therefore Γ $\mathcal{R}_2$-attacks Θ.

Suppose now that Γ $\mathcal{R}_2$-attacks Θ. Thus, there are $A, B \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S})$ such that $\mathsf{Supp}(A) = \Gamma$, $\mathsf{Supp}(B) = \Theta$ and $\mathsf{C}_{\mathcal{R}_2}(\Gamma, \mathsf{Conc}A), \Theta)$ holds. So, A $\mathcal{R}_2$-attacks B. By **Att2** there is an argument $A' \in \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$ with $\mathsf{Supp}(A') = \mathsf{Supp}(A)$ and A' $\mathcal{R}_1^{\downarrow}$-attacks B. Let $B^{\downarrow} = \langle \Theta, \phi \rangle$, where $\phi \in \Theta$ (Note that $\Theta \neq \emptyset$, so by the reflexivity of $\vdash_{\mathfrak{L}_1}$, $B^{\downarrow} \in \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$). Then, $\mathsf{C}_{\mathcal{R}_1}(\Gamma, \mathsf{Conc}(A'), \Theta)$ also holds, and therefore Γ $\mathcal{R}_1$-attacks Θ. ☐

[7] In Condition **Att2**, the requirement A' $\mathcal{R}_1^{\downarrow}$-attacks B denotes that $\mathsf{C}_{\mathcal{R}_1}(\mathsf{Supp}(A'), \mathsf{Con}(A'), \mathsf{Supp}(B))$ holds. We do not require that A' $\mathcal{R}_1$-attacks B, since it may happen that $B \notin \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$.

Note 3. When $\mathfrak{L}_1$ is strictly included in $\mathfrak{L}_2$ (i.e., $\vdash_{\mathfrak{L}_1} \subsetneq \vdash_{\mathfrak{L}_2}$), there are sets $\mathcal{S}$ for which $\mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S}) \subsetneq \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S})$, in which case the corresponding logical argumentation frameworks are *not* the same (thus $\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}) \neq \mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})$). Yet, what Proposition 2 indicates is that when the sets of attacks of the two logical frameworks are corresponding, the *compact representations* of these frameworks are the same.

The next proposition relates logical inclusion to argumentative inclusion when the condition in Definition 9 is met.

Proposition 3. *Suppose that $\mathfrak{L}_1$ is included in $\mathfrak{L}_2$ and that $\langle \mathfrak{L}_1, \mathcal{A}_1 \rangle$ and $\langle \mathfrak{L}_2, \mathcal{A}_2 \rangle$ have corresponding attacks. Then, for every $\mathcal{S}$ and* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$, *it holds that $\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S})$ is argumentatively included in $\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})$.*

Proof. We show **Inc1** (the proof of **Inc2** is similar): Let $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}))$. By Item 1 of Proposition 1,

$$\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\} \in \mathsf{Sem}(\mathcal{SAF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S})).$$

By Proposition 2,

$$\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\} \in \mathsf{Sem}(\mathcal{SAF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})).$$

By Item 2 of Proposition 1,

$$(\dagger)\ \begin{array}{l}\{A \in \mathsf{Arg}_{\mathfrak{L}_2}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\}\} \\ \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})).\end{array}$$

The first line in $(\dagger)$ is $\mathcal{E}^{\uparrow}$, thus $\mathcal{E}^{\uparrow} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S}))$. ☐

As a corollary of Proposition 3, we have the next results:

Corollary 1. *If $\mathfrak{L}_1$ is included in $\mathfrak{L}_2$ and $\langle \mathfrak{L}_1, \mathcal{A}_1 \rangle$ and $\langle \mathfrak{L}_2, \mathcal{A}_2 \rangle$ have corresponding attacks, then for every $\mathcal{S}$ and* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$,

- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}_2,\mathcal{A}_2}_{\cap\mathsf{Sem}} \psi$ *if* $\{\phi \mid \mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}_1,\mathcal{A}_1}_{\cap\mathsf{Sem}} \phi\} \vdash_{\mathfrak{L}_2} \psi$
- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}_2,\mathcal{A}_2}_{\Cap\mathsf{Sem}} \psi$ *if* $\{\phi \mid \mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}_1,\mathcal{A}_1}_{\Cap\mathsf{Sem}} \phi\} \vdash_{\mathfrak{L}_2} \psi$
- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}_2,\mathcal{A}_2}_{\cup\mathsf{Sem}} \psi$ *if* $\psi \in \bigcup_{\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}))} \mathsf{Cn}_{\mathfrak{L}_2}\{\phi \mid \exists(\Gamma, \phi) \in \mathcal{E}\})$
- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}_1,\mathcal{A}_1}_{\cap\mathsf{Sem}} \psi$ *if* $\exists\langle \Gamma, \psi \rangle \in \bigcap \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S})) \cap \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$
- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}_1,\mathcal{A}_1}_{\Cap\mathsf{Sem}} \psi$ *if* $\forall \mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_2,\mathcal{A}_2}(\mathcal{S}))\ \exists\langle \Gamma, \psi \rangle \in \mathcal{E} \cap \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$
- $\mathcal{S} \mathrel{|\!\sim}^{\mathfrak{L}_1,\mathcal{A}_1}_{\cup\mathsf{Sem}} \psi$ *if* $\exists\langle \Gamma, \psi \rangle \in \left(\bigcup \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L}_1,\mathcal{A}_1}(\mathcal{S}))\right) \cap \mathsf{Arg}_{\mathfrak{L}_1}(\mathcal{S})$

Next, we demonstrate the results above in three cases. In each case one starts with a framework based on a 3-valued logic: Bochvar (1938), Kleene (1952), Priest (1989). This framework is used for generating essential conclusions from a concise setting, and only then a transformation is made to a more standard framework, based on classical logic. As guaranteed by our results, a careful choice of (corresponding) attack rules in each case allows to preserve the argumentative inclusion between the resulting logical frameworks.

4.1 Application I: Bochvar B3 and Classical Logic

Bochvar's 3-valued logic B3 (Bochvar 1938) (also known as *weak* Kleene logic, as opposed to *strong* Kleene logic, considered in the next section) can be represented by the two classical truth values t, f and a third intermediate element i, together with the following truth tables for disjunction, conjunction, and negation:

$\vee$	t	f	i
t	t	t	i
f	t	f	i
i	i	i	i

$\wedge$	t	f	i
t	t	f	i
f	f	f	i
i	i	i	i

	$\neg$
t	t
f	f
i	i

Thus, on $\{t, f\}$ the truth table coincide with those of classical logic CL, while the third element i has an "infectious" effect: compound formulas are assigned the value i iff at least one of their subformulas has the value i. Accordingly, $\langle \mathcal{S}, \psi \rangle$ is a B3-argument (that is, $\mathcal{S} \vdash_{\mathsf{B3}} \psi$), if every B3-interpretation that assigns t to every formula in $\mathcal{S}$, also assigns t to ψ.

Note 4. Some remarks and illustrations are in-order here:

(a) If $\mathsf{Atoms}(\psi) \subseteq \mathsf{Atoms}(\mathcal{S})$ and if $\mathcal{S}$ is classically consistent, then $\mathcal{S} \vdash_{\mathsf{B3}} \psi$ iff $\mathcal{S} \vdash_{\mathsf{CL}} \psi$. In this case, then, B3-arguments and CL-arguments coincide. Moreover, $\mathcal{S}$ is classically consistent iff it is B3-consistent.

(b) If $\mathsf{Atoms}(\psi) \not\subseteq \mathsf{Atoms}(\mathcal{S})$ then $\mathcal{S} \not\vdash_{\mathsf{B3}} \psi$. Thus, for instance, $\langle \{p\}, p \vee q \rangle$ is *not* a B3-argument (consider a B3-interpretation that assigns t to p and i to q).

(c) The properties in Items (a) and (b) above render B3 particularly interesting for applications in argumentation. B3-inferences are classical as long as the reasoner "stays on topic", while it disallows arguments that go "off-topic" (Beall 2016). The example in Item (b) constitutes such a case: the disjunct q in the conclusion $p \vee q$ has nothing to do with the given support p. In this interpretation, the third truth value i expresses that a sentence is "off-topic".

(d) Generally, CL and B3 have different arguments and therefore may lead to different, non-corresponding, argumentative selections. Therefore, unless the attack rules are corresponding w.r.t. these logics, argumentative inclusion might get lost, as illustrated in the next example.

Example 5. Consider $\mathcal{S} = \{p \wedge q, \neg p\}$. Then,

- $A = \langle \{p \wedge q\}, \neg\neg p \rangle \in \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}) \cap \mathsf{Arg}_{\mathsf{B3}}(\mathcal{S})$, but
- $B = \langle \{\neg p\}, \neg(p \wedge q) \rangle \in \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}) \setminus \mathsf{Arg}_{\mathsf{B3}}(\mathcal{S})$.

Thus, for instance, in frameworks that are induced either from B3 or from CL, and having Direct Defeat (DirDef) as the sole attack rule, A DirDef-attacks $C = \langle \neg p, \neg p \rangle$, but only in frameworks that are induced from CL, C can be defended (e.g., by B) from this attack. In fact, it holds that $\mathsf{Arg}_{\mathsf{CL}}(\{\neg p\}) \in \mathsf{Stb}(\mathcal{AF}_{\mathsf{CL},\{\mathrm{DirDef}\}}(\mathcal{S}))$, while $\mathsf{Arg}_{\mathsf{B3}}(\{\neg p\}) \notin \mathsf{Stb}(\mathcal{AF}_{\mathsf{B3},\{\mathrm{DirDef}\}}(\mathcal{S}))$.

The situation in Example 5 seems undesired. However, as indicated in Note 4(c), B3 is an attractive logic for applications in argumentation that is very close to classical logic. So, the question arises, whether B3 can serve as a base logic in logical argumentation in such a way that argumentative inclusion (Definition 8) holds for classical logic.

For the purpose of utilizing B3 as a base logic for argumentation frameworks, we enhance B3 with a verum ($\vdash_{\mathsf{B3}}$-truth) constant T that is always interpreted as t. We call the resulting logic $\mathsf{B3_T}$.

Next, we consider cases where logical inclusion is preserved when trading $\mathsf{B3_T}$ by CL. For this, we introduce the reductio attacks in Table 2.

Acronym	Attacking Argument	Attacked Argument	Conditions
Red	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$	$\{\psi_1\} \cup \mathcal{S}_2 \vdash \neg\psi_1$
FullRed	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2, \psi_2 \rangle$	$\{\psi_1\} \cup \mathcal{S}_2 \vdash \neg\psi_1$
DirRed	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \{\varphi\} \cup \mathcal{S}_2', \psi_2 \rangle$	$\{\varphi, \psi_1\} \vdash \neg\psi_1$

Table 2: Reductio attacks. We again abbreviate the rules' names: 'red' and 'dir' stand, respectively, for 'reductio' and 'direct'.

These attacks have the form of an 'argumentum ad absurdum' (also known as reductio). To see this, consider the direct variant where $A = \langle \mathcal{S}, \psi \rangle$ attacks $B = \langle \{\phi\} \cup \mathcal{S}', \psi' \rangle$ and $\{\phi, \psi\} \vdash \neg\psi$ holds. A establishes that ψ is true, while $\{\phi, \psi\} \vdash \neg\psi$ expresses that from one of the premises of argument B a contradiction follows, namely that ψ is false. Therefore B has to be rejected.

Lemma 3. *Consider the following two cases: (i)* $\mathfrak{L}_1 = \mathsf{B3_T}$ *and* $\mathfrak{L}_2 = \mathsf{CL}$*, (ii)* $\mathfrak{L}_1 = \mathsf{CL}$ *and* $\mathfrak{L}_2 = \mathsf{CL}$*. In both cases,*

1. *Reductio and Defeat are corresponding attacks (recall Definition. 9) relative to* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$*,*
2. *Full Reductio and Full Defeat are corresponding attacks relative to* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$*.*
3. *Direct Reductio and Direct Defeat are corresponding attacks relative to* $\mathfrak{L}_1$ *and* $\mathfrak{L}_2$*,*

Proof. Let $\mathcal{S}$ be a set of $\mathcal{L}_{\mathsf{T}}$-formulas and let $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}$. In the proof we rely on the following two fact:

F1 If $\mathsf{Atoms}(\phi) \subseteq \mathsf{Atoms}(\mathcal{S})$, then $\mathcal{S} \vdash_{\mathsf{CL}} \phi$ iff $\mathcal{S} \vdash_{\mathsf{B3_T}} \phi$.

F2 $\mathcal{S}$ is classically inconsistent iff $\mathcal{S}$ is $\mathsf{B3_T}$-inconsistent. In this case, $\mathcal{S} \vdash_{\mathsf{B3_T}} \phi$ for every $\phi \in \mathcal{L}_{\mathsf{T}}$.

Now, we paradigmatically show the lemma for Item 1 and Case (i) (respectively, Case (ii)). For **Att1**, suppose that $A = \langle \mathcal{S}_1, \psi_1 \rangle$ DirRed-attacks $B = \langle \mathcal{S}_2' \cup \{\varphi\}, \psi_2 \rangle$, where $A, B \in \mathsf{Arg}_{\mathsf{B3_T}}(\mathcal{S})$ (respectively, where $A, B \in \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S})$). Then $\{\varphi, \psi_1\} \vdash_{\mathsf{B3_T}} \neg\psi_1$ (respectively, $\{\varphi, \psi_1\} \vdash_{\mathsf{CL}} \neg\psi_1$). In any case, by **F1**, $\{\varphi, \psi_1\} \vdash_{\mathsf{CL}} \neg\psi_1$, and so $\psi_1 \vdash_{\mathsf{CL}} \neg\varphi$. Thus, A directly defeats B.

For **Att2**, suppose that $A = \langle \mathcal{S}_1, \psi_1 \rangle$ directly defeats $B = \langle \{\varphi\} \cup \mathcal{S}_2, \phi \rangle$. Then $\psi_1 \vdash_{\mathsf{CL}} \neg\varphi$. We distinguish between two cases: (a) $\mathcal{S}_1 \neq \emptyset$ and (b) $\mathcal{S}_1 = \emptyset$.

- Suppose first (a). Then $\bigwedge\mathcal{S}_1 \vdash_{\mathsf{CL}} \neg\varphi$, and hence $\bigwedge\mathcal{S}_1, \varphi \vdash_{\mathsf{CL}} \neg\bigwedge\mathcal{S}_1$. By **F2**, $\bigwedge\mathcal{S}_1, \varphi \vdash_{\mathsf{B3_T}} \neg\bigwedge\mathcal{S}_1$. Thus, $A' = \langle \mathcal{S}_1, \bigwedge\mathcal{S}_1 \rangle \in \mathsf{Arg}_{\mathsf{B3_T}}(\mathcal{S})$ (respectively, $A' = \langle \mathcal{S}_1, \bigwedge\mathcal{S}_1 \rangle \in \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S})$) DirRed-attacks B. Hence, $\mathsf{C}_{\mathrm{DirRed}}(\mathcal{S}_1, \bigwedge\mathcal{S}_1, \mathsf{Supp}(B))$ holds for $\mathsf{B3_T}$ (respectively, for CL), which assures **Att2**.

- Suppose now (b). So, $\vdash_{\mathsf{CL}} \neg\varphi$ and thus $\{\mathsf{T}, \varphi\} \vdash_{\mathsf{CL}} \neg\mathsf{T}$. By **F2**, $\{\mathsf{T}, \varphi\} \vdash_{\mathsf{B3}_\mathsf{T}} \neg\mathsf{T}$. So, $\langle\emptyset, \mathsf{T}\rangle$ DirRed-attacks B. Hence, $\mathsf{C}_{\mathrm{DirRed}}(\emptyset, \mathsf{T}, \mathsf{Supp}(B))$ holds for $\mathsf{B3}_\mathsf{T}$ (respectively, for CL), which again assures **Att2**. □

Note 5. The reason we enhanced B3 with T is to obtain **Att2**. Note that, for instance, $\langle\emptyset, \mathsf{T}\rangle$ DirRed-attacks $\langle\{p \wedge \neg p\}, q\rangle$ (since $\mathsf{T}, p \wedge \neg p \vdash_{\mathsf{B3}_\mathsf{T}} \neg\mathsf{T}$), but $\langle\emptyset, \mathsf{T}\rangle$ is *not* an argument according to B3. Thus Lemma 3 fails for B3.

By Proposition 3, and since $\mathsf{B3}_\mathsf{T}$ is included in CL, we have the following corollary:

Corollary 2. *Let* $\mathcal{A}_1 \subseteq \{\mathrm{Red}, \mathrm{FullRed}, \mathrm{DirRed}\}$ *and* $\mathcal{A}_2 \subseteq \{\mathrm{Def}, \mathrm{FullDef}, \mathrm{DirDef}\}$ *be two non-empty sets of attacks that correspond relative to* $\mathsf{B3}_\mathsf{T}$ *and* CL *as described in Lemma 3. Then, for every set of formulas* $\mathcal{S}$ *and semantics* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$*, the framework* $\mathcal{AF}_{\mathsf{B3}_\mathsf{T},\mathcal{A}_1}(\mathcal{S})$ *is* Sem*-argumentatively included in* $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_2}(\mathcal{S})$.

Example 6. Consider again Example 5, where $\mathsf{B3}_\mathsf{T}$ is the underlying logic, but this time DirRed is the attack rule (instead of DirDef). We still have that A and C are in $\mathsf{Arg}_{\mathsf{B3}_\mathsf{T}}(\mathcal{S})$, but now C defends itself from the attack of A, since it DirRed-attacks A. It follows that $\mathsf{Arg}_{\mathsf{B3}_\mathsf{T}}(\{\neg p\}) \in \mathsf{Stb}(\mathcal{AF}_{\mathsf{B3}_\mathsf{T},\{\mathrm{DirDef}\}}(\mathcal{S}))$, as intuitively expected (and as is the case when $\mathsf{B3}_\mathsf{T}$ is traded by CL). As shown in the last corollary, this is not a coincidence.

In sum, argumentative reasoning with CL can be preserved when switching to a logic that enforces relevance of a specific kind, namely staying "on-topic". This coheres with insights from informal argumentation (Blair 1992).

We conclude this case study by noting another corollary of Proposition 3 and Lemma 3: The reductio-based attacks are also argumentatively equivalent to defeat-based attacks in the context of classical logic.

Corollary 3. *Suppose that* $\mathcal{A}_1 \subseteq \{\mathrm{Red}, \mathrm{FullRed}, \mathrm{DirRed}\}$ *and* $\mathcal{A}_2 \subseteq \{\mathrm{Def}, \mathrm{FullDef}, \mathrm{DirDef}\}$ *are two non-empty sets of attacks that correspond relative to* CL *and* CL *as described in Lemma 3. For every set of formulas* $\mathcal{S}$ *and semantics* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$*, the frameworks* $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_1}(\mathcal{S})$ *and* $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_2}(\mathcal{S})$ *are* Sem*-argumentatively equivalent, namely: each one of them is* Sem*-argumentatively included in the other.*

4.2 Application II: Kleene K3 and Classical Logic

(Strong) Kleene's logic K3 (Kleene 1952) is perhaps the best-known 3-valued logic. Its negation connective is the same as that of Bochvar's logic, while the conjunction $\wedge$ and the disjunction $\vee$ are defined by the minimum and the maximum relative to the ordering $\mathsf{f} < \mathsf{i} < \mathsf{t}$.

$\vee$	t	f	i
t	t	t	t
f	t	f	i
i	t	i	i

$\wedge$	t	f	i
t	t	f	i
f	f	f	f
i	i	f	i

	$\neg$
t	t
f	f
i	i

Again, $\langle\mathcal{S}, \psi\rangle$ is a K3-argument (that is, $\mathcal{S} \vdash_{\mathsf{K3}} \psi$), if every K3-interpretation that assigns t to every formula in $\mathcal{S}$ also assigns t to ψ. In particular (like B3), K3 does not have tautologies (and so there are no tautological K3-arguments), and it is *paradefinite*: the rule of excluded middle does *not* hold in it ($\nvdash_{\mathsf{K3}} \psi \vee \neg\psi$).

In order to allow tautological arguments, and better accommodate K3 for argumentative inclusion in classical logic, we again add the propositional constant T (with the usual meaning of representing truth) to the language. The resulting logic is denoted $\mathsf{K3}_\mathsf{T}$.

The logic K3 (respectively, $\mathsf{K3}_\mathsf{T}$) lays strictly between classical logic and B3 (respectively, $\mathsf{B3}_\mathsf{T}$). For instance, we have that $p \vdash_{\mathsf{K3}} p \vee q$ while $p \nvdash_{\mathsf{B3}} p \vee q$. Similarly for $\mathsf{K3}_\mathsf{T}$ and $\mathsf{B3}_\mathsf{T}$.

Lemma 4. *Relative to* $\mathsf{K3}_\mathsf{T}$ *and* CL*, we have the following correspondences:*

1. *Reductio corresponds to Defeat,*
2. *Full Reductio corresponds to Full Defeat.*
3. *Direct Reductio corresponds to Direct Defeat,*

By Proposition 3, and since $\mathsf{K3}_\mathsf{T}$ is included in CL, we have the following result:

Corollary 4. *Suppose that* $\mathcal{A}_1 \subseteq \{\mathrm{Red}, \mathrm{FullRed}, \mathrm{DirRed}\}$ *and* $\mathcal{A}_2 \subseteq \{\mathrm{Def}, \mathrm{FullDef}, \mathrm{DirDef}\}$ *are two corresponding non-empty sets of attacks relative to* $\mathsf{K3}_\mathsf{T}$ *and* CL*, as described in Lemma 4. For every set of formulas* $\mathcal{S}$ *and semantics* $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$*, it holds that* $\mathcal{AF}_{\mathsf{K3}_\mathsf{T},\mathcal{A}_1}(\mathcal{S})$ *is* Sem*-argumentatively included in* $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_2}(\mathcal{S})$.

4.3 Application III: Priest LP and Classical Logic

Priest's 3-valued logic LP (Priest 1989)[8] has the same truth tables for the basic connectives $\{\neg, \wedge, \vee\}$ as those of strong Kleene's 3-valued logic. The difference is that in LP the middle element (i) is designated. Thus, $\langle\mathcal{S}, \psi\rangle$ is an LP-argument (and so $\mathcal{S} \vdash_{\mathsf{LP}} \psi$), if every LP-interpretation that assigns either t or i to every formula in $\mathcal{S}$ also assigns t or i to ψ. This implies, in particular, that LP (unlike K3 and B3) is not paracomplete ($\vdash_{\mathsf{LP}} \psi \vee \neg\psi$)[9] but it is *paraconsistent*, i.e., avoids logical explosion: $p, \neg p \nvdash_{\mathsf{LP}} q$ (consider for this a valuation in which p is assigned i, while q is assigned f).

Example 7. Consider $\mathcal{S} = \{p \vee q, \neg p, \neg q\}$. Note that $\langle\{p \vee q, \neg p\}, q\rangle, \langle\{p \vee q, \neg q\}, p\rangle \notin \mathsf{Arg}_{\mathsf{LP}}(\mathcal{S})$. This is due to the fact that disjunctive syllogism does not hold for LP. For instance, when Direct Defeat is the sole attack rule, the only stable set of the corresponding LP-based argumentation framework for $\mathcal{S}$ is $\mathsf{Arg}_{\mathsf{LP}}(\{\neg p, \neg q\})$ since $\langle\{\neg p, \neg q\}, \neg(p \vee q)\rangle$ attacks every argument with $p \vee q$ among its premises. This is an undesired asymmetry since one also expects $\mathsf{Arg}_{\mathsf{LP}}(\{p \vee q, \neg p\})$ and $\mathsf{Arg}_{\mathsf{LP}}(\{p \vee q, \neg q\})$ to be stable sets.

To avoid the problem in the last example, we introduce in Table 3 another family of attack rules, called LP-defeats.

Lemma 5. *Consider the following two cases: (i)* $\mathfrak{L}_1 = \mathsf{LP}$ *and* $\mathfrak{L}_2 = \mathsf{CL}$*, (ii)* $\mathfrak{L}_1 = \mathsf{CL}$ *and* $\mathfrak{L}_2 = \mathsf{CL}$*. In both cases, we have that:*

[8] Also attributed to Asenjo (1966).

[9] In fact, the theorems of LP are exactly those of CL; See for instance (Avron, Arieli, and Zamansky 2018).

Acronym	Attacking	Attacked	Attack Condition
LPDef	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$	(†)
FullLPDef	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2, \psi_2 \rangle$	(†)
DirLPDef	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \{\varphi\} \cup \mathcal{S}_2', \psi_2 \rangle$	(‡)

Table 3: LP-Defeats. Condition (†) is $\psi_1 \vdash \neg \bigwedge \mathcal{S}_2 \vee \bigvee\{(\varphi \wedge \neg\varphi) \mid \varphi \in \mathcal{S}_1\}$ and Condition (‡) is $\psi_1 \vdash \neg\varphi \vee \bigvee\{(\varphi \wedge \neg\varphi) \mid \varphi \in \mathcal{S}_1\}$.

1. *LP-Defeat corresponds to Defeat, relative to $\mathfrak{L}_1$ and $\mathfrak{L}_2$,*
2. *Full LP-Defeat corresponds to Full Defeat, relative to $\mathfrak{L}_1$ and $\mathfrak{L}_2$,*
3. *Direct LP-Defeat corresponds to Direct Defeat, relative to $\mathfrak{L}_1$ and $\mathfrak{L}_2$.*

By Proposition 3, and since LP is included in CL, we then have the next corollary:

Corollary 5. *Let $\mathcal{A}_1 \subseteq \{\text{LPDef}, \text{LPFullDef}, \text{LPDirDef}\}$ and $\mathcal{A}_2 \subseteq \{\text{Def}, \text{FullDef}, \text{DirDef}\}$ be two non-empty attack sets that correspond relative to LP and CL as described in Lemma 5. Then for every set of formulas $\mathcal{S}$ and semantics $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$, it holds that $\mathcal{AF}_{\mathsf{LP},\mathcal{A}_1}(\mathcal{S})$ is Sem-argumentatively included in $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_2}(\mathcal{S})$.*

Example 8. Consider again the set $\mathcal{S} = \{p \vee q, \neg p, \neg q\}$ from Example 7, where LP is the underlying logic. When DirLPDef is the attack rule we avoid the problem in Example 7, since this time, as followed from the last corollary, $\mathsf{Arg}_{\mathsf{LP}}(\{\neg p, \neg q\})$, $\mathsf{Arg}_{\mathsf{LP}}(\{p \vee q, \neg p\})$ and $\mathsf{Arg}_{\mathsf{LP}}(\{p \vee q, \neg q\})$ are all stable extensions of $\mathcal{AF}_{\mathsf{LP},\{\text{DirLPDef}\}}(\mathcal{S})$.

Finally, we note that, by Proposition 3 and Lemma 5, LP-defeat-based attacks are also argumentatively equivalent to defeat-based attacks in the context of classical logic:

Corollary 6. *Let $\mathcal{A}_1 \subseteq \{\text{LPDef}, \text{LPFullDef}, \text{LPDirDef}\}$ and $\mathcal{A}_2 \subseteq \{\text{Def}, \text{FullDef}, \text{DirDef}\}$ be two non-empty attack sets that correspond relative to CL and CL as described in Lemma 5. For every set of formulas $\mathcal{S}$ and semantics $\mathsf{Sem} \in \{\mathsf{Adm}, \mathsf{Cmp}, \mathsf{Grd}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Prf}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$, the frameworks $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_1}(\mathcal{S})$ and $\mathcal{AF}_{\mathsf{CL},\mathcal{A}_2}(\mathcal{S})$ are Sem-argumentatively equivalent.*

5 Links to Assumption-Based Argumentation

In what follows we show that the results in this paper may be used for another purpose: converting logic-based frameworks to simple contrapositive extensions (Heyninck and Arieli 2020) of assumption-based frameworks (Bondarenko et al. 1997) (See, e.g., (Čyras et al. 2018) for a survey on such frameworks and their applications).

Definition 10. An *assumption-based framework* (an ABF, for short) is a tuple $\mathcal{ABF} = \langle \mathfrak{L}, \mathcal{X}, \mathcal{S}, \sim \rangle$, where:

- $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is a propositional Tarskian logic.
- $\mathcal{X}$ (the *strict assumptions*) and $\mathcal{S}$ (the *defeasible assumptions*) are distinct (countable) sets of $\mathcal{L}$-formulas, where the former is $\vdash$-consistent and the latter is nonempty.
- $\sim : \mathcal{S} \to \wp(\mathcal{L})$ is a *contrariness operator*, assigning a finite set of $\mathcal{L}$-formulas to every defeasible assumption in $\mathcal{S}$, such that for every consistent and non-tautological formula $\psi \in \mathcal{S}$ it holds that $\psi \not\vdash \bigwedge \sim\psi$ and $\bigwedge \sim\psi \not\vdash \psi$.

A *simple contrapositive* ABF is an assumption-based framework $\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S}) = \langle \mathfrak{L}, \mathcal{X}, \mathcal{S}, \sim \rangle$, where

- for every $\psi \in \mathcal{S}$ it holds that $\sim\psi = \{\neg\psi\}$, and
- the logic $\mathfrak{L}$ is an explosive (i.e., for every $\mathcal{L}$-formula ψ, the set $\{\psi, \neg\psi\}$ is $\vdash$-inconsistent) and contrapositive (i.e., for every nonempty set of formulas Γ and formula ψ, it holds that $\Gamma \vdash \neg\psi$ iff for every $\phi \in \Gamma$ we have that $\Gamma \setminus \{\phi\}, \psi \vdash \neg\phi$).

Let $\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S})$ be a (simple contrapositive) ABF. Let also $\Delta, \Theta \subseteq \mathcal{S}$, and $\psi \in \mathcal{S}$. We say that Δ *attacks* ψ iff $\mathcal{X}, \Delta \vdash \phi$ for $\phi \in \sim\psi$. Also, Δ attacks Θ if Δ attacks some $\psi \in \Theta$. Dung semantics for (simple contrapositive) ABFs is defined just as in Definition 4 (see (Heyninck and Arieli 2020)).

Example 9. The (simple contrapositive) assumption-based argumentation framework $\langle \mathsf{CL}, \emptyset, \{p, \neg p, q\}, \neg \rangle$ is the same as the support-induced framework in Example 3 and has the same extensions as of the latter, as specified in Example 4. Proposition 4 below shows that this is not a coincidence.

Suppose now that q is a strict assumption. The revised ABF is then $\langle \mathsf{CL}, \{q\}, \{\neg p, p\}, \neg \rangle$. Its attack diagram is represented in Figure 3.

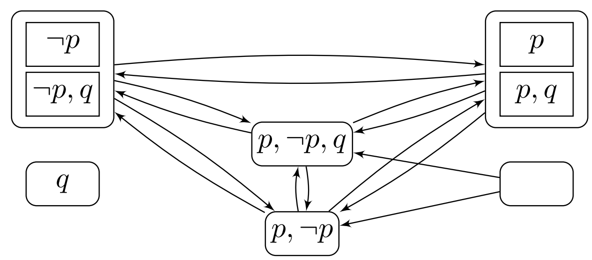

Figure 3: Assumption-based framework (Example 9).

Note that, since q appeared in every extension of the ABF as a defeasible assumption, treating it as a strict assumption does not alter the set of conclusions derived from the ABF.

To relate ABFs and logic-based argumentation frameworks, we need to extend the latter with strict (non-attackable) set of assumptions $\mathcal{X}$, in addition to the defeasible assumptions in $\mathcal{S}$. Next, we do so (cf. Definition 3):

Definition 11. Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and $\mathcal{A}$ a set of attack rules with respect to $\mathfrak{L}$. Let also $\mathcal{X}$ and $\mathcal{S}$ be two distinct sets of $\mathcal{L}$-formulas, where $\mathcal{X}$ is $\vdash$-consistent. The *(logical) argumentation framework* for $\mathcal{X}$ and $\mathcal{S}$, induced by $\mathfrak{L}$ and $\mathcal{A}$, is the pair $\mathcal{AF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}^{\mathcal{X}}(\mathcal{A}) \rangle$, where $\mathsf{Arg}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S}) = \{\langle \mathcal{S}', \psi \rangle \mid \mathcal{X}, \mathcal{S}' \vdash \psi \text{ and } \mathcal{S}' \subseteq \mathcal{S}\}$ and $\mathit{Attack}^{\mathcal{X}}(\mathcal{A})$ is defined by $(A_1, A_2) \in \mathit{Attack}^{\mathcal{X}}(\mathcal{L})$ iff there is some $\mathcal{R}_{\mathcal{X}} \in \mathcal{A}$ such that A_1 $\mathcal{R}_{\mathcal{X}}$-attacks A_2.

[9]Again, for simplifying the figure, nodes sharing identical incoming and outgoing edges are grouped as inner nodes in a single outer node.

In $\mathcal{AF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$, an argument remains a pair $A = \langle \mathcal{S}', \psi \rangle$ with $\mathcal{S}' \subseteq \mathcal{S}$, but ψ now follows from $\mathcal{S}' \cup \mathcal{X}$, where $\mathcal{X}$ is a set of strict assumptions. These assumptions are protected from attacks, as defined by the rules in $Attack^{\mathcal{X}}(\mathcal{A})$. For instance, a variation $\mathrm{DirDef}_{\mathcal{X}}$ of DirDef appears in Table 4:

Acronym	Attacking	Attacked	Attack Conditions
$\mathrm{DirDef}_{\mathcal{X}}$	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \{\varphi\} \cup \mathcal{S}'_2, \psi_2 \rangle$	$\mathcal{X}, \psi_1 \vdash \neg\varphi,\ \varphi \notin \mathcal{X}$

Table 4: Direct defeat with strict assumptions.

Thus, based on the formulas in $\mathcal{X}$, the conclusion ψ_1 of the attacking argument entails the negation of some formula in the support of the attacked argument, provided that this formula is not a strict assumption (that cannot be attacked). Clearly, Definition 3 is a particular case of Definition 11 when $\mathcal{X} = \emptyset$ (and DirDef is the same as $\mathrm{DirDef}_{\emptyset}$).

Now, we are ready to relate logical argumentation framework and assumption-based argumentation frameworks.

Proposition 4. *Let $\mathfrak{L}$ be explosive and contrapositive logic, and let $\mathcal{A} = \{\mathrm{DirDef}_{\mathcal{X}}\}$. Given a logical argumentation framework $\mathcal{AF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ (Definition 11), let $\mathcal{SAF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ be the corresponding support-induced argumentation framework (Definition 7), and let $\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S})$ be the corresponding simple contrapositive ABF (Definition 10). Then, for every* $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}, \mathsf{SStb}, \mathsf{Idl}, \mathsf{Egr}, \mathsf{Stg}\}$,

$$\Xi \in \mathsf{Sem}(\mathcal{SAF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})) \textit{ iff } \Xi \in \mathsf{Sem}(\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S})).$$

Moreover, for every such Ξ, it holds that:

$$\{A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid \mathsf{Supp}(A) \in \Xi\} \in \mathsf{Sem}(\mathcal{AF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})).$$

Additionally, for every $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$, we have:

$$\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\} \in \mathsf{Sem}(\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S})),$$
$$\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\} \in \mathsf{Sem}(\mathcal{SAF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})).$$

Proof. By the definitions of SAFs and ABFs, as since the attack relation of the latter is DirDef, it is easy to see that

$$(\dagger)\ \mathsf{Sem}(\mathcal{SAF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})) = \mathsf{Sem}(\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S}))$$

for every semantics Sem. In fact, these structures are isomorphic, since they have the same nodes (arguments) and edges (attacks). Indeed, for every $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}$,

$\mathcal{S}_1$ attacks $\mathcal{S}_2$ in $\mathcal{SAF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ iff
$(\mathcal{S}_1, \mathcal{S}_2) \in S\text{-}Attack(\mathcal{A})$, iff
$\exists \psi_1$ s.t. $\langle \mathcal{S}_1, \psi_1 \rangle \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and $\mathsf{C}_{\mathrm{DirDef}_{\mathcal{X}}}(\mathcal{S}_1, \psi_1, \mathcal{S}_2)$, iff
$\mathcal{X}, \mathcal{S}_1 \vdash \psi_1$ and $\mathcal{X}, \psi_1 \vdash \neg\varphi$ for some $\varphi \in \mathcal{S}_2$, iff
$\mathcal{X}, \mathcal{S}_1 \vdash \neg\varphi$ for some $\varphi \in \mathcal{S}_2$, iff
$\mathcal{S}_1$ attacks $\mathcal{S}_2$ in $\mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S})$.

Let now $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$. By Item 1 of Proposition 1, $\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\} \in \mathsf{Sem}(\mathcal{SAF}^{\mathcal{X}}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$,[10] and by (†), also $\{\mathsf{Supp}(A) \mid A \in \mathcal{E}\} \cup \{\emptyset\} \in \mathcal{ABF}^{\mathcal{X}}_{\mathfrak{L}}(\mathcal{S}))$. The converse follows similarly from Item 2 in Proposition 1. □

[10] Proposition 1 does not take into account strict assumptions, but it is not difficult to extend the result to this case as well.

Example 10. Consider again the two ABFs in Example 9 (i.e., where q is either defeasible or strict assumption). By Example 4 and Proposition 4 we get that the grounded, ideal and eager extension of these ABFs is $\{\emptyset, \{q\}\}$, while the preferred, stable, semi-stable and stage extensions are $\{\emptyset, \{q\}, \{p\}, \{q, p\}\}$ and $\{\emptyset, \{q\}, \{\neg p\}, \{q, \neg p\}\}$.

To summarize the results in this section, we have obtained a correspondence among three forms of argumentative frameworks:

1. logic-based frameworks with strict assumptions,
2. the related support-induced frameworks, and
3. the corresponding assumption-based frameworks.

This correspondence is shown with respect to direct defeat, since this is the rule traditionally used for attacks in ABFs. However, under some straightforward modifications it is not difficult to show further results, similar to those of Proposition 4, with respect to other attack rules.

6 Conclusion

In this paper, we addressed two key and interconnected issues in the representation of logical argumentation frameworks: how to represent them compactly by finite frameworks, and how to translate a framework based on a logic that captures the core reasoning into a broader and more conventional representation without losing logical inferences. In process, we introduced some new attack rules that allow to bridge between argumentation frameworks based on specific logics, and provided a conversion method between two well-established argumentative settings: logic-base frameworks and assumption-based frameworks (see (Borg 2020)).

The problem of reducing the size of logic-based argumentation frameworks has already been considered in the literature (see, e.g., (Amgoud, Besnard, and Vesic 2011)). Such reductions are often formulated by equivalence classes (cf. Note 2 and (Arieli et al. 2022, Section 4.3)), but are applied to specific cases.[11] Here, we consider broader settings and stricter reductions, yielding frameworks with a finite number of arguments (and not only finite number of attacks per argument, as in (Amgoud, Besnard, and Vesic 2011)).

The issues considered in this work bring up a bunch of new questions. One such question is how to adjust the attack rules when changing the base logics. The interplay between the nature of the underlying logic and the formulation of the attack rules has already been considered in the literature (see, e,.g., (Corsi and Fermüller 2017; Shi, Smets, and Velázquez-Quesada 2018; Arieli and Straßer 2020; Corsi 2025)). However, all these works assume a fixed setting from which base logics and attack rules need to be correlated. The adaptation of given attack rules when the setting itself changes (e.g., for having a more compact representation), is a topic that, to our knowledge, has not yet been investigated, and remains a subject for future research.

[11] For example, (Amgoud, Besnard, and Vesic 2011) focuses on classical logic, direct undercut attacks, stable semantics, and arguments with subset-minimal and classically consistent supports.

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