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An Analysis of The Role of Syntax in Inductive Inference

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Abstract

Inductive inference is a well-studied form of nonmonotonic reasoning in which various forms of inference are based on conditional belief bases rather than belief bases consisting of classical logic statements. Given its nonmonotonic nature, many important logical properties that are taken for granted in the classical case do not necessarily carry over to inference involving conditionals. In this paper we consider two such properties—*equivalence* and *language-independence*. More specifically, we provide different notions of equivalence in the conditional case, and show which of these are satisfied by which forms of conditional inference. Similarly, we consider two different versions of language independence, and test various forms of conditional inference against these. As its main overall contribution, the paper provides deeper theoretical insights into the field of inductive inference.

1 Introduction

Nonmonotonic reasoning is one of the original areas of Knowledge Representation and Reasoning, and a driving force behind its early development. It remains a major area of KR today (Delgrande et al. 2024). Because of its departure from classical (monotonic) reasoning, many fundamental properties that are taken for granted in classical logics do not carry over to nonmonotonic formalisms. In this paper we focus on two such properties involving the interplay between syntax and semantics, *equivalence* and *language-independence*, and consider their role within the context of *inductive inference*, a well-studied form of nonmonotonic reasoning in which inference is based on conditional belief bases rather than belief bases consisting of classical logic statements (Kern-Isberner, Beierle, and Brewka 2020). More specifically, we make the following contributions:

- We introduce different versions of *equivalence* for inductive inference, point out the relationships between them, and test some well-known forms of inductive inference against them.
- We introduce properties constraining model-based inductive operators to be tightly coupled to the conditional statements provided in a belief base, and show that this is incompatible with one of the notions of equivalence and the property of Syntax Splitting (Kern-Isberner, Beierle, and Brewka 2020).
- Based on the failure of the well-studied form of inductive inference known as *lexicographic inference* (Lehmann 1995) to satisfy one of our notions of equivalence, we propose a variant of lexicographic inference that satisfies it.
- We introduce two versions of *language independence* for inductive inference and show that properties referred to as *conditional-functional* and *conditional-relational* ensures these forms of language independence.

The paper is organized as follows. Section 2 provides the various preliminaries required to present our contributions. Section 3 presents versions of equivalence for inductive inference, and tests this against the newly-added property of being conditional-based. Section 4 presents a version of lexicographic inference that satisfies the notion of pairwise equivalence introduced in the previous section. Section 5 introduces and studies two versions of language independence. Section 6 considers related work. Finally, Section 7 concludes and considers future work.

2 Preliminaries

In the following we recall preliminaries on propositional logic, and technical details on inductive inference.

2.1 Propositional Logic

For a finite set Σ of atoms, let $\mathcal{L}(\Sigma)$ be the corresponding propositional language constructed using the usual connectives $\wedge$ (*and*), $\vee$ (*or*), $\neg$ (*negation*), $\rightarrow$ (*material implication*) and $\leftrightarrow$ (*material equivalence*). A (classical) *interpretation* (also called *possible world*) ω for a propositional language $\mathcal{L}(\Sigma)$ is a function $\omega : \Sigma \rightarrow \{1, 0\}$ where 1 is understood to denote truth, and 0 to denote falsity. Let $\Omega(\Sigma)$ denote the set of all interpretations for Σ. We simply write Ω if the set of atoms is implicitly given, and similarly for $\mathcal{L}$. An interpretation ω *satisfies* (or is a *model* of) an atom $a \in \Sigma$, denoted by $\omega \models a$, if and only if $\omega(a) = 1$. The satisfaction relation $\models$ is extended to formulas in the usual way. As an abbreviation we sometimes identify an interpretation ω with its *complete conjunction*, i. e., if $a_1, \ldots, a_n \in \Sigma$ are those atoms that are assigned 1 by ω and $a_{n+1}, \ldots, a_m \in \Sigma$ are those propositions that are assigned 0 by ω we identify ω with $a_1 \wedge \ldots \wedge a_n \wedge \neg a_{n+1} \ldots \wedge \neg a_m$, and will often abbreviate this as $a_1 \ldots a_n \overline{a_{n+1}} \ldots \overline{a_m}$ (or any permutation of this). For $X \subseteq \mathcal{L}(\Sigma)$ we also define $\omega \models X$ if and only if $\omega \models A$ for every $A \in X$. Define the set of models $\mathsf{Mod}_\Sigma(X) = \{\omega \in \Omega(\Sigma) \mid \omega \models X\}$ for every formula or set of formulas X (and use $\mathsf{Mod}(X)$ if Σ is clear). A formula or set of formulas X_1 *entails* another formula or set of formulas X_2, denoted by $X_1 \models X_2$, if $\mathsf{Mod}(X_1) \subseteq \mathsf{Mod}(X_2)$. Where $\theta \subseteq \Sigma$, and $\omega \in \Omega(\Sigma)$, we denote by ω^θ the restriction of ω to θ, i.e. ω^θ is the interpretation over Σ^θ that agrees with ω on all atoms in θ. Where $\Sigma_i, \Sigma_j \subseteq \Sigma$, $\Omega(\Sigma_i)$ will also be denoted by Ω_i for any $i \in \mathbb{N}$, and likewise $\Omega_{i,j}$ will denote $\Omega(\Sigma_i \cup \Sigma_j)$ (for $i, j \in \mathbb{N}$). Likewise, for some $X \subseteq \mathcal{L}(\Sigma_i)$, we define $\mathsf{Mod}_i(X) = \{\omega \in \Omega_i \mid \omega \models X\}$.

2.2 Reasoning with Nonmonotonic Conditionals

Given a language $\mathcal{L}$, conditionals are objects of the form $(B|A)$ where $A, B \in \mathcal{L}$. The set of all conditionals based on a language $\mathcal{L}$ is defined as: $(\mathcal{L}|\mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}$. We follow the approach of de Finetti (1937) who considers conditionals as *generalized indicator functions* for possible worlds or propositional interpretations ω:

$$(B|A)(\omega) = \begin{cases} 1 & : \quad \omega \models A \wedge B \\ 0 & : \quad \omega \models A \wedge \neg B \\ u & : \quad \omega \models \neg A \end{cases} \tag{1}$$

where u stands for *unknown* or *indeterminate*. In other words, a possible world ω *verifies* a conditional $(B|A)$ iff it satisfies both antecedent and conclusion ($(B|A)(\omega) = 1$); it *falsifies, or violates* it iff it satisfies the antecedent but not the conclusion ($(B|A)(\omega) = 0$); otherwise the conditional is *not applicable*, i. e., the interpretation does not satisfy the antecedent ($(B|A)(\omega) = u$). We say that ω *satisfies* a conditional $(B|A)$ iff it does not falsify it, i.e., iff ω satisfies its *material counterpart* $A \to B$. We will look at the semantics of conditionals given by total preorders (TPOs)[1] $\preceq \subseteq \Omega(\Sigma) \times \Omega(\Sigma)$. As is usual, given a preorder $\preceq$, we denote $\omega \preceq \omega'$ and $\omega' \preceq \omega$ by $\omega \approx \omega'$ and $\omega \preceq \omega'$ and $\omega' \not\preceq \omega$ by $\omega \prec \omega'$. Given a TPO $\preceq$ on possible worlds, representing relative plausibility, we define $A \prec B$ iff for every $\omega' \in \min_\preceq(\mathsf{Mod}(B))$ there is an $\omega \in \min_\preceq(\mathsf{Mod}(A))$ such that $\omega \prec \omega'$. This allows for expressing the validity of conditional inferences via stating that $A \mid\!\sim_\preceq B$ iff $(A \wedge B) \prec (A \wedge \neg B)$ (Makinson 1988) for a TPO. We say that a set $\Delta \subseteq (\mathcal{L}(\Sigma)|\mathcal{L}(\Sigma))$ of conditionals is *consistent* if there is an TPO $\preceq$ over $\Omega(\Sigma)$ s.t. $A \mid\!\sim_\preceq B$ for every $(B|A) \in \Delta$. In what follows, we will, for simplicity, always assume a set of conditionals is finite and consistent, and call such a set a *conditional belief base*. Δ is said to be valid w.r.t. a TPO if every element of Δ is valid w.r.t. to the TPO.

We can *marginalize* total preorders and even inference operators, i.e., restricting them to sublanguages, in a natural way: If $\Theta \subseteq \Sigma$ then any TPO $\preceq$ on $\Omega(\Sigma)$ uniquely induces a *marginalized TPO* $\preceq_{|\Theta}$ on $\Omega(\Theta)$ by setting

$$\omega_1^\Theta \preceq_{|\Theta} \omega_2^\Theta \text{ iff } \omega_1^\Theta \preceq \omega_2^\Theta. \tag{2}$$

Note that on the right hand side of the *iff* condition above $\omega_1^\Theta, \omega_2^\Theta$ are considered as propositions in the super-language $\mathcal{L}(\Omega)$. Hence $\omega_1^\Theta \preceq \omega_2^\Theta$ is well defined (Kern-Isberner and Brewka 2017). Similarly, any inference relation $\mid\!\sim$ on $\mathcal{L}(\Sigma)$ induces a *marginalized inference relation* $\mid\!\sim_{|\Theta}$ on $\mathcal{L}(\Theta)$ by setting $A \mid\!\sim_{|\Theta} B$ iff $A \mid\!\sim B$ for any $A, B \in \mathcal{L}(\Theta)$.

An obvious generalisation of total preorders are *ordinal conditional functions (OCFs)*, (also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$. (Spohn 1988). They express degrees of (im)plausibility of possible worlds and propositional formulas A by setting $\kappa(A) := \min\{\kappa(\omega) \mid \omega \models A\}$. A conditional $(B|A)$ is accepted by κ iff $A \mid\!\sim_\kappa B$ iff $\kappa(A \wedge B) < \kappa(A \wedge \neg B)$. Notice that any OCF induces a TPO on Ω, defined by $\omega_1 \preceq \omega_2$ iff $\kappa(\omega_1) \leqslant \kappa(\omega_2)$.

2.3 Inductive Inference Operators

In this paper we will be interested in inference operators $\mid\!\sim_\Delta$ parametrized by a finite conditional belief base Δ. Such inference operators are *induced by* Δ, in the sense that Δ serves as a starting point for the inferences in $\mid\!\sim_\Delta$. We call such operators *inductive inference operators*:

Definition 1 ((Kern-Isberner, Beierle, and Brewka 2020)). *An* inductive inference operator (from conditional belief bases) *is a mapping* $\mathbf{C}$ *that assigns to each* Σ *and each conditional belief base* $\Delta \subseteq (\mathcal{L}(\Sigma)|\mathcal{L}(\Sigma))$ *an inference relation* $\mid\!\sim_\Delta$ *on* $\mathcal{L}(\Sigma)$ *that satisfies the following basic requirement of* direct inference*:*

(DI) *If Δ is a conditional belief base and $\mid\!\sim_\Delta$ is an inference relation that is induced by Δ, then $(B|A) \in \Delta$ implies $A \mid\!\sim_\Delta B$.*

As already indicated in the previous subsection, inference operators can be obtained on the basis of TPOs, and OCFs, respectively (among others):

Definition 2. *A* model-based inductive inference operator for TPOs *(on Ω) is a mapping $\mathbf{C}^{tpo}$ that assigns to each conditional belief base Δ a TPO $\preceq_\Delta$ on Ω s.t. $A \mid\!\sim_{\preceq_\Delta} B$ for every $(B|A) \in \Delta$ (i.e., s.t.* **(DI)** *is ensured).*

A model-based inductive inference operator for OCFs *(on Ω) is a mapping $\mathbf{C}^{ocf}$ that assigns to each conditional belief base Δ an OCF κ_Δ on Ω s.t. Δ is accepted by κ_Δ (i.e., s.t.* **(DI)** *is ensured).*

Examples of inductive inference operators for OCFs include System Z (also called rational closure, (Goldszmidt and Pearl 1996; Pearl 1990), see Sec. 2.4) and c-representations with selection strategies ((Kern-Isberner 2002), see Sec. 2.6), whereas lexicographic inference ((Lehmann 1995), see Sec. 2.5) is an example of an inductive inference operator for TPOs.

This paper extends the study of properties of inductive inference operators. We first recall a property that has been recently introduced and studied, *syntax splitting* (Kern-Isberner, Beierle, and Brewka 2020). To define the property of syntax splitting we assume a conditional belief base Δ that can be split into subbases Δ^1, Δ^2 s.t. $\Delta^i \subset (\mathcal{L}_i|\mathcal{L}_i)$ with $\mathcal{L}_i = \mathcal{L}(\Sigma_i)$ for $i = 1, 2$ s.t. $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\Sigma_1 \cup \Sigma_2 = \Sigma$, writing $\Delta = \Delta^1 \bigcup_{\Sigma_1,\Sigma_2} \Delta^2$ whenever this is the case.

[1]A total preorder is a binary relation that is transitive and complete (and therefore reflexive), i.e., $\omega_1 \preceq \omega_2$ or $\omega_2 \preceq \omega_1$ for all ω_1, ω_2.

Definition 3 (Independence (**Ind**), (Kern-Isberner, Beierle, and Brewka 2020))**.** *An inductive inference operator* $\mathbf{C}$ *satisfies (***Ind***) if for any* $\Delta = \Delta^1 \bigcup_{\Sigma_1,\Sigma_2} \Delta^2$ *and for any* $A, B \in \mathcal{L}_i$, $C \in \mathcal{L}_j$ *(*$i, j \in \{1,2\}$, $j \neq i$*),*

$$A \mid\!\sim_{\Delta} B \text{ iff } A \wedge C \mid\!\sim_{\Delta} B$$

Definition 4 (Relevance (**Rel**), (Kern-Isberner, Beierle, and Brewka 2020))**.** *An inductive inference operator* $\mathbf{C}$ *satisfies (***Rel***) if for any* $\Delta = \Delta^1 \bigcup_{\Sigma_1,\Sigma_2} \Delta^2$ *and for any* $A, B \in \mathcal{L}_i$ *(*$i \in \{1,2\}$*),*

$$A \mid\!\sim_{\Delta} B \text{ iff } A \mid\!\sim_{\Delta^i} B.$$

Definition 5 (Syntax splitting (**SynSplit**), (Kern-Isberner, Beierle, and Brewka 2020))**.** *An inductive inference operator* $\mathbf{C}$ *satisfies (***SynSplit***) if it satisfies (***Ind***) and (***Rel***).*

Thus, **(Ind)** requires that inferences from one sublanguage are independent from formulas over the other sublanguage, if the belief base splits over the respective sublanguages. In other words, information on the basis of one sublanguage does not influence inferences made in the other sublanguage. **(Rel)**, on the other hand, restricts the scope of inferences, by requiring that inferences in a sublanguage can be made on the basis of the conditionals in a conditional belief base formulated on the basis of that sublanguage. **(SynSplit)** combines these two properties. System Z satisfies **(Rel)** but not **(Ind)** (Kern-Isberner, Beierle, and Brewka 2020), while lexicographic inference (Heyninck, Kern-Isberner, and Meyer 2022) and c-inference (Kern-Isberner, Beierle, and Brewka 2020) satisfy full **(SynSplit)**.

2.4 System Z

We present system Z as defined by Goldszmidt and Pearl (1996) as follows. A conditional $(B|A)$ is *tolerated* by a finite set of conditionals Δ if there is a possible world ω with $(B|A)(\omega) = 1$ and $(B'|A')(\omega) \neq 0$ for all $(B'|A') \in \Delta$, i.e. ω verifies $(B|A)$ and does not falsify any (other) conditional in Δ. The Z-partitioning (or ordered partition) $OP(\Delta) = (\Delta_0, \ldots, \Delta_n)$ of Δ is defined as: $\Delta_0 = \{\delta \in \Delta \mid \Delta \text{ tolerates } \delta\}$; and $OP(\Delta \setminus \Delta_0) = \Delta_1, \ldots, \Delta_n$.

For $\delta \in \Delta$ we define: $Z_\Delta(\delta) = i$ iff $\delta \in \Delta_i$ and $OP(\Delta) = (\Delta_0, \ldots, \Delta_n)$. Finally, the ranking function κ^Z_Δ is defined via: $\kappa^Z_\Delta(\omega) = \max\{Z_\Delta(\delta) \mid \delta(\omega) = 0, \delta \in \Delta\} + 1$, with $\max \emptyset = -1$. The resulting inductive inference operator $\mathbf{C}^{ocf}_{\kappa^Z_\Delta}$ is denoted by $\mathbf{C}^Z$. System Z is equivalent to *rational closure* (Lehmann and Magidor 1992), and the two terms are used interchangeably in the literature.

We now illustrate OCFs in general and system Z in particular with the well-known “Tweety the penguin”-example.

Example 1. *Consider the conditional belief base* $\Delta = \{(f|b), (b|p), (\neg f|p)\}$*, where* b *is intended to represent being a bird,* f *represents being able to fly, and* p *represents being a penguin.* Δ *has the following Z-partitioning:* $\Delta_0 = \{(f|b)\}$ *and* $\Delta_1 = \{(b|p), (\neg f|p)\}$*. This gives rise to the following* κ^Z_Δ*-ordering over the worlds based on the signature* $\{b, f, p\}$*:*

ω	κ^Z_Δ	ω	κ^Z_Δ	ω	κ^Z_Δ	ω	κ^Z_Δ
pbf	2	$pb\overline{f}$	1	$p\overline{b}f$	2	$p\overline{b}\,\overline{f}$	2
$\overline{p}bf$	0	$\overline{p}b\overline{f}$	1	$\overline{p}\overline{b}f$	0	$\overline{p}\overline{b}\,\overline{f}$	0

As an example of a (non-)inference, observe that e.g. $\top \mid\!\sim^Z_\Delta \neg p$ *and* $p \wedge f \not\mid\!\sim^Z_\Delta b$*.*

2.5 Lexicographic Inference

We recall lexicographic inference as introduced by Lehmann (1995). For some conditional belief base Δ, the order $\preceq^{\mathsf{lex}}_\Delta$ is defined as follows: Given $\omega \in \Omega$ and $\Delta' \subseteq \Delta$, $V(\omega, \Delta') = |\{(B|A) \in \Delta' \mid (B|A)(\omega) = 0\}|$. Given a set of conditionals Δ partitioned in $OP(\Delta) = (\Delta_0, \ldots, \Delta_n)$, the *lexicographic vector* for a world $\omega \in \Omega$ is the vector $\mathsf{lex}(\omega) = (V(\omega, \Delta_0), \ldots, V(\omega, \Delta_n))$. Given two vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, $(x_1, \ldots, x_n) \preceq^{\mathsf{lex}} (y_1, \ldots, y_n)$ iff there is some $j \leqslant n$ s.t. $x_k = y_k$ for every $k > j$ and $x_j \leqslant y_j$. $\omega \preceq^{\mathsf{lex}}_\Delta \omega'$ iff $\mathsf{lex}(\omega) \preceq^{\mathsf{lex}} \mathsf{lex}(\omega')$. The resulting inductive inference operator $\mathbf{C}^{tpo}_{\preceq^{\mathsf{lex}}}$ will be denoted by $\mathbf{C}^{\mathsf{lex}}$ to avoid clutter.

Example 2 (Example 1 ctd.)**.** *For the Tweety belief base* Δ *as in Example 1 we obtain the following* $\mathsf{lex}(\omega)$*-vectors:*

ω	$\mathsf{lex}(\omega)$	ω	$\mathsf{lex}(\omega)$	ω	$\mathsf{lex}(\omega)$	ω	$\mathsf{lex}(\omega)$
pbf	(0,1)	$pb\overline{f}$	(1,0)	$p\overline{b}f$	(0,2)	$p\overline{b}\,\overline{f}$	(0,1)
$\overline{p}bf$	(0,0)	$\overline{p}b\overline{f}$	(1,0)	$\overline{p}\overline{b}f$	(0,0)	$\overline{p}\overline{b}\,\overline{f}$	(0,0)

The lex*-vectors are ordered as follows:*

$$(0,0) \prec^{\mathsf{lex}} (1,0) \prec^{\mathsf{lex}} (0,1) \prec^{\mathsf{lex}} (0,2).$$

Observe that e.g. $\top \mid\!\sim^{\mathsf{lex}}_\Delta \neg p$ *and* $p \wedge f \mid\!\sim^{\mathsf{lex}}_\Delta b$*.*

2.6 Reasoning with c-Representations

c-Representations (Kern-Isberner 2001b; Kern-Isberner 2004) are special ranking models obtained by assigning individual integer impacts to the conditionals in Δ and generating the world ranks as the sum of impacts of falsified conditionals. A related concept is that of ranking constructions (Weydert 1996).

Definition 6 (c-representation (Kern-Isberner 2001b; Kern-Isberner 2004))**.** *A* c-representation *of* $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$ *is an OCF* κ *constructed from non-negative integer impacts* $\eta_j \in \mathbb{N}_0$ *assigned to each* $(B_j|A_j)$ *such that* κ *accepts* Δ *and is given by:*

$$\kappa(\omega) = \sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j \overline{B}_j}} \eta_j \qquad (3)$$

c-Representations can conveniently be specified using a constraint satisfaction problem:

Definition 7 ($CR(\Delta)$, (Kern-Isberner 2001b; Beierle et al. 2018))**.** *Let* $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$*. The* constraint satisfaction problem for c-representations of Δ*, denoted by*

$CR(\Delta)$, is given by the conjunction of the constraints, for all $j \in \{1, \dots, n\}$:

$$\eta_j \geqslant 0 \tag{4}$$

$$\eta_j > \min_{\omega \models A_j B_j} \sum_{\substack{k \neq j \\ \omega \models A_k \overline{B_k}}} \eta_k - \min_{\omega \models A_j \overline{B}_j} \sum_{\substack{k \neq j \\ \omega \models A_k \overline{B_k}}} \eta_k \tag{5}$$

Note that (4) expresses that falsification of conditionals should not make worlds more plausible, and (5) ensures that κ as specified by (3) accepts Δ. A solution of $CR(\Delta)$ is a vector $\overrightarrow{\eta} = (\eta_1, \dots, \eta_n)$ of natural numbers. $Sol(CR(\Delta))$ denotes the set of all solutions of $CR(\Delta)$. For $\overrightarrow{\eta} \in Sol(CR(\Delta))$ and κ as in Equation (3), κ is the *OCF induced by $\overrightarrow{\eta}$* and is denoted by $\kappa_{\overrightarrow{\eta}}$. $CR(\Delta)$ is sound and complete (Kern-Isberner 2001b; Beierle et al. 2018): For every $\overrightarrow{\eta} \in Sol(CR(\Delta))$, $\kappa_{\overrightarrow{\eta}}$ is a c-representation with $\kappa_{\overrightarrow{\eta}} \models \Delta$, and for every c-representation κ with $\kappa \models \Delta$, there is $\overrightarrow{\eta} \in Sol(CR(\Delta))$ such that $\kappa = \kappa_{\overrightarrow{\eta}}$.

c-Inference was introduced by Beierle et al. (2016; 2018) as the skeptical inference relation obtained by taking all c-representations of a belief base Δ into account.

Definition 8 (c-inference, $\mid\!\sim^{c\text{-}sk}_{\boldsymbol{\Delta}}$, (Beierle, Eichhorn, and Kern-Isberner 2016))**.** *Let Δ be a conditional belief base and let A, B be formulas. B is* a (skeptical) c-inference from A in the context of Δ*, denoted by $A \mid\!\sim^{c\text{-}sk}_{\Delta} B$, iff $A \mid\!\sim_{\kappa} B$ holds for all c-representations κ of Δ, yielding the inductive inference operator*

$$\mathbf{C}^{c\text{-}sk} : \Delta \mapsto \mid\!\sim^{c\text{-}sk}_{\Delta}$$

Additionally to taking all c-representations of a belief base into account, an inductive inference operator can also be defined based on single c-representations. This can be done by employing the concept of selection strategies (Kern-Isberner, Beierle, and Brewka 2020).

Definition 9 (selection strategy γ, (Kern-Isberner, Beierle, and Brewka 2020))**.** *A* selection strategy (for c-representations) *is a function $\gamma : \Delta \mapsto \overrightarrow{\eta}$ assigning to each conditional belief base Δ an impact vector $\overrightarrow{\eta} \in Sol(CR(\Delta))$.*

Each selection strategy yields an inductive inference operator for OCFs via $\mathbf{C}^{c\text{-}rep}_{\gamma} : \Delta \mapsto \kappa_{\gamma(\Delta)}$.

Because every OCF κ can be uniquely represented by a TPO $\preceq_\kappa$ where, for all $\omega_1, \omega_2 \in \Omega$, $\omega_1 \preceq_\kappa \omega_2$ iff $\kappa(\omega_1) \leqslant \kappa(\omega_2)$, this also yields an inductive inference operator for TPOs via $\mathbf{C}^{c\text{-}rep}_{\gamma,\preceq} : \Delta \mapsto \preceq_{\kappa_{\gamma(\Delta)}}$.

Example 3. *For the belief base Δ from Ex. 1 a selection strategy γ for c-representations could yield the OCF $\kappa_{\overrightarrow{\eta}}$ via $\gamma(\Delta) = \overrightarrow{\eta} = (1, 2, 2)$, yielding the following ordering of worlds:*

ω	$\kappa_{\overrightarrow{\eta}}$	ω	$\kappa_{\overrightarrow{\eta}}$	ω	$\kappa_{\overrightarrow{\eta}}$	ω	$\kappa_{\overrightarrow{\eta}}$
pbf	2	$pb\overline{f}$	1	$p\overline{b}f$	4	$p\overline{b}\,\overline{f}$	2
$\overline{p}bf$	0	$\overline{p}b\overline{f}$	1	$\overline{p}\overline{b}f$	0	$\overline{p}\overline{b}\,\overline{f}$	0

We have $pb \mid\!\sim_{\kappa} \overline{f}$, because $\kappa_{\overrightarrow{\eta}}(pb\overline{f}) = 1 < 2 = \kappa_{\overrightarrow{\eta}}(pbf)$ (and thus also $pb\overline{f} \preceq_{\kappa_{\overrightarrow{\eta}}} pbf$). In fact, this holds for all c-representations of Δ which means $pb \mid\!\sim^{c\text{-}sk}_{\Delta} \overline{f}$.

3 Equivalence in Conditional Reasoning

We first define some preliminaries regarding equivalence. Two conditionals $(B_1|A_1)$ and $(B_2|A_2)$ are *equivalent* iff every world has the same attitude to both conditionals: i.e. $(B_1|A_1)(\omega) = (B_2|A_2)(\omega)$ for every $\omega \in \Omega$. This is equivalent to $A_1 \equiv A_2$ and $B_1 \wedge A_1 \equiv B_2 \wedge A_2$. Notice that this implies that for any TPO $\preceq$, $A_1 \mid\!\sim_{\preceq} B_1$ iff $A_2 \mid\!\sim_{\preceq} B_2$. We write $(B_1|A_1) \equiv (B_2|A_2)$ in that case.

We now define the following kinds of equivalence for conditional knowledge bases:

Definition 10. *Two conditional knowledge bases Δ_1 and Δ_2 are:*

- bijective pairwise equivalent *if there is a bijection $f : \Delta_1 \to \Delta_2$ s.t. $\delta \equiv f(\delta)$ for every $\delta \in \Delta_1$;*
- pairwise equivalent *if for every $\delta_1 \in \Delta_1$ there is some $\delta_2 \in \Delta_2$ s.t. $\delta_1 \equiv \delta_2$, and vice versa;*
- globally equivalent *if for every TPO $\preceq$, Δ_1 is valid w.r.t. $\preceq$ iff Δ_2 is valid w.r.t. $\preceq$.*

The intuition behind these notions is the following: bijective pairwise equivalence requires that two sets of conditionals have the same size, and that every conditional in one set is equivalent to a conditional in the other set. Pairwise equivalence requires that for every conditional in the first set Δ_1, we can find an equivalent conditional in Δ_2, and vice versa, but does not require these sets to have the same size. Finally, global equivalence merely requires that Δ_1 and Δ_2 have the same *semantic* structure, in the sense that they are valid w.r.t. the same TPOs.

These notions are strictly hierarchical:

Proposition 1. *If Δ_1 and Δ_2 are bijectively pairwise equivalent, they are pairwise equivalent, and if they are pairwise equivalent, they are globally equivalent.*

The following example shows global equivalence does not imply pairwise equivalence:

Example 4. *Consider $\Delta_1 = \{(q|p), (r|p)\}$ and $\Delta_2 = \{(q \wedge r|p)\}$. Then clearly Δ_1 and Δ_2 are globally equivalent but not pairwise equivalent.*

The following example shows pairwise equivalence does not imply bijective pairwise equivalence:

Example 5. *Consider $\Delta_1 = \{(q|p)\}$ and $\Delta_2 = \{(q|p), (q \wedge p|p)\}$. Then clearly Δ_1 and Δ_2 are pairwise equivalent but not bijectively so.*

The following properties express that an inductive inference operator satisfies a given notion of equivalence:

Definition 11. *Let an inductive inference operator $\mathbf{C}$ be given. Then $\mathbf{C}$:*

- satisfies bijective pairwise equivalence *if for any two bijective pairwise equivalent knowledge bases Δ_1 and Δ_2, $\mathbf{C}(\Delta_1) = \mathbf{C}(\Delta_2)$.*
- satisfies pairwise equivalence *if for any two pairwise equivalent knowledge bases Δ_1 and Δ_2, $\mathbf{C}(\Delta_1) = \mathbf{C}(\Delta_2)$.*
- satisfies global equivalence *if for any two pairwise globally equivalent knowledge bases Δ_1 and Δ_2, $\mathbf{C}(\Delta_1) = \mathbf{C}(\Delta_2)$.*

In other words, an inductive inference operator $\mathbf{C}$ satisfies [bijective] pairwise [global] equivalence if for any [bijective] pairwise [globally] equivalent knowledge bases Δ_1 and Δ_2, $A \mid\!\sim_{\Delta_1} B$ iff $A \mid\!\sim_{\Delta_2} B$. Bijective pairwise equivalence guarantees that replacing a conditional in the knowledge base with another that imposes identical constraints on possible worlds does not significantly alter the inferences made. Pairwise equivalence builds in the additional requirement that removing duplicate (modulo equivalence) conditionals should not impact the conclusions drawn. Global equivalence is the strongest of the three properties, and bijective pairwise equivalence the weakest (this is an immediate consequence of Proposition 1).

We commence the study of the satisfaction of equivalence for the inductive inference operators, moving from the strongest to the weakest result. We start with System Z:

Proposition 2. *System Z satisfies global equivalence.*

We show C-inference satisfies pairwise equivalence:

Proposition 3. *c-Inference satisfies pairwise equivalence.*

The proof is based on constructing for each c-representation of Δ_1 a c-representation of Δ_2 and vice versa, such that the sum of impacts of all equivalent conditionals is the same.

Regarding inference with single c-representations, we introduce the following postulate for selection strategies.

(PPE) A selection strategy σ satisfies preservation of pairwise equivalence if, for any two pairwise equivalent belief bases Δ_1, Δ_2 and every $\delta \in \Delta_1$:

$$\sum \sigma(\Delta_1)|_{\{\delta_1 | \delta_1 \in \Delta_1; \delta_1 \equiv \delta\}} = \sum \sigma(\Delta_2)|_{\{\delta_2 | \delta_2 \in \Delta_2; \delta_2 \equiv \delta\}}$$

(PPE) demands that for each conditional the sum of impacts over the set of equivalent conditionals must be the same in Δ_1 and Δ_2. Utilizing **(PPE)** we obtain the following result.

Proposition 4. *If σ satisfies* **(PPE)** *then $\mathbf{C}_{\gamma}^{c\text{-}rep}$ satisfies pairwise equivalence.*

We will see in Section 3.1 that global equivalence is not satisfied for both inferences based on c-representations, but first turn to lexicographic inference. Surprisingly, we observe that it does not even satisfy pairwise equivalence:

Proposition 5. *Lexicographic inference does not satisfy pairwise equivalence.*

Proof. Consider the following conditional knowledge bases:

$$\Delta_1 = \{(p|q), (r|q)\}$$
$$\Delta_2 = \Delta_1 \cup \{(r \wedge q|q)\}.$$

Δ_1 and Δ_2 are pairwise equivalent. It is easy to see that $OP(\Delta_1) = (\Delta_1)$ and $OP(\Delta_2) = (\Delta_2)$. Thus, the lexicographic vectors for the worlds $\overline{p}qr$ and $pq\overline{r}$ are determined as follows:

$V(\overline{p}qr, \Delta_1) = 1, V(\overline{p}qr, \Delta_2) = 1$, since $\overline{p}qr \models q \wedge \neg p$;
$V(pq\overline{r}, \Delta_1) = 1, V(pq\overline{r}, \Delta_2) = 2$, since $pq\overline{r} \models q \wedge \neg r \wedge \neg(r \wedge q)$.

This means that $\overline{p}qr \approx^{\mathsf{lex}}_{\Delta_1} pq\overline{r}$ whereas $\overline{p}qr \prec^{\mathsf{lex}}_{\Delta_2} pq\overline{r}$. □

Notice that the example used in this proof is also a violation of the property of Cut when applied to conditionals: if $A \mid\!\sim_{\Delta} B$ then $\mathbf{C}(\Delta \cup \{(B|A)\}) \subseteq \mathbf{C}(\Delta)$. This rule says that if we add a conditional to Δ that is already inferred on the basis of Δ - such as $(r \wedge q|q)$ being added to Δ_1 above - then this should not lead to the inference of any new conditionals. In the above we have, e.g., $q \wedge (p \leftrightarrow \neg r) \not\mid\!\sim_{\Delta_1} r$ but $q \wedge (p \leftrightarrow \neg r) \mid\!\sim_{\Delta_2} r$. Despite violating pairwise equivalence, lexicographic inference satisfies bijective pairwise equivalence:

Proposition 6. *Lexicographic inference satisfies bijective pairwise equivalence.*

Arguably, the failure of lexicographic inference to satisfy pairwise equivalence is undesirable, as it means that the number of (equivalent) conditionals in a knowledge base has an effect on the inferences from the knowledge base. To overcome this defect, we will propose a variant of lexicographic inference that avoids this in Section 4.

3.1 Satisfaction of Global Equivalence and Syntax Splitting

We now show a more general result that shows that the satisfaction of global equivalence is perhaps too strong of a requirement, in the sense that it is incompatible with another property deemed desirable for inductive inference operators, namely syntax splitting. To show this, we will assume another property, namely *conditional-basedness*, which expresses that worlds that have exactly the same attitudes w.r.t. the inducing set of conditionals should not be distinguished. Intuitively, in inductive inference, the only information that is relevant is the set of conditionals the inductive inference operator is based on.

Definition 12. *A model-based inductive inference operator* $\mathbf{C}$ *for TPOs is* conditional-based *if, for any $\omega_1, \omega_2 \in \Omega$, if $(\delta)(\omega_1) = (\delta)(\omega_2)$ for every $\delta \in \Delta$ then $\omega_1 \approx_{\Delta} \omega_2$.*

A similar property can be defined for model-based inductive inference operators on OCFs.

Notice that this is a rather harmless property, in the sense that any of the inductive inference relations studied in this paper satisfy it:

Proposition 7. *System Z, lexicographic inference and inference with a single c-representation are conditional-based.*

We can now show that, in the context of conditional-based inductive inference operators, global equivalence and **Ind** are jointly incompatible.

Proposition 8. *There exists no conditional-based inductive inference operation that satisfies global equivalence and satisfies* **(Ind)**.

Proof. Suppose that $\mathbf{C}$ satisfies global equivalence and satisfies syntax splitting.

Consider first $\Delta_1 = \{(a|\top), (b|\top)\}$. With **(DI)**, $\top \mid\!\sim^{\mathbf{C}}_{\Delta_1} a$ (which implies $ab \prec \omega$ for any $\omega \in \Omega \setminus \{ab\}$), and likewise, $\top \mid\!\sim^{\mathbf{C}}_{\Delta_1} b$. Then since $\Delta_1 = \{(a|\top)\} \bigcup_{\{a\},\{b\}} \{(b|\top)\}$, $\top \wedge \neg b \mid\!\sim^{\mathbf{C}}_{\Delta_1} a$ by **(Ind)**. This means that $a\overline{b} \prec^{\mathbf{C}}_{\Delta_1} \overline{a}\overline{b}$. With symmetry, we establish that $\overline{a}b \prec^{\mathbf{C}}_{\Delta_1} \overline{a}\overline{b}$.

Consider now $\Delta_2 = \{(a \wedge b|\top)\}$. Notice that Δ_1 and Δ_2 are globally equivalent. Thus, since $\mathbf{C}$ satisfies global equivalence, $\prec^{\mathbf{C}}_{\Delta_1} = \prec^{\mathbf{C}}_{\Delta_2}$. However, as $((a \wedge b|\top))(a\bar{b}) = ((a \wedge b|\top))(\bar{a}b) = ((a \wedge b|\top))(\bar{a}\bar{b}) = 0$, we see that $a\bar{b} \approx^{\mathbf{C}}_{\Delta_2} \bar{a}b \approx^{\mathbf{C}}_{\Delta_2} \bar{a}\bar{b}$, contradiction. □

Notice that global equivalence is only incompatible with the **(Ind)** sub- property of **(SynSplit)**. Indeed, as system Z satisfies only **(Rel)** (Kern-Isberner and Brewka 2017), we see it is possible to satisfy global equivalence and **(Rel)**.

We can utilize Proposition 8 to see that inference with a single c-representation does not satisfy global equivalence.

Proposition 9. $\mathbf{C}^{c\text{-}rep}_{\gamma}$ *does not satisfy global equivalence.*

Because c-inference is not OCF-based, Proposition 8 is not applicable. Nevertheless, the following result holds.

Proposition 10. *c-Inference does not satisfy global equivalence.*

Proof. Consider the globally equivalent belief bases $\Delta_1 = \{(a|\top), (b|\top)\}$ and $\Delta_2 = \{(a \wedge b|\top)\}$. Then it can be verified that $\bar{b} \mid\!\sim^{c\text{-}sk}_{\Delta_1} a$ but $\bar{b} \not\mid\!\sim^{c\text{-}sk}_{\Delta_2} a$ □

4 A Variant of Lexicographic Inference that Satisfies Pairwise Equivalence

Obtaining a variant of lexicographic inference that satisfies pairwise equivalence is rather straightforward. Instead of counting which conditionals are violated by a world, we count which conditionals are violated *up to equivalence*. In more detail, we observe that equivalence of conditionals is an equivalence relation over $(\mathcal{L}|\mathcal{L})$, and thus, as usual, we define the equivalence class of a conditional $(B|A)$ as $[(B|A)] = \{(D|C) \in (\mathcal{L}|\mathcal{L}) \mid A \equiv C \text{ and } A \wedge B \equiv C \wedge D\}$. We can now count the violations of conditionals in Δ by ω up to equivalence as:

$$V^{\equiv}(\omega, \Delta) := |\{[(B|A)] \mid (B|A)(\omega) = 0, (B|A) \in \Delta\}$$

It is easy to observe that $V^{\equiv}(\omega, \Delta) \leqslant V(\omega, \Delta)$ for any ω and set of conditionals Δ. We can now define, for Δ with $OP(\Delta) = (\Delta_0, \ldots, \Delta_n)$, $\mathtt{lex}^{\equiv}(\omega) = (V^{\equiv}(\omega, \Delta_0), \ldots, V^{\equiv}(\omega, \Delta_n))$. We furthermore let $\omega_1 \preceq^{\mathsf{lex},\equiv}_{\Delta} \omega_2$ iff $\mathtt{lex}^{\equiv}(\omega_1) \preceq^{\mathsf{lex}} \mathtt{lex}^{\equiv}(\omega_2)$. We denote the corresponding inductive inference relation by $C^{\mathsf{lex},\equiv}$. We illustrate this with an adapted Tweety-Example:

Example 6. *Let* $\Delta = \{(f|b), (b|p), (\neg f|p), (\neg f \wedge p|p)\}$. *Notice that* $(\neg f \wedge p|p) \equiv (\neg f|p)$ *We have the following* $\mathtt{lex}$- *and* $\mathtt{lex}^{\equiv}$-*vectors:*

ω	$\mathsf{lex}(\omega)$	$\mathsf{lex}^{\equiv}(\omega)$	ω	$\mathsf{lex}(\omega)$	$\mathsf{lex}^{\equiv}(\omega)$
pbf	(0,2)	(0,1)	$pb\overline{f}$	(1,0)	(1,0)
$p\overline{b}f$	(0,3)	(0,2)	$p\overline{b}\,\overline{f}$	(0,1)	(0,1)
$\overline{p}bf$	(0,0)	(0,0)	$\overline{p}b\overline{f}$	(1,0)	(1,0)
$\overline{p}\overline{b}f$	(0,0)	(0,0)	$\overline{p}\overline{b}\,\overline{f}$	(0,0)	(0,0)

We see that e.g. $p\overline{bf} \prec^{\mathsf{lex}}_{\Delta} pbf$ *yet* $p\overline{bf} \approx^{\mathsf{lex},\equiv}_{\Delta} pbf$. *This means that* $p \wedge \neg(b \wedge \neg f) \mid\!\sim^{\mathsf{lex}}_{\Delta} \overline{bf}$ *whereas* $p \wedge \neg(b \wedge \neg f) \not\mid\!\sim^{\mathsf{lex},\equiv}_{\Delta} \overline{bf}$.

We note firstly that this inference relation lies between System Z and lexicographic inference:

Proposition 11. *For any conditional knowledge base* Δ, $A \mid\!\sim^{Z}_{\Delta} B$ *implies* $A \mid\!\sim^{\mathsf{lex},\equiv}_{\Delta} B$ *and* $A \mid\!\sim^{\mathsf{lex},\equiv}_{\Delta} B$ *implies* $A \mid\!\sim^{\mathsf{lex}}_{\Delta} B$.

The next proposition shows that this inference relation is well-behaved in the sense that it satisfies pairwise equivalence and syntax splitting.

Proposition 12. $\mathbf{C}^{\mathsf{lex},\equiv}$ *satisfies pairwise equivalence.*

Proposition 13. $\mathbf{C}^{\mathsf{lex},\equiv}$ *satisfies SynSplit.*

Furthermore, it should be noticed that, as $\mathbf{C}^{\mathsf{lex},\equiv}$ is a model-based inductive inference operator for TPOs, it satisfies all the so-called KLM-postulates, including rational monotony.

5 Language Independence

In this section we consider the property of *language independence* for inductive inference operators. The property intuitively states that inductive inference should be independent of how exactly atoms are expressed. For example, it should not matter whether we represent two atoms as a and b or p and q. More generally, in many cases, atoms can be equivalently represented as complex formulas and vice versa. For example, one can represent “I don’t have a dog” by a or $\neg a$. In more formal detail, we follow Marquis and Schwind (2014) by defining a symbol translation as any mapping $\sigma : \Sigma \to \mathcal{L}(\Sigma')$. We can extend such a translation to formulas by simply defining $\sigma(A)$ as the formula obtained by substituting any $p \in \Sigma$ by $\sigma(p)$ in A, and for any conditional knowledge base Δ we denote by $\sigma(\Delta)$ the knowledge base obtained by replacing each $(B \mid A)$ in Δ by $(\sigma(B) \mid \sigma(A))$. As Marquis and Schwind already observed, it is sensible to restrict attention to to a specific classes of symbol translations. We look at two such classes, namely *belief-amount preserving symbol translations* (BAP) (Section 5.1) and the more general *atom independence preserving* (AIP) symbol translations (Section 5.2). Section 5.3 contains results on language independence for concrete inference operators.

5.1 Belief-Amount Preserving Symbol Translations

Following Marquis and Schwind (2014), a BAP-translation is defined as a translation that induces a bijection between interpretations in the two signatures:

Definition 13. *A mapping* $\sigma : \Sigma \to \mathcal{L}(\Sigma')$ *is a* belief-amount preserving symbol translation *from* Σ *to* Σ' *(in short, a BAP-translation) if there is a bijection* $\gamma : \Omega(\Sigma) \to \Omega(\Sigma')$ *s.t. for every* $A \in \mathcal{L}(\Sigma)$, $\mathsf{Mod}(\sigma(A)) = \{\gamma(\omega) \mid \omega \in \mathsf{Mod}(A)\}$.

The idea is that a symbol translation is a way of translating every atom to a formula such that the images of atoms are semantically equivalent (in the new language) to their originals: every world in the original language corresponds to exactly one world in the translated language.

Example 7. *Consider $\Sigma = \{a, b\}$ and the symbol translation from Σ to itself given by $\sigma(a) = a$ and $\sigma(b) = a \leftrightarrow b$. Then we have the following translations of $\Omega(\Sigma)$:*

$$\begin{array}{rcll} \sigma(ab) & = & a \wedge a \leftrightarrow b & (= ab) \\ \sigma(a\overline{b}) & = & a \wedge \neg(a \leftrightarrow b) & (= a\overline{b}) \\ \sigma(\overline{a}b) & = & \overline{a} \wedge (a \leftrightarrow b) & (= \overline{a}\overline{b}) \\ \sigma(\overline{a}\overline{b}) & = & \overline{a} \wedge \neg(a \leftrightarrow b) & (= \overline{a}b) \end{array}$$

We thus see that the bijection γ with $\gamma(\overline{a}b) = \overline{a}\overline{b}$, $\gamma(\overline{a}\overline{b}) = \overline{a}b$ and that maps ab and $a\overline{b}$ to their selves is a bijection that ensures σ is a BAP-translation.

We are now ready to state what it means for an inductive inference operator to satisfy BAP-language independence—it should be invariant under BAP symbol translation.

Definition 14. *An inductive inference operator* $\mathbf{C}$ *satisfies* BAP-language independence *if for every Σ, Σ', every BAP-translation $\sigma : \Sigma \to \mathcal{L}(\Sigma')$, and every conditional belief base $\Delta \subseteq (\mathcal{L}(\Sigma)|\mathcal{L}(\Sigma))$ we have $A \mid\!\sim_{\Delta} B$ iff $\sigma(A) \mid\!\sim_{\sigma(\Delta)} \sigma(B)$.*

On the level of TPOs, we obtain the following representation of BAP-language independence (a variant for OCFs is obtained similarly):

Proposition 14. *A model-based inductive inference operator for TPOs $\mathbf{C}$ satisfies BAP-language independence if for every BAP-translation $\sigma : \Sigma_1 \to \Sigma_2$, and any conditional belief base Δ over $\mathcal{L}(\Sigma_1)$, $\omega_1 \preceq_\Delta \omega_2$ iff $\sigma(\omega_1) \preceq_{\sigma(\Delta)} \sigma(\omega_2)$.*

We now delineate a condition that ensures BAP-language independence (as well as generalising the property of being conditional-based), which we call conditional-functional. Intuitively, this property requires that an induced consequence relation $\mathbf{C}(\Delta)$ only depends on the attitudes worlds have w.r.t. conditionals. Formally defining this property turned out to be rather intricate, and we do so below in full detail. Intuitively, the idea is that we are interested in inference operators that only depend on vectors $\langle i_1, \ldots, i_n \rangle$ of attitudes of worlds to conditionals.

In more formal detail, we let an *n-dimensional vector mass distribution* (nVMD) for a signature Σ be a function $F : \{1, 0, u\}^n \mapsto \mathbb{N}$ s.t. $\Sigma_{\vec{\alpha} \in \{1,0,u\}^n} F(\vec{\alpha}) = 2^{|\Sigma|}$. Intuitively, a VMD F is a function that keeps track of how many times every vector of attitudes occurs. This can be seen as a placeholder for a conditional knowledge base, in the sense that this is the only information about a conditional knowledge base that should be of interest for a conditional-functional inductive inference operator that looks solely at the attitudes of worlds w.r.t. conditionals. Before we define conditional-functionality formally, we need to define the notion of a *vector function*.

Definition 15. *A* vector function *is a function D that returns, for any n and any nVMD F, a pair $(V_F^D, \sqsubseteq_F^D)$ where:*

1. $V_F^D \subseteq \{1, 0, u\}^n$,
2. $\sqsubseteq_F^D$ *is a TPO on* V_F^D.

such that:

- $\vec{\alpha} \in V_F^D$ *implies* $F(\vec{\alpha}) > 0$, *and*
- *for any permutation σ on $\{1, \ldots, n\}$, $V_{\sigma(F)}^D = \sigma(V_F^D)$ and $\vec{\alpha} \sqsubseteq_{\sigma(F)}^D \vec{\beta}$ iff $\sigma^{-1}(\vec{\alpha}) \sqsubseteq_F^D \sigma^{-1}(\vec{\beta})$.*

This places us in the position to be able to define conditional-functionality.

Definition 16. *A TPO-based inductive inference operator* $\mathbf{C}$ *is* conditional-functional *iff there is a vector function D such that, for any Δ, $\omega_1 \preceq_\Delta \omega_2$ iff $\langle \delta_1(\omega_1), \ldots, \delta_n(\omega_1) \rangle \sqsubseteq_{F_\Delta}^D \langle \delta_1(\omega_2), \ldots, \delta_n(\omega_2) \rangle$, where $\Delta = \{\delta_1, \ldots, \delta_n\}$ and $F_\Delta(\vec{\alpha}) = |\{\omega \mid \vec{\alpha} = \langle \delta_1(\omega), \ldots, \delta_n(\omega) \rangle\}|$.*

The intuition behind this definition is the following: $\mathbf{C}$ should depend only on attitudes of worlds w.r.t. conditionals. That is, we should be able to formulate it on basis of the VMD alone. This is formalized by the condition that $\omega_1 \preceq_\Delta \omega_2$ iff $\langle \delta_1(\omega_1), \ldots, \delta_n(\omega_1) \rangle \sqsubseteq_{F_\Delta}^D \langle \delta_1(\omega_2), \ldots, \delta_n(\omega_2) \rangle$ for a vector function D generating TPOs over the attitude-vectors that depends only on the VMD. Furthermore, the exact ordering of the conditionals in a conditional knowledge base should not matter. Hence the requirement of invariance under permutations for vector functions.

Example 8. *We start with the conditional belief base from Example 1 and show how it can be interpreted in terms of a 3VMD. We first recall that the worlds have the following attitudes w.r.t. the conditionals $\delta_1 = (f|b)$, $\delta_2 = (b|p)$ and $\delta_3 = (\neg f|p)$:*

ω	$\omega(\delta_1)$	$\omega(\delta_2)$	$\omega(\delta_3)$	ω	$\omega(\delta_1)$	$\omega(\delta_2)$	$\omega(\delta_3)$
pbf	*1*	*1*	*0*	$pb\overline{f}$	*0*	*1*	*1*
$p\overline{b}f$	*u*	*0*	*0*	$p\overline{b}\,\overline{f}$	*u*	*0*	*1*
$\overline{p}bf$	*1*	*u*	*u*	$\overline{p}b\overline{f}$	*0*	*u*	*u*
$\overline{p}\overline{b}f$	*u*	*u*	*u*	$\overline{p}\overline{b}\,\overline{f}$	*u*	*u*	*u*

This means that we can view this knowledge base as the 3VMD F defined by:

$\vec{\alpha}$	$F(\vec{\alpha})$	$\vec{\alpha}$	$F(\vec{\alpha})$
$\langle 1, 1, 0 \rangle$	*1*	$\langle u, 0, 0 \rangle$	*1*
$\langle 1, u, u \rangle$	*1*	$\langle u, u, u \rangle$	2
$\langle 0, 1, 1 \rangle$	*1*	$\langle u, 0, 1 \rangle$	*1*
$\langle 0, u, u \rangle$	*1*		

and $F(\vec{\alpha}) = 0$ for all remaining $\vec{\alpha} \in \{1, 0, u\}^n$.

Furthermore, system Z for this instance is captured by $D(F) = (V_F^D, \sqsubseteq_F^D)$ with $V_F^D = \{\vec{\alpha} \mid F(\vec{\alpha} > 0\}$ and $\langle u, u, u \rangle, \langle 1, u, u \rangle \sqsubseteq_F^D \langle 0, u, u \rangle, \langle 0, 1, 1 \rangle \sqsubseteq_F^D$ $\langle 1, 1, 0 \rangle, \langle u, 0, 0 \rangle, \langle u, 0, 1 \rangle$.

Notice that while conditional-basedness (Definition 12) also requires that ranking of worlds only depend on the attitudes of those worlds w.r.t. conditionals, it does not require that this attitude is invariant under variations in the language (e.g. symbol-translations). It is thus not hard to see that any conditional-functional operator is also conditional-based, but not vice-versa. The latter is shown by the following example:

Example 9. *Consider $\Delta_1 = \{(q|p)\}$ and $\Delta_2 = \{(p|q)\}$. Then there are conditional-based operators that assign*

$pq, \overline{p}q, \overline{pq} \prec_{\Delta_1} p\overline{q}$ (corresponding to the ordering over vectors $1, u \sqsubset 0$) and $pq \prec_{\Delta_2} \overline{p}q, \overline{pq}, p\overline{q}$ (corresponding to the ordering over vectors $1 \sqsubset u, 0$). However, such an inductive inference operator is not conditional-functional.

For TPO-based inductive inference operators, being conditional-functional implies satisfying bijective pairwise-equivalence and BAP-language independence.

Proposition 15. *If a TPO-based inductive inference operator $\mathbf{C}$ is conditional-functional then it satisfies bijective pairwise equivalence and BAP-language independence.*

The final result in this section shows the converse—for TPO-based inductive inference operators, satisfying bijective pairwise equivalence and language independence implies being conditional-functional.

Proposition 16. *If a TPO-based inductive inference operator $\mathbf{C}$ satisfies bijective pairwise equivalence and BAP-language independence then it is conditional-functional.*

5.2 AIP Language Independence

In the previous section we considered the notion of language independence, which was based on the concept of BAP-translations. We now consider a stronger form of language independence (stronger in the sense that the two languages are "more" independent), which is based on a more general class of symbol translations which Marquis and Schwind called *atom independence preservation (AIP)* symbol translations. These are defined in terms of *relations* between valuations of different languages, rather than bijections as with BAP-translations. In what follows, given a relation $R \subseteq \Omega(\Sigma) \times \Omega(\Sigma')$ and $\omega \in \Omega(\Sigma)$, we use $R(\omega)$ to denote the set $\{\omega' \in \Omega(\Sigma') \mid R(\omega, \omega')\}$. Then an *AIP relation between $\Omega(\Sigma) \times \Omega(\Sigma')$* is a relation that satisfies the following three properties:

- $R(\omega) \neq \emptyset$ for all $\omega \in \Omega(\Sigma)$
- $R(\omega_1) \cap R(\omega_2) = \emptyset$ for all $\omega_1, \omega_2 \in \Omega(\Sigma)$ s.t. $\omega_1 \neq \omega_2$
- $\bigcup_{\omega \in \Omega(\Sigma)} R(\omega) = \Omega(\Sigma')$

Definition 17. *A mapping $\sigma : \Sigma \to \mathcal{L}(\Sigma')$ is a* AIP-translation *if there is an AIP relation between $\Omega(\Sigma)$ and $\Omega(\Sigma')$ s.t. for every $A \in \mathcal{L}(\Sigma)$, $\mathsf{Mod}(\sigma(A)) = \{R(\omega) \mid \omega \in \mathsf{Mod}(A)\}$.*

Example 10. *Suppose $\Sigma = \{a, b\}$ and $\Sigma' = \{b, c, d\}$. Then a possible AIP-translation would be $\sigma(a) = c \vee d$, $\sigma(b) = b$. This corresponds to the AIP relation R such that $R(ab) = \{bcd, bc\overline{d}, b\overline{c}d\}$, $R(a\overline{b}) = \{\overline{b}cd, \overline{b}c\overline{d}, \overline{b}\overline{c}d\}$, $R(\overline{a}b) = \{b\overline{c}\overline{d}\}$ and $R(\overline{a}\overline{b}) = \{\overline{b}\overline{c}\overline{d}\}$.*

Definition 18. *An inductive inference operator $\mathbf{C}$ satisfies* AIP language independence *if for every Σ, Σ', every AIP-translation $\sigma : \Sigma \to \mathcal{L}(\Sigma')$, and every conditional belief base $\Delta \subseteq (\mathcal{L}(\Sigma)|\mathcal{L}(\Sigma))$ we have $A \mid\!\sim_\Delta B$ iff $\sigma(A) \mid\!\sim_{\sigma(\Delta)} \sigma(B)$.*

A special case of AIP language independence, termed *Atomicity*, has been investigated in the context of inference from probabilistic knowledge bases by Paris and Vencovská (1990). This special case restricts to AIP-translations σ and bases Δ such that σ assigns each atom to either itself or a formula over atoms that do not appear in Δ.

Since every BAP-translation is an AIP-translation (they are the special case in which $R(\omega)$ always consists of a unique element), AIP language independence implies BAP language independence. We note the following characterisation of AIP languange independence in terms of TPOs:

Proposition 17. *A model-based inductive inference operator for TPOs $\mathbf{C}$ satisfies AIP language independence if for every AIP-translation $\sigma : \Sigma_1 \to \Sigma_2$, and any conditional belief base Δ over $\mathcal{L}(\Sigma_1)$, $\omega_1 \preceq_\Delta \omega_2$ iff $\min_{\preceq_{\sigma(\Delta)}} R(\omega_1) \preceq_{\sigma(\Delta)} \min_{\preceq_{\sigma(\Delta)}} R(\omega_2)$.*

To be able to characterise AIP language independence, we need to restrict conditional-functionality by considering a (strict) subset of the vector functions. Two nVMDs F_1 and F_2 are said to be equivalent, denoted as $F_1 \simeq F_2$, whenever $\{\vec{\alpha} \mid F_1(\vec{\alpha}) > 0\} = \{\vec{\alpha} \mid F_2(\vec{\alpha}) > 0\}$. We refer to a vector function D as *weight-independent* if $(V^D_{F_1}, \sqsubseteq^D_{F_1}) = (V^D_{F_2}, \sqsubseteq^D_{F_2})$ whenever $F_1 \simeq F_2$. Intuitively, a vector function is weight-independent if the weights of vectors do not matter, only if they are zero or non-zero. On the level of attitudes of worlds w.r.t. conditionals, this means that we disregard *how many* worlds have a certain attitude w.r.t. a set of conditionals.

Definition 19. *A TPO-based inductive inference operator $\mathbf{C}$ is* conditional-relational *iff there is a* weight-independent *vector function D such that, for any Δ, $\omega_1 \preceq_\Delta \omega_2$ iff $\langle \delta_1(\omega_1), \ldots, \delta_n(\omega_1) \rangle \sqsubseteq^D_{F_\Delta} \langle \delta_1(\omega_2), \ldots, \delta_n(\omega_2) \rangle$, where $\Delta = \{\delta_1, \ldots, \delta_n\}$ and $F_\Delta(\vec{\alpha}) = |\{\omega \mid \vec{\alpha} = \langle \delta_1(\omega), \ldots, \delta_n(\omega) \rangle\}|$.*

Clearly, conditional-relationality implies conditional-functionality, as any weight-independent vector function is also a vector function. However, the converse does not hold:

Example 11. *Consider the following inductive inference operator $\mathbf{C}^{Z\#}$ defined by: $\omega_1 \prec^{Z\#}_\Delta \omega_2$ iff $\kappa^Z_\Delta(\omega_1) < \kappa^Z_\Delta(\omega_2)$ or $\kappa^Z_\Delta(\omega_1) = \kappa^Z_\Delta(\omega_2)$ and $|\{\omega \in \Omega \mid \forall \delta \in \Delta : \delta(\omega) = \delta_1(\omega)\}| > |\{\omega \in \Omega \mid \forall \delta \in \Delta : \delta(\omega) = \delta_1(\omega)\}|$. Intuitively, this operator refines system Z by also comparing the number of worlds that have exactly the same attitude towards conditionals as the world under consideration. For example, for the Tweety belief base (Example 1), we obtain:*

$$\overline{p}\overline{b}f, \overline{p}\overline{b}\,\overline{f} \prec^{Z\#}_\Delta \overline{p}bf \prec^{Z\#}_\Delta \overline{p}b\overline{f}, pb\overline{f} \prec^{Z\#}_\Delta pbf, p\overline{b}f, p\overline{b}\,\overline{f}, p\overline{b}\,\overline{f}$$

This refines system Z e.g. by positing $\overline{p}\overline{b}f$, $\overline{p}\overline{b}\,\overline{f}$ are more preferred than $\overline{p}bf$, as the former both are indifferent with respect to each conditional, whereas $\overline{p}bf$ verifies $(f|b)$ but is indifferent w.r.t. the two other conditionals in Δ.

Conditional-relationality characterizes AIP-language independence (given bijective pairwise equivalence):

Proposition 18. *If a TPO-based inductive inference operator $\mathbf{C}$ is conditional-relational then it satisfies bijective pairwise equivalence and AIP-language independence.*

Proposition 19. *If a TPO-based inductive inference operator $\mathbf{C}$ satisfies bijective pairwise equivalence and AIP-language independence then it is conditional-relational.*

5.3 Conditional-Relationality for Concrete Operators

We show that all TPO-based inductive inference operators considered in this paper are conditional-relational.

Proposition 20. *System Z, Lexicographic inference and its variant* $\mathbf{C}^{\mathsf{lex},\equiv}$ *are conditional-relational.*

Regarding inference with single c-representations, we first state the following postulate, guaranteeing that a selection strategy is consistent with AIP language translations.

(IPAIP) A selection strategy γ is *impact preserving w.r.t. AIP language translations* if, for every belief base Δ and every AIP language translation σ, we have $\gamma(\Delta) = \gamma(\sigma(\Delta))$.

Utilizing **(IPAIP)** we obtain the following proposition.

Proposition 21. *If* γ *satisfies* **(IPAIP)**, *then* $\mathbf{C}^{\textit{c-rep}}_{\gamma}$ *satisfies AIP-language independence.*

Beierle and Kern-Isberner (2021) introduced the postulate *syntax independendence* (**(SI)**) for selection strategies and showed that inference with a single c-representation based on a **(SI)** -selection strategy satisfies bijective pairwise equivalence. We make use of this by combining **(SI)** and **(IPAIP)** and applying Proposition 19 to obtain the following result.

Proposition 22. *If* γ *satisfies* **(IPAIP)** *and* **(SI)**, *then* $\mathbf{C}^{\textit{c-rep}}_{\gamma,\preceq}$ *is conditional-relational.*

6 Related Work

While there are several works related to the work we have done in this paper (Weydert 2003; Kern-Isberner and Brewka 2017; Kern-Isberner, Heyninck, and Beierle 2022), the work of Weydert (2003) is perhaps the closest. Weydert suggested several properties in his study of default reasoning that share strong commonalities with some of the properties discussed in our work. E.g. *global logical invariance* is similar to the satisfaction of global equivalence, even though he does not define (as far as we could see) in a formally precise manner when two sets of conditionals are equivalent. Likewise, pairwise equivalence is closely related to his local semanticality (Weydert 2003). Furthermore, *strong irrelevance* is very similar to *relevance* and *representation independence* is very similar to language independence (with the main differences being induced by the differences in the assumptions of the framework of Weydert, such as allowing for languages generated on the basis of infinite Boolean algebras, and allowing for rankings over rational numbers). Furthermore, he shows, in his *exceptional inheritance paradox*, that no consistent default inference notion (his version of an inductive operator) can satisfy logicality, exceptional inheritance and global logical invariance. This is a slightly different but still quite similar result to our Proposition 8. Essentially, we assume syntax splitting whereas he assumes logicality (which means that the basic KLM-properties are satisfied) and exceptional inheritance. Exceptional inheritance states that $\{\phi, \neg\psi\} \mid\!\sim_{\{(\psi|\phi),(\psi'|\phi)\}} \psi'$ if "ψ and ψ' are logically independent given ϕ", although the concept of logical independence is not precisely formalised. We note as a further difference that he does not study System Z, lexicographic inference and c-representations, as we do.

Another related line of work is the work on *conditional structures* by Kern-Isberner (2001a; 1998; 1999). Conditional structures are algebraic structures that represent the attitudes of worlds w.r.t. a set of conditionals. This allows Kern-Isberner to, for example, define the notion of *indifference* w.r.t. a set of conditionals, which is essentially a special case of conditional-basedness, but restricted to OCFs. Conditional structures have been a fruitful framework, for example for characterizing so-called minimum cross-entropy and conditional preservation in belief revision. In this paper, we look at structures beyond OCFs, namely TPOs, and consider additional postulates (e.g. language independence and conditional relationality). We plan to look at formal connections between our work and conditional structures in future work.

Finally, notions of equivalence have also been used by Beierle and co-authors to investigate the expressivity of conditionals (2023) and normal forms (2019).

7 Conclusion

The focus of this paper is on properties of inference within classical logic-based inference that do not necessarily apply in the nonmonotonic case. Specifically, we consider the cases of *equivalences* and *language-independence* within the context of inductive inference. Table 1 summarizes the main findings of our work. More specifically, it considers the inductive inference operators System Z, lexicographic inference and its variation (that was introduced in Section 4), inference based on a single c-representation, and c-inference. It shows for each of them, whether or not they satisfy the listed properties. The numbers refer to the initial reference of the result: 1 refers to Kern-Isberner et al. (2020) 2 to Heyninck et al. (2022), 3 to Beierle and Kern-Isberner (2021), and 4 to Kern-Isberner (2001a).

	System Z	Lex	$\text{Lex}^{\equiv}$	c-reps.	c-inf.
Independence	$\times^{1}$	$\checkmark^{2}$	$\checkmark$	$\checkmark^{1}$	$\checkmark^{1}$
Relevance	$\checkmark^{1}$	$\checkmark^{2}$	$\checkmark$	$\checkmark^{*,1}$	$\checkmark^{1}$
Global Eq.	$\checkmark$	$\times$	$\times$	$\times$	$\times$
Pairwise Eq.	$\checkmark$	$\times$	$\checkmark$	$\checkmark^{*}$	$\checkmark$
Bij. Pairwise Eq.	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark^{*,3}$	$\checkmark$
Cond.-based	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark^{4}$	n.a.
Cond.-funct.	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark^{*}$	n.a.
Cond.-relat.	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark^{*}$	n.a.

Table 1: Summary of the properties studied in this paper. An * indicates that the underlying selection strategy needs to satisfy a fitting postulate, while 'n.a.' means the notion is not applicable.

An obvious direction for future work is to study other inductive inference operators, such as relevant closure (Casini et al. 2014), disjunctive rational closure (Booth and Varzinczak 2021), system W (Haldimann and Beierle 2022; Komo and Beierle 2022) or Weydert's System J-variants (2003). Another avenue is to see whether these postulates can also be helpful in characterising inductive inference operators. Finally, it may be useful to broaden this study to include other properties inherent to classical logic-based reasoning that do not carry over to nonmonotonic formalisms.

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