Reasoning in Defeasible Description Logics with System W and Lexicographic Inference

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Abstract

Description Logics (DLs) are widely applied in AI and database systems. However, like other classical logics, they cannot adequately handle *defeasible* information. Building on the notion of *rational closure* - a form of defeasible reasoning originally developed for the propositional setting and later adapted to DLs - we extend this approach by incorporating two further forms of defeasible reasoning: *System W* and *lexicographic closure*. Both are well-established entailment relations in the propositional case and are known to satisfy several desirable properties. In this paper, we provide model-theoretic definitions of these extensions for DLs, analyze their behaviour by relating them to their propositional counterparts, and present algorithms for their computation.

1 Introduction

Description logics (DLs) (Baader et al. 2007) are central to many modern AI and database applications (Delgrande et al. 2024). Yet, as classical formalisms, DLs do not allow for the proper representation of and reasoning with *defeasible information*, as demonstrated by the following example, adapted from Giordano et al. (2007): Students do not get tax invoices; employed students do; employed students who are also parents do not. Classically, the conclusion from this would be that the concept of an employed student is unsatisfiable. While concept unsatisfiability has been investigated extensively in ontology debugging and repair (Moodley, Meyer, and Varzinczak 2011; Schlöbach and Cornet 2003), a different solution to this problem is to incorporate *defeasibility* into DLs. Preferential extensions of DLs turn out to be particularly promising, mainly because they are based on an elegant, comprehensive and well-studied framework for non-monotonic reasoning in the propositional case proposed by Kraus, Lehmann and Magidor (1990; 1992) and often referred to as the *KLM approach*. Indeed, several extensions to the KLM approach to DLs have been proposed (Giordano et al. 2007; Britz, Heidema, and Meyer 2008; Giordano et al. 2008; Britz, Meyer, and Varzinczak 2011; Casini and Straccia 2010; Giordano et al. 2013; Giordano et al. 2015; Britz et al. 2021). With some notable exceptions (Casini and Straccia 2013; Casini et al. 2014; Giordano and Gliozzi 2018; Pensel and Turhan 2018; Hahn, Kern-Isberner, and Meyer 2024), such extensions have focused mostly on a basic form of propositional defeasible reasoning known as *rational closure* (RC) (Lehmann and Magidor 1992) or *System Z* (Pearl 1990). The focus of this paper is the lifting of two other forms of propositional defeasible reasoning known as *System W* (Komo and Beierle 2022) and *lexicographic closure* (LC) (Lehmann 1995) to defeasible DLs. Our primary motivation for doing so is that both forms of reasoning retain the desirable properties of RC (in fact, they both extend RC in terms of what can be concluded) while avoiding well-known issues with RC, such as the drowning problem (Benferhat et al. 1993; Komo and Beierle 2022).

Our main contributions are the following: Building on the variant of RC for $\mathcal{DALC}$ (a defeasible extension of the description logic $\mathcal{ALC}$) developed by Britz et al. (2021), we define appropriate variants of both System W and LC for $\mathcal{DALC}$. We then investigate the core properties of the DL variants of System W and LC, and present algorithms for computing both System W and LC.

The rest of the paper is structured in the following way. Section 2 is a brief introduction to KLM-style defeasible reasoning for propositional logic, while Section 3 introduces defeasible reasoning for DLs. Section 4 introduces the DL variant of System W with the associated algorithms. Section 5 does the same for the LC variant of DLs. Section 6 is a discussion of related work. Finally, Section 7 concludes and points to future work.

2 Propositional Defeasible Entailment

Assuming an underlying (finite) signature Σ, we denote the propositional language constructed from the default connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$ by $\mathcal{L}^{prop}$. We may denote a conjunction $A \wedge B$ by AB for brevity of notation. The set of interpretations over the underlying signature is denoted as Ω. An interpretation $\omega \in \Omega$ is a model of a formula $A \in \mathcal{L}^{prop}$ if A holds in ω, denoted as $\omega \Vdash A$. The set of models of a formula is denoted as $\Omega_A = \{\omega \in \Omega \mid \omega \Vdash A\}$, while the set of models of a set of formulas S is denoted as $\Omega_S = \{\omega \in \Omega \mid \omega \Vdash A \text{ for every } A \in S\}$. A set of formulas S *entails* a formula A, written $S \models A$, if $\Omega_S \subseteq \Omega_A$. By slight abuse of notation, we sometimes consider interpretations as the corresponding complete conjunction of all elements in the signature in either positive or negated form.

In the framework proposed by Kraus et al. (1990), often referred to as the *KLM approach*, we are interested in *defea-*

sible implications (DIs), or *conditionals* of the form $A \mathrel{|\!\sim} B$, read as "if A then usually B". The resulting conditional language is $\mathcal{L}^{def} = \{A \mathrel{|\!\sim} B \mid A, B \in \mathcal{L}^{prop}\}$. A finite set $\mathcal{K}$ of conditionals is called a *knowledge base*. Given a knowledge base $\mathcal{K}$, an intelligent agent should be able to draw inferences, or entailments, from it. While propositional logic has a mostly agreed upon semantics, different methods for reasoning from knowledge bases have been proposed.

One well-known form of entailment from knowledge bases is *rational closure* (RC) (Lehmann 1989; Lehmann and Magidor 1992), that was shown to be equivalent to an extended version of System Z (Pearl 1990) by Goldszmidt and Pearl (1990). To define RC, we first consider the notion of the *rank* of a formula. The *materialisation* of a conditional $A \mathrel{|\!\sim} B$ is defined as $A \to B$, and the materialisation of a knowledge base $\mathcal{K}$ is defined as $\overrightarrow{\mathcal{K}} = \{A \to B \mid A \mathrel{|\!\sim} B \in \mathcal{K}\}$. A propositional formula A is *exceptional* w.r.t. $\mathcal{K}$ if $\overrightarrow{\mathcal{K}} \models \neg A$. Let $\varepsilon(\mathcal{K}) = \{A \mathrel{|\!\sim} B \in \mathcal{K} \mid \overrightarrow{\mathcal{K}} \models \neg A\}$ indicate the exceptional conditionals in a knowledge base $\mathcal{K}$. Define a sequence of knowledge bases $\mathcal{E}_0^{\mathcal{K}}, \dots, \mathcal{E}_\infty^{\mathcal{K}}$ as follows: $\mathcal{E}_0^{\mathcal{K}} = \mathcal{K}$, $\mathcal{E}_i^{\mathcal{K}} = \varepsilon(\mathcal{E}_{i-1}^{\mathcal{K}})$, for $i > 0$ and $\mathcal{E}_\infty = \mathcal{E}_n^{\mathcal{K}}$, where n is the smallest k for which $\mathcal{E}_k^{\mathcal{K}} = \mathcal{E}_{k+1}^{\mathcal{K}}$ (since $\mathcal{K}$ is finite, such an n must exist). The *rank* $r_{\mathcal{K}}(A)$ of a propositional formula A w.r.t. a knowledge base $\mathcal{K}$ is defined to be the smallest r for which A is *not* exceptional w.r.t. $\mathcal{E}_r^{\mathcal{K}}$, i.e., $r_{\mathcal{K}}(A) = \min\{r \mid \overrightarrow{\mathcal{E}_r^{\mathcal{K}}} \not\models \neg A\}$. For A with $\mathcal{K}_\infty \models \neg A$ we let $r_{\mathcal{K}}(A) = \infty$. From this we define the *ranked partition* of $\mathcal{K}$ as $RP(\mathcal{K}) = \langle \mathcal{K}_0, \dots, \mathcal{K}_{n-1}, \mathcal{K}_\infty \rangle$ where $\mathcal{K}_i = \{A \mathrel{|\!\sim} B \in \mathcal{K} \mid r_{\mathcal{K}}(A) = i\}$. To define RC semantically we construct an ordering $<_{\mathcal{K}}^{\mathsf{RC}}$ over Ω. First, we assign a *height* $h_{\mathcal{K}}(\omega)$ to each $\omega \in \Omega$ as follows:

- $h_{\mathcal{K}}(\omega) = 0$ iff $\omega \Vdash A \to B$ for every $A \mathrel{|\!\sim} B \in \mathcal{K}$;
- $h_{\mathcal{K}}(\omega) = i$ with $0 < i < n$ iff $\omega \Vdash A \to B$ for every $A \mathrel{|\!\sim} B \in \bigcup_{i \le j < n} \mathcal{K}_j \cup \mathcal{K}_\infty$, and $\omega \nVdash A \to B$ for some $A \mathrel{|\!\sim} B \in \mathcal{K}_{i-1}$;
- $h_{\mathcal{K}}(\omega) = \infty$ iff $\omega \nVdash A \to B$ for some $A \mathrel{|\!\sim} B \in \mathcal{K}_\infty$.

Let $\Omega^{feas} = \{\omega \mid h_{\mathcal{K}}(\omega) \in \{0, \dots, n-1\}\}$ be the set of worlds with finite height, and let $\Omega_A^{feas} = \Omega^{feas} \cap \Omega_A$ for $A \in \mathcal{L}^{prop}$. The *RC-ordering w.r.t.* $\mathcal{K}$ is the ordering $<_{\mathcal{K}}^{\mathsf{RC}}$ over Ω^{feas} defined by $\omega <_{\mathcal{K}}^{\mathsf{RC}} \omega'$ iff $h_{\mathcal{K}}(\omega) < h_{\mathcal{K}}(\omega')$.

Definition 1 (RC)**.** *Let $\mathcal{K}$ be a knowledge base and $A, B \in \mathcal{L}^{prop}$. Then $A \mathrel{|\!\sim} B$ is an* RC entailment *of $\mathcal{K}$, written $\mathcal{K} \models_{RC} A \mathrel{|\!\sim} B$, if $\min_{<_{\mathcal{K}}^{\mathsf{RC}}}(\Omega_A^{feas}) \subseteq \Omega_B$.*

It is known (Casini and Straccia 2010) that RC can also be obtained from the ranked partition of a knowledge base using classical entailment.

Proposition 1. *Let $\mathcal{K}$ be a knowledge base with $RP(\mathcal{K}) = \langle \mathcal{K}_0, \dots, \mathcal{K}_{n-1}, \mathcal{K}_\infty \rangle$. Let $r = \min\{i \mid \overrightarrow{\mathcal{K}_\infty} \cup \bigcup_{j=i}^{j<n} \overrightarrow{\mathcal{K}_k} \not\models \neg A\}$.*[1] *Then $A \mathrel{|\!\sim} B$ is an RC entailment of $\mathcal{K}$ iff $\overrightarrow{\mathcal{K}_\infty} \cup \bigcup_{j=r}^{j<n} \overrightarrow{\mathcal{K}_j} \models A \to B$.*

[1] If $\overrightarrow{\mathcal{K}_\infty} \cup \overrightarrow{\mathcal{K}_{n-1}} \models \neg A$, then $r = n$ in case $\overrightarrow{\mathcal{K}_\infty} \not\models \neg A$; $r = \infty$ otherwise.

That is, RC can be obtained by finding the smallest rank r for which the antecedent A is not exceptional w.r.t. the conditionals in the union $\mathcal{K}_r \cup \dots \mathcal{K}_n \cup \mathcal{K}_\infty$, and take $A \mathrel{|\!\sim} B$ to be an RC entailment of $\mathcal{K}$ if its materialisation is entailed by the materialisation of that union.

System W is another form of entailment that is based on an ordering $<_{\mathcal{K}}^{\mathsf{W}}$ over Ω^{feas} w.r.t. every knowledge base $\mathcal{K}$ (Komo and Beierle 2022). System W allows more entailments than RC, and satisfies additional desirable properties for nonmonotonic entailment such as avoiding the drowning problem (Benferhat et al. 1993; Komo and Beierle 2022), fully complying with syntax splitting (Kern-Isberner, Beierle, and Brewka 2020; Haldimann and Beierle 2022c; Haldimann and Beierle 2022b; Haldimann et al. 2023), and also with conditional syntax splitting (Heyninck et al. 2023). Here, we use an extended version of System W introduced by Haldimann et al. (2023) that does not require the knowledge bases to satisfy a notion of consistency. A conditional $A \mathrel{|\!\sim} B$ is said to be *falsified* w.r.t. an interpretation ω if $\omega \nVdash A \to B$. For a knowledge base $\mathcal{K}$ and its ranked partition $RP(\mathcal{K}) = \langle \mathcal{K}_0, \dots, \mathcal{K}_{n-1}, \mathcal{K}_\infty \rangle$, and for $j \in \{0, \dots, n-1, \infty\}$, we define the $\xi_{\mathcal{K}}^j$ to be the function mapping interpretations to the set of falsified conditionals in $\mathcal{K}_j$. That is, $\xi_{\mathcal{K}}^j(\omega) = \{A \mathrel{|\!\sim} B \in \mathcal{K}_j \mid \omega \nVdash A \to B\}$ for $j \in \{0, \dots, n-1, \infty\}$.

Definition 2 (W-ordering w.r.t. $\mathcal{K}$)**.** *For a knowledge base $\mathcal{K}$ and its ranked partition $RP(\mathcal{K}) = \langle \mathcal{K}_0, \dots, \mathcal{K}_{n-1}, \mathcal{K}_\infty \rangle$, the* W-ordering w.r.t. $\mathcal{K}$ *is the ordering $<_{\mathcal{K}}^{\mathsf{W}}$ over Ω^{feas} defined by*

$$\begin{aligned} \omega <_{\mathcal{K}}^{\mathsf{W}} \omega' \;\; & \textit{iff } \exists r \in \{0, \dots, n-1\} \textit{ s.t.} \\ & \xi_{\mathcal{K}}^i(\omega) = \xi_{\mathcal{K}}^i(\omega') \forall i \in \{r+1, \dots, n-1\} \\ & \textit{and } \xi_{\mathcal{K}}^r(\omega) \subsetneqq \xi_{\mathcal{K}}^r(\omega'). \end{aligned}$$

Thus, $\omega <_{\mathcal{K}}^{\mathsf{W}} \omega'$ iff ω falsifies strictly fewer conditionals than ω' in the partition with the biggest index r where the conditionals falsified by ω and ω' differ.

Definition 3 (System W)**.** *Let $\mathcal{K}$ be a knowledge base and $A, B \in \mathcal{L}^{prop}$. Then $A \mathrel{|\!\sim} B$ is a* System W entailment*, written $\mathcal{K} \models_W A \mathrel{|\!\sim} B$, if $\min_{<_{\mathcal{K}}^{\mathsf{W}}} \Omega_A^{feas} \subseteq \Omega_B$.*

It is known that $<_{\mathcal{K}}^{\mathsf{W}}$ is a refinement of $<_{\mathcal{K}}^{\mathsf{RC}}$ (Komo and Beierle 2022). This allows us to define the semantic characterisation of System W in terms of the semantic characterisation of RC by refining the order $<_{\mathcal{K}}^{\mathsf{RC}}$ into the order $<_{\mathcal{K}}^{\mathsf{W}}$, slightly extending the previous result.

Proposition 2. *Let $\mathcal{K}$ be a knowledge base with the ranked partition $RP(\mathcal{K}) = \langle \mathcal{K}_0, \dots, \mathcal{K}_n, \mathcal{K}_\infty \rangle$. Then*

$$\begin{aligned} \omega <_{\mathcal{K}}^{\mathsf{W}} \omega' \;\; & \textit{iff } \omega <_{\mathcal{K}}^{\mathsf{RC}} \omega', \textit{or} \\ & h_{\mathcal{K}}(\omega) = h_{\mathcal{K}}(\omega'), \textit{ and } \exists r \in \{0, \dots, n-1\} \\ & \textit{s.t. } \xi_{\mathcal{K}}^i(\omega) = \xi_{\mathcal{K}}^i(\omega') \forall i \in \{r+1, \dots, n-1\} \textit{ and} \\ & \xi_{\mathcal{K}}^r(\omega) \subsetneqq \xi_{\mathcal{K}}^r(\omega'). \end{aligned}$$

We conclude this section with the description of *lexicographic closure* (LC) (Lehmann 1995), another well-known form of entailment from conditionals that extends both RC and System W (Haldimann and Beierle 2022c). It is based on an ordering of interpretations that considers the number of falsified conditionals in each part of the ranked partition.

Definition 4 (LC-ordering w.r.t. $\mathcal{K}$)**.** *Let $\mathcal{K}$ be a knowledge base with the ranked partition $RP(\mathcal{K}) = \langle \mathcal{K}_0, \dots, \mathcal{K}_{n-1}, \mathcal{K}_\infty \rangle$. The* LC-ordering w.r.t. $\mathcal{K}$ *is the ordering $<^{\mathtt{LC}}_{\mathcal{K}}$ over Ω^{feas} defined by*

$$\omega <^{\mathtt{LC}}_{\mathcal{K}} \omega' \text{ iff } \exists r \in \{0, \dots, n-1\} \text{ s.t. } |\xi^i_{\mathcal{K}}(\omega)| = |\xi^i_{\mathcal{K}}(\omega')| \ \forall i \in \{r+1, \dots, n-1\} \text{ and } |\xi^r_{\mathcal{K}}(\omega)| < |\xi^r_{\mathcal{K}}(\omega')| .$$

LC is then defined in terms of the LC-ordering w.r.t. KB.

Definition 5 (LC)**.** *Let $\mathcal{K}$ be a knowledge base and $A, B \in \mathcal{L}^{prop}$. Then $A \mid\!\sim B$ is an* LC entailment*, written $\mathcal{K} \models_{LC} A \mid\!\sim B$, if $\min_{<^{\mathtt{LC}}_{\mathcal{K}}}(\Omega^{feas}_A) \subseteq \Omega_B$.*

It is known (Haldimann and Beierle 2022c) that $<^{\mathtt{LC}}_{\mathcal{K}}$ refines $<^{\mathtt{W}}_{\mathcal{K}}$ which, as we have seen, refines $<^{\mathtt{RC}}_{\mathcal{K}}$. In fact, we can also obtain a result similar to Proposition 2 to show that $<^{\mathtt{LC}}_{\mathcal{K}}$ refines $<^{\mathtt{RC}}_{\mathcal{K}}$.

Proposition 3. *Let $\mathcal{K}$ be a knowledge base with the ranked partition $RP(\mathcal{K}) = \langle \mathcal{K}_0, \dots, \mathcal{K}_n, \mathcal{K}_\infty \rangle$. Then*

$$\omega <^{\mathtt{LC}}_{\mathcal{K}} \omega' \text{ iff } \omega <^{\mathtt{RC}}_{\mathcal{K}} \omega', \text{ or } h_{\mathcal{K}}(\omega) = h_{\mathcal{K}}(\omega'), \text{ and } \exists r \in \{0, \dots, n-1\} \text{ s.t. } |\xi^i_{\mathcal{K}}(\omega)| = |\xi^i_{\mathcal{K}}(\omega')| \forall i \in \{r+1, \dots, n-1\} \text{ and } |\xi^r_{\mathcal{K}}(\omega)| < |\xi^r_{\mathcal{K}}(\omega')|.$$

As can be seen from this overview, System W and LC are very similar. Actually, they both rely on the principles that Lehmann has indicated as the rationale behind LC (1995, Section 9.2): given two sets of falsified conditionals, the *seriousness* of the infringements has to be decided based on two principles:

- The size of the set: the smaller the set of falsified conditionals, the better it is.
- The priority among the elements of the sets: it is preferable to violate more generic conditionals w.r.t. violating more specific ones.

Both the approaches considered here follow these principles, the main difference being how the comparison between the sizes of the sets is done: while System W relies on the subset relation, LC relies on cardinality.

3 Defeasible Description Logics

In this section we provide a brief introduction to the description logic $\mathcal{ALC}$ (Baader et al. 2007), and its extension to defeasible $\mathcal{ALC}$ ($\mathcal{DALC}$) (Britz et al. 2021). Let Σ_C, Σ_R be finite, disjoint sets of symbols. These symbols will be our *concept names* and *role names*, respectively. Based on this we define the concept language $\mathcal{L}^{con}$ of $\mathcal{ALC}$ by

$$\mathcal{L}^{con} ::= A \mid \neg A \mid C \sqcap D \mid C \sqcup D \mid \exists r.C \mid \forall r.C \mid \bot \mid \top$$

for $A \in \Sigma_C, r \in \Sigma_R$, and $C, D \in \mathcal{L}^{con}$. A TBox $\mathcal{T}$ is a finite set of *concept inclusions* of the form $C \sqsubseteq D$ for $C, D \in \mathcal{L}^{con}$.

Interpretations give a semantics to concept inclusions. An interpretation is a tuple $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ where $\Delta^{\mathcal{I}}$ is a nonempty set called the *domain* and $\cdot^{\mathcal{I}}$ is a function that maps every $r \in \Sigma_R$ to a set of ordered pairs $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, and every $C \in \Sigma_C$ to a set $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$. Interpretations are lifted to complex concepts by $\bot^{\mathcal{I}} = \emptyset$, $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$, $(\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$, $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$, $(C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$, $(\exists r.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \text{there is an } (x, y) \in r^{\mathcal{I}} \text{ s.t. } y \in C^{\mathcal{I}}\}$, $(\forall r.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid y \in C^{\mathcal{I}} \text{ for all } (x, y) \in r^{\mathcal{I}}\}$.

An interpretation $\mathcal{I}$ is a *model* of a concept inclusion $C \sqsubseteq D$, denoted $\mathcal{I} \Vdash (C \sqsubseteq D)$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. The notion of a model is canonically lifted to sets of concept inclusions. A TBox $\mathcal{T}$ *entails* a concept inclusion $C \sqsubseteq D$, written $\mathcal{T} \models C \sqsubseteq D$, if every model of $\mathcal{T}$ is also a model of $C \sqsubseteq D$.

Concept inclusions $C \sqsubseteq D$ are the main building blocks of DL terminologies, but do not allow for exceptions. To do so, we define *defeasible concept inclusions* (defeasible inclusions, or DCIs, for short) of the form $C \mathrel{\underset{\sim}{\sqsubset}} D$ (Britz, Heidema, and Meyer 2008; Britz et al. 2021; Giordano et al. 2015; Giordano and Gliozzi 2020). The intuition of a defeasible inclusion is that the objects in C are typically, but not necessarily, also in D. Defeasible inclusions can be thought of as the description logic analogues of defeasible conditionals. A DBox $\mathcal{D}$ is a set of defeasible inclusions.

The defeasible description logic $\mathcal{DALC}$ allows for knowledge bases of the form $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ where $\mathcal{T}$ is a TBox and $\mathcal{D}$ is a DBox. To obtain a semantics for $\mathcal{DALC}$, we consider *preferential interpretations*. In addition to including the structure of a classical DL interpretation, a preferential interpretation also includes an ordering on the objects of the domain, representing how typical (or exceptional) an object is (with objects more typical, or less exceptional, lower down in the ordering).

Definition 6 (preferential interpretation)**.** *A* preferential interpretation *is a tuple $\mathcal{P} = \langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}}, \prec^{\mathcal{P}} \rangle$ such that $\mathcal{I}^{\mathcal{P}} = \langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}} \rangle$ is a DL interpretation and $\prec^{\mathcal{P}}$ is a strict partial order on $\Delta^{\mathcal{P}}$ satisfying the* smoothness condition *(for every $C \in \mathcal{L}^{con}$, if $C^{\mathcal{P}} \neq \emptyset$ then $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \neq \emptyset$).*

A preferential interpretation $\mathcal{P} = \langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}}, \prec^{\mathcal{P}} \rangle$ is a model for a TBox $\mathcal{T}$ if $\mathcal{I}^{\mathcal{P}}$ is a model for $\mathcal{T}$, and $\mathcal{P}$ is a model for a DBox $\mathcal{D}$ if $\min_{\prec^{\mathcal{P}}} C^{\mathcal{P}} \subseteq D^{\mathcal{P}}$ for every $C \mathrel{\underset{\sim}{\sqsubset}} D \in \mathcal{D}$. A knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ is *consistent* iff it has a preferential model.

Example 1. *Consider a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ that models in $\mathcal{T}$ that penguins are birds ($P \sqsubseteq B$); and in $\mathcal{D}$ models the following defeasible inclusions: birds fly ($B \mathrel{\underset{\sim}{\sqsubset}} F$), penguins do not fly ($P \mathrel{\underset{\sim}{\sqsubset}} \neg F$), and feathers are a part of a bird ($B \mathrel{\underset{\sim}{\sqsubset}} \exists part.Fe$).*

A possible preferential model of $\mathcal{K}$ is $\mathcal{P} = \langle \Delta^{\mathcal{P}}, \cdot^{\mathcal{P}}, \prec^{\mathcal{P}} \rangle$ with $\Delta^{\mathcal{I}} = \{s, t, u\}$, $P^{\mathcal{P}} = \{t\}, B^{\mathcal{P}} = \{s, t\}$, $F^{\mathcal{P}} = \{s, u\}, Fe^{\mathcal{P}} = \{u\}, part = \{(s, u)\}$, and $\prec^{\mathcal{P}} = \{(s, t), (u, t)\}$. Observe that all strict and defeasible inclusions in $\mathcal{K}$ are satisfied by $\mathcal{P}$.

For a defeasible inclusion $C \mathrel{\underset{\sim}{\sqsubset}} D$, the concept $\neg C \sqcup D$ is the *material counterpart* of the inclusion, denoted by $\overline{C \mathrel{\underset{\sim}{\sqsubset}} D}$. For a DBox $\mathcal{D}$, $\overline{\mathcal{D}}$ is the set of material counterparts of each of the defeasible inclusions in $\mathcal{D}$ (Britz et al. 2021).

Analogous to the propositional case, a ranked partition for $\mathcal{DALC}$ can be defined. A concept C is *exceptional* for a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ if $\mathcal{T} \models \bigsqcap \overline{\mathcal{D}} \sqsubseteq \neg C$. Analogous to the propositional case, let $\varepsilon^{\mathcal{K}}(\mathcal{D}) = \{C \mathrel{\substack{\sqsubset\\\sim}} D \in \mathcal{D} \mid \mathcal{T} \models \bigsqcap \overline{\mathcal{D}} \sqsubseteq \neg C\}$, and define a sequence of DCIs (w.r.t. $\mathcal{K}$) $\mathcal{E}^{\mathcal{K}}_0, \ldots, \mathcal{E}^{\mathcal{K}}_{n-1}, \mathcal{E}^{\mathcal{K}}_\infty$ as follows: $\mathcal{E}^{\mathcal{K}}_0 = \mathcal{D}$, $\mathcal{E}^{\mathcal{K}}_i = \varepsilon(\mathcal{E}^{\mathcal{K}}_{i-1})$, for $i > 0$, and $\mathcal{E}^{\mathcal{K}}_\infty = \mathcal{E}^{\mathcal{K}}_n$, where n is the smallest k for which $\mathcal{E}^{\mathcal{K}}_k = \mathcal{E}^{\mathcal{K}}_{k+1}$ (since $\mathcal{D}$ is finite, such an n must exist). The *rank* $r_{\mathcal{K}}(C)$ of a concept C w.r.t. a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ is defined to be the smallest r ($0 \leq r \leq (n-1)$) for which C is *not* exceptional w.r.t. $\langle \mathcal{T}, \mathcal{E}^{\mathcal{K}}_r \rangle$, i.e., $r_{\mathcal{K}}(C) = \min\{r \mid \mathcal{T} \not\models \bigsqcap \overline{\mathcal{E}^{\mathcal{K}}_r} \sqsubseteq \neg C\}$; If $\mathcal{T} \models \bigsqcap \overline{\mathcal{E}^{\mathcal{K}}_{n-1}} \sqsubseteq \neg C$, then $r_{\mathcal{K}}(C) = n$ if $\mathcal{T} \not\models \top \sqsubseteq \neg C$, and $r_{\mathcal{K}}(C) = \infty$ otherwise. The notion of a rank can be extended to DCIs by letting the rank of a DCI $C \mathrel{\substack{\sqsubset\\\sim}} D$ (w.r.t. $\mathcal{K}$) be the same as the rank of its antecedent C. From this we define the *ranked partition* of $\mathcal{K}$ as $RP(\mathcal{K}) = \langle \mathcal{D}^{\mathcal{K}}_0, \ldots, \mathcal{D}^{\mathcal{K}}_{n-1}, \mathcal{D}^{\mathcal{K}}_\infty \rangle$ where $\mathcal{D}_i = \{C \mathrel{\substack{\sqsubset\\\sim}} D \in \mathcal{D} \mid r_{\mathcal{K}}(C) = i\}$.

Regarding entailment from defeasible knowledge bases, both Giordano et al. (2015) and Britz et al. (2021) have developed semantic versions of rational closure (RC) for $\mathcal{DALC}$. Both are based on the definition of RC for the propositional case as outlined in Definition 1, but with the ordering defined over objects in the domains of interpretations, instead of on propositional interpretations. These approaches have been shown to be inferentially equivalent (Casini, Straccia, and Meyer 2019). Below we describe the approach by Britz et al. (2021) which makes use of a construction referred to as the *Big Ranked Model*.[2] A *ranked* interpretation $\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle$ is a preferential interpretation such that $\prec^{\mathcal{R}}$ is *modular*: its associated incomparability relation $\sim^{\mathcal{R}}$, defined by $x \sim^{\mathcal{R}} y$ if neither $x \prec^{\mathcal{R}} y$ nor $y \prec^{\mathcal{R}} x$, is transitive. Modularity means that $\prec^{\mathcal{R}}$ partitions the elements of $\Delta^{\mathcal{R}}$ into (non-empty) layers $L^{\mathcal{R}}_0, \ldots, L^{\mathcal{R}}_n$ such that for $p \in L^{\mathcal{R}}_i, q \in L^{\mathcal{R}}_j$ we have $p \prec^{\mathcal{R}} q$ iff $i < j$. As a result, it is possible to define the *height* $h_{\mathcal{R}}(x)$ of an object x in $\Delta^{\mathcal{R}}$ as $h_{\mathcal{R}}(x) = i$ where $x \in L^{\mathcal{R}}_i$.

Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ be a knowledge base and let Δ be any fixed countably infinite set. Then

$$Mod_\Delta(\mathcal{K}) = \{\mathcal{R} = \langle \Delta^{\mathcal{R}}, \cdot^{\mathcal{R}}, \prec^{\mathcal{R}} \rangle \mid \mathcal{R} \Vdash \mathcal{K}, \mathcal{R} \text{ is ranked, and } \Delta^{\mathcal{R}} = \Delta\}.$$

To define the *Big Ranked Model* (Britz et al. 2021, Definition 21), we refer to the notion of ranked union (Britz et al. 2021, Definition 20).

Definition 7 (Big Ranked Model)**.** *Let $\mathcal{K}$ be a defeasible knowledge base. The* Big Ranked Model *of $\mathcal{K}$ is the ranked model $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \prec^{\mathcal{O}} \rangle$ that is the ranked union of the models in $Mod_\Delta(\mathcal{K})$. That is,*

- $\Delta^{\mathcal{O}} = \coprod_{\mathcal{R} \in Mod_\Delta(\mathcal{K})} \Delta^{\mathcal{R}}$ *is the disjoint union of the domains from $Mod_\Delta(\mathcal{K})$, where each $\mathcal{R} \in Mod_\Delta(\mathcal{K})$ has the elements $x, y, \ldots$ of its domain renamed as $x_{\mathcal{R}}, y_{\mathcal{R}}, \ldots$ so that they are all distinct in $\Delta^{\mathcal{O}}$;*
- $x_{\mathcal{R}} \in A^{\mathcal{O}}$ *iff* $x \in A^{\mathcal{R}}$, *for all concept name A;*
- $(x_{\mathcal{R}}, y_{\mathcal{R}'}) \in r^{\mathcal{O}}$ *iff* $\mathcal{R} = \mathcal{R}'$ *and* $(x, y) \in r^{\mathcal{R}}$*;*
- $x_{\mathcal{R}} \prec^{\mathcal{O}} y_{\mathcal{R}'}$ *iff* $h_{\mathcal{R}}(x) < h_{\mathcal{R}'}(y)$.

The last condition requires that for every $x_{\mathcal{R}} \in \Delta^{\mathcal{O}}$, $h_{\mathcal{O}}(x_{\mathcal{R}}) = h_{\mathcal{R}}(x)$.
Now we use the Big Ranked Model to define RC for $\mathcal{DALC}$.

Definition 8. *For a $\mathcal{DALC}$ knowledge base $\mathcal{K}$ and a defeasible inclusion $C \mathrel{\substack{\sqsubset\\\sim}} D$, we say that $C \mathrel{\substack{\sqsubset\\\sim}} D$ is an* RC entailment *of $\mathcal{K}$ (written $\mathcal{K} \models_{RC} C \mathrel{\substack{\sqsubset\\\sim}} D$) iff $\mathcal{O} \Vdash C \mathrel{\substack{\sqsubset\\\sim}} D$, where $\mathcal{O}$ is the Big Ranked Model of $\mathcal{K}$.*

The properties of RC for $\mathcal{DALC}$ are explored by Britz et al. (2021). In the next two sections we present two alternative approaches for reasoning with $\mathcal{DALC}$ knowledge bases.

4 System W for $\mathcal{DALC}$

We now turn to the first main contribution of the paper—a version of System W for $\mathcal{DALC}$. We start with the provision of a semantic definition.

4.1 Semantic Definition: The Big W-Model

As shown in Section 2, at the propositional level, System W has a semantic definition that is a refinement of the semantic definition of RC. Hence, moving to System W for $\mathcal{DALC}$, we adapt the same approach by starting with the definition of RC for $\mathcal{DALC}$ outlined in the previous section, and apply on top of it a refinement of the preference ordering analogous to the way $<^{\mathsf{W}}_{\mathcal{K}}$ refines $<^{\mathsf{RC}}_{\mathcal{K}}$ for the propositional case.

Given that the Big Ranked Model characterises RC in $\mathcal{DALC}$, we proceed by building a similar model for System W on top of the Big Ranked Model, using the same approach outlined in Proposition 2.

Definition 9 (Big W-Model)**.** *For a $\mathcal{DALC}$ knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, let $\mathcal{O}$ be the Big Ranked Model of $\mathcal{K}$ and let $RP(\mathcal{K}) = \langle D^{\mathcal{K}}_0, \ldots, D^{\mathcal{K}}_{n-1}, D^{\mathcal{K}}_\infty \rangle$ be the ranked partition of $\mathcal{K}$. The* Big W-Model *of $\mathcal{K}$ is the model $W(\mathcal{O}) = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \prec^W \rangle$, where $\prec^W$ can be defined as follows. For $x, y \in \Delta^{\mathcal{O}}$, $x \prec^W y$ iff*

$$
\begin{aligned}
& x \prec^{\mathcal{O}} y; \textit{ or} \\
& h_{\mathcal{O}}(x) = h_{\mathcal{O}}(y) \textit{ and there is an } m \in \{0, \ldots, n-1\} \\
& \textit{s.t. } \xi^i(x) = \xi^i(y) \quad \forall i \in \{m+1, \ldots, n-1\} \textit{ and} \\
& \xi^m(x) \subsetneqq \xi^m(y)
\end{aligned}
$$

where, for any $x \in \Delta^{\mathcal{O}}$, $\xi^m(x) = \{C \mathrel{\substack{\sqsubset\\\sim}} D \in \mathcal{D}^j \mid x \in (C \sqcap \neg D)^{W(\mathcal{O})}\}$.

Note that the preference order $\prec^W$ is obtained as a refinement of $\prec^{\mathcal{O}}$ that exactly mirrors the propositional case, as presented in Proposition 2. Proposition 4 below shows that such a definition is in line with the original propositional formulation of of the preference $<^{\mathsf{W}}_{\mathcal{K}}$ for System W as introduced in Definition 2.

Proposition 4. *For a $\mathcal{DALC}$ knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, let $\langle D^{\mathcal{K}}_0, \ldots, D^{\mathcal{K}}_{n-1}, D^{\mathcal{K}}_\infty \rangle$ be the ranked partition of $\mathcal{K}$ and let $W(\mathcal{O}) = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \prec^W \rangle$ be the Big W-Model of $\mathcal{K}$. Then $\prec^W$ is equivalent to the relation $\prec^*$: $\Delta^{\mathcal{O}} \times \Delta^{\mathcal{O}}$, defined as*

[2]The Big Ranked Model is appropriate for modelling RC in all DLs satisfying the so-called *Disjoint Union Property*. We refer the reader to (Bonatti 2019) for a semantic definition of RC that also covers more expressive DLs.

follows. For any pair $x, y \in \Delta^{\mathcal{O}}$, $x \prec^* y$ *iff there exists an* $m \in \{0, \ldots, k\}$ *such that*

$$\xi^i(o) = \xi^i(o') \quad \forall i \in \{m+1, \ldots, n-1\} \text{ and}$$
$$\xi^m(o) \subsetneqq \xi^m(o')$$

where, for any $x \in \Delta^{\mathcal{O}}$, $\xi^m(x) = \{C \mathrel{\sqsubset\!\!\!\sim} D \in \mathcal{D}^j \mid x \in (C \sqcap \neg D)^{W(\mathcal{O})}\}$.

As for the Big Ranked Model and RC, we use the Big W-Model to define System W entailment.

Definition 10. *For a* $\mathcal{DALC}$ *knowledge base* $\mathcal{K}$ *and a defeasible inclusion* $C \mathrel{\sqsubset\!\!\!\sim} D$, *we say that* $C \mathrel{\sqsubset\!\!\!\sim} D$ *is a* System W entailment of $\mathcal{K}$ *(written* $\mathcal{K} \models_W C \mathrel{\sqsubset\!\!\!\sim} D$*) iff* $W(\mathcal{O}) \Vdash C \mathrel{\sqsubset\!\!\!\sim} D$ *where* $W(\mathcal{O})$ *is the* Big W-model *of* $\mathcal{K}$.

To understand the properties of System W entailment, let us first show that the Big W-Model is indeed a preferential interpretation.

Proposition 5. *Let* $W(\mathcal{O}) = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \prec^W \rangle$ *be the Big W-Model of some knowledge base* $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$. $W(\mathcal{O})$ *is a preferential model of* $\mathcal{K}$.

Since the Big W-Model is a preferential interpretation we can conclude that System W for $\mathcal{DALC}$ satisfies the System P postulates (Ref), (LLE), (RW), (AND), (OR), (CM), ($\text{CM}_\exists$), and ($\text{CM}_\forall$) for defeasible DL (Britz et al. 2021).

We already saw that $<^{\mathtt{W}}_{\mathcal{K}}$ is a refinement of $<^{\mathtt{RC}}_{\mathcal{K}}$. Furthermore, we can show that System W for $\mathcal{DALC}$ extends RC.

Proposition 6. *For any knowledge base* $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ *and any defeasible subsumption* $C \mathrel{\sqsubset\!\!\!\sim} D$, *if* $\mathcal{K} \models_{RC} C \mathrel{\sqsubset\!\!\!\sim} D$, *then* $\mathcal{K} \models_W C \mathrel{\sqsubset\!\!\!\sim} D$.

The reverse of Proposition 6 is not true, i.e., there are knowledge bases for which W-entailment licenses strictly more queries than RC. Example 2 below will illustrate an instace where System W strictly extends RC.

Note that every consistent $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ also has a Big W-Model.

Proposition 7. *A knowledge base* $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ *is consistent iff it has a Big W-Model* $W(\mathcal{O})$.

Finally, we want to point out that our definition of System W for $\mathcal{DALC}$ properly generalizes the original System W, i.e., both definitions coincide for propositional conditionals. Observe, that propositional logic corresponds to a fragment of $\mathcal{ALC}$.

Definition 11. *Let* Σ *be a propositional signature and* Σ_C *be a set of concept names such that* $\Sigma \subseteq \Sigma_C$. *The mapping* ALC *from propositional formulas to* $\mathcal{ALC}$ *concepts is recursively defined as follows:*

$$ALC(\neg E) = \neg ALC(E) \qquad ALC(a) = a$$
$$ALC(E \wedge F) = ALC(E) \sqcap ALC(F)$$
$$ALC(E \vee F) = ALC(E) \sqcup ALC(F)$$

for any formulas $E, F \in \mathcal{L}^{prop}$ *and any* $a \in \Sigma$.

First, we observe that the notions of exceptionality for DCIs and conditionals coincide. Therefore, the ranked partition of a conditional knowledge base corresponds to the ranked partition of a DBox.

Lemma 1. *Let* $\mathcal{K}$ *be a set of (propositional) conditionals, and let* $\mathcal{D} = \{ALC(C) \mathrel{\sqsubset\!\!\!\sim} ALC(D) \mid C \mathrel{|\!\!\sim} D \in \mathcal{K}\}$. *Then a formula* C *is exceptional for* $\mathcal{K}$ *iff* $ALC(C)$ *is exceptional for* $\mathcal{D}$. *Furthermore, the ranked partition* $RP(\mathcal{K}) = (\mathcal{K}_0, \ldots, \mathcal{K}_k, \mathcal{K}_\infty)$ *corresponds to the ranked partition* $RP(\mathcal{D}) = (\mathcal{D}^{\mathcal{K}}_0, \ldots, \mathcal{D}^{\mathcal{K}}_l, \mathcal{D}^{\mathcal{K}}_\infty)$ *in the sense that* $k = l$ *and* $ALC(\mathcal{K}^i) = \mathcal{D}^{\mathcal{K}}_i$ *for* $i \in \{0, \ldots, k, \infty\}$.

With this we can show that the preferential System W coincides with System W for $\mathcal{DALC}$.

Proposition 8. *Let* $\mathcal{K}$ *be a set of (propositional) conditionals, and let* $\mathcal{D} = \{ALC(C) \mathrel{\sqsubset\!\!\!\sim} ALC(D) \mid C \mathrel{|\!\!\sim} D \in \mathcal{K}\}$. *Then for formulas* E, F *we have* $\mathcal{K} \models_W E \mathrel{|\!\!\sim} F$ *iff* $\mathcal{D} \models_W ALC(E) \mathrel{\sqsubset\!\!\!\sim} ALC(F)$.

Thus, System W for $\mathcal{DALC}$ is a proper generalisation of its propositional counterpart.

4.2 An Algorithm for W-Entailment in $\mathcal{DALC}$

In this section we present an algorithm for deciding W-entailment. As it was the case for RC (Britz et al. 2021), the decision procedure is based on a series of decision problems for classical $\mathcal{ALC}$. Because of that, the overall procedure can be implemented on top of classical reasoners for $\mathcal{ALC}$.

As we have seen in Section 4.1, given a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, modelling W-entailment relies on the functions ξ^i, that in turn are defined using the ranked partition $\langle \mathcal{D}^{\mathcal{K}}_0, \ldots, \mathcal{D}^{\mathcal{K}}_n, \mathcal{D}^{\mathcal{K}}_\infty \rangle$ of $\mathcal{D}$ associated to RC. In order to obtain such a partition, we rely on the procedure `ComputeRanking`($\mathcal{K}$) that in turn relies on the procedure `Exceptional`($\mathcal{T}, \mathcal{D}$), both described in (Britz et al. 2021). The procedure `Exceptional`($\mathcal{T}, \mathcal{D}$) (Algorithm 1) takes a set $\mathcal{T}$ of strict inclusions and a set $\mathcal{D}$ of DCIs, and returns the set $\mathcal{E}$ of DCIs in $\mathcal{D}$ that are exceptional w.r.t. $\langle \mathcal{T}, \mathcal{D} \rangle$. The procedure `ComputeRanking`($\mathcal{K}$) (Algorithm 2) takes as input a defeasible $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ and returns the partition $\langle \mathcal{D}^{\mathcal{K}}_0, \ldots, \mathcal{D}^{\mathcal{K}}_n, \mathcal{D}^{\mathcal{K}}_\infty \rangle$. To be more precise, `ComputeRanking`($\mathcal{K}$) gives back

- a sequence $\mathcal{E} = (\mathcal{E}_0, \ldots, \mathcal{E}_n)$ where, for each $i \in \{0, \ldots, n\}$, we have $\mathcal{E}_i = \bigcup_{i \leq j \leq n} \mathcal{D}^{\mathcal{K}}_j$;
- a knowledge base $\mathcal{K}^* = \langle \mathcal{T}^*, \mathcal{D}^* \rangle$, where $\mathcal{D}^*$ has been obtained by removing from $\mathcal{D}$ all the axioms $C \mathrel{\sqsubset\!\!\!\sim} D$ that have rank ∞, while $\mathcal{T}^*$ has been obtained by adding to $\mathcal{T}$ the correspondent axioms $C \sqsubseteq D$.

Britz et al. (2021) proved that the following holds for the output of `ComputeRanking`($\mathcal{K}$):

- The sequence $\mathcal{E} = (\mathcal{E}_0, \ldots, \mathcal{E}_n)$ is correct in the sense that for each $i \in \{0, \ldots, n-1\}$ we have $\mathcal{D}^{\mathcal{K}}_i = \mathcal{E}_i \setminus \mathcal{E}_{i+1}$, and $\mathcal{D}_n = \mathcal{E}_n$.
- $\mathcal{K}$ and $\mathcal{K}^*$ are equivalent w.r.t. RC, that is, they have the same Big Ranked Model $\mathcal{O}$, and, consequently, for every $C \mathrel{\sqsubset\!\!\!\sim} D$, $\mathcal{K} \models_{RC} C \mathrel{\sqsubset\!\!\!\sim} D$ iff $\mathcal{K}^* \models_{RC} C \mathrel{\sqsubset\!\!\!\sim} D$.

The transformation from $\mathcal{K}$ to $\mathcal{K}^*$ properly separates the DCIs that can have exceptions from the inclusions that cannot have exceptions. The knowledge base $\mathcal{K}^* = \langle \mathcal{T}^*, \mathcal{D}^* \rangle$ is defined by the procedure `ComputeRanking`($\mathcal{K}$) by eliminating all the defeasible axioms $C \mathrel{\sqsubset\!\!\!\sim} D$ in $\mathcal{D}$ with rank ∞ and adding the corresponding axioms $C \sqsubseteq D$ to $\mathcal{T}$. If $C \sqsubsim D$ has rank ∞, it follows that its antecedent C cannot be populated in any preferential model of $\mathcal{K}$. In particular, we have both $\mathcal{K} \models_{RC} C \sqsubseteq \bot$ and $\mathcal{K} \models_{W} C \sqsubseteq \bot$. It turns out that we obtain exactly the same result, both from the semantical and the inferential point of view, by eliminating such $C \sqsubsim D$ from $\mathcal{D}$ and adding $C \sqsubseteq D$ (or equivalently $C \sqsubseteq \bot$) to $\mathcal{T}$ – both for RC and System W. We refer the reader to Britz et al. (2021, Section 5) for more details.

Proposition 9. *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ be any $\mathcal{ALC}$ defeasible KB and $\mathcal{K}^* = \langle \mathcal{T}^*, \mathcal{D}^* \rangle$ the KB obtained from* `ComputeRanking`$(\mathcal{K})$. *$\mathcal{K}$ and $\mathcal{K}^*$ are equivalent w.r.t. $\models_W$, that is, for any DI $C \sqsubsim D$, $\mathcal{K} \models_W C \sqsubsim D$ iff $\mathcal{K}^* \models_W C \sqsubsim D$.*

From now on, when dealing with some knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$, we will often refer also to the knowledge base $\mathcal{K}^* = \langle \mathcal{T}^*, \mathcal{D}^* \rangle$ obtained from `ComputeRanking`$(\mathcal{K})$, the latter being associated with a sequence $\mathcal{E} = (\mathcal{E}_0, \ldots, \mathcal{E}_n)$ and with $\mathcal{E}_\infty = \emptyset$. Such a move is justified by Proposition 9.

From Proposition 9, Proposition 7, and a result by Britz et al. (2021, Corollary 7), we immediately obtain an easy way to check whether a knowledge base is consistent.

Proposition 10. *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$. $\mathcal{K}$ has a Big W-Model iff $\mathcal{T}^* \not\models \top \sqsubseteq \bot$.*

Using `ComputeRanking`$(\mathcal{K})$, we now introduce the procedure `WClosure`$(\mathcal{K}, C \sqsubsim D)$ (Algorithm 3) to decide whether the inclusion $C \sqsubsim D$ is W-entailed by $\mathcal{K}$. As indicated above, `ComputeRanking`$(\mathcal{K})$ returns a sequence $\mathcal{E} = (\mathcal{E}_0, \ldots, \mathcal{E}_n)$ such that for each i, $0 \leq i < n$, $\mathcal{D}_i^{\mathcal{K}} = \mathcal{E}_i \setminus \mathcal{E}_{i+1}$ while $\mathcal{D}_n^{\mathcal{K}} = \mathcal{E}_n$. The latter operation is implemented in Lines 2–4 of the procedure `WClosure`$(\mathcal{K}, C \sqsubsim D)$.

The minimal objects in $C^{\mathcal{O}}$ w.r.t. $<_{\mathcal{K}}^{\mathsf{w}}$ are exactly those objects in $C^{\mathcal{O}}$ that satisfy an inclusion-maximal set of DCIs, where preference is given to DCIs in the later part of the ranked partition $\langle \mathcal{D}_0^{\mathcal{K}}, \ldots, \mathcal{D}_n^{\mathcal{K}} \rangle$. Because of such a preference we can construct the inclusion-maximal sets characterizeing minimal objects by considering the ranks individually and extending the sets step by step.

The set γ_j will contain the restrictions of all inclusion maximal sets to the DCIs of rank i and above. For $j = n+1$ it is initialized by $\gamma_{n+1} := \{\emptyset\}$ in Line 5. The for-loop beginning in Line 6 iterates through the indices of the ranked partition from n down to 0. In iteration j, for each set r in γ_{j+1} all inclusion maximal sets p with $\overline{r} \cup \overline{p}$ being consistent with C are listed in $\mathcal{P}_j^{max}$ (Lines 10–14). Then, in Lines 15–16, the extended sets from γ_{j-1} are added to γ_j. After the while loop, the objects in $C^{\mathcal{O}}$ that are minimal w.r.t. $<_{\mathcal{K}}^{\mathsf{w}}$ are exactly the objects in $\Delta^{\mathcal{O}}$ that are in $C^{\mathcal{O}}$ and satisfy all DCIs in one of the sets in γ_0.

Finally, in Lines 17–20, the algorithm checks if the strict inclusions in $\mathcal{T}^*$ imply that C in conjunction with the sets in γ_0 is always subsumed by D, i.e., if all minimal objects in $C^{\mathcal{O}}$ are in $D^{\mathcal{O}}$.

To summarize, we can show that the procedure `WClosure` is correct and complete w.r.t. the entailment relation $\models_W$.

Theorem 1. *For a knowledge base $\mathcal{K}$ and concepts C, D,* `WClosure`$(\mathcal{K}, C \sqsubsim D)$ *returns* `True` *iff $\mathcal{K} \models_W C \sqsubsim D$.*

Algorithm 1: `Exceptional`$(\mathcal{T}, \mathcal{D})$

Input: $\mathcal{T}$ and $\mathcal{D}$
Output: $\mathcal{E} \subseteq \mathcal{D}$ s.t. $\mathcal{E}$ is exceptional w.r.t. $(\mathcal{T}, \mathcal{D})$

$\mathcal{E} := \emptyset$;
foreach $C \sqsubsim D \in \mathcal{D}$ **do**
  if $\mathcal{T} \models \bigsqcap \overline{\mathcal{D}} \sqsubseteq \neg C$ **then**
    $\mathcal{E} := \mathcal{E} \cup \{C \sqsubsim D\}$
return $\mathcal{E}$;

Algorithm 2: `ComputeRanking`$(\mathcal{K})$

Input: $\mathcal{K} = (\mathcal{T}, \mathcal{D})$
Output: $\mathcal{K}^* = (\mathcal{T}^*, \mathcal{D}^*)$; exceptionality ranking $\mathcal{E}$

$\mathcal{T}^* := \mathcal{T}, \quad \mathcal{D}^* := \mathcal{D}$;
repeat
  $i := 0$;
  $\mathcal{E}_0 := \mathcal{D}^*$;
  $\mathcal{E}_1 :=$ `Exceptional`$(\mathcal{T}^*, \mathcal{E}_0)$;
  while $\mathcal{E}_{i+1} \neq \mathcal{E}_i$ **do**
    $i := i + 1$;
    $\mathcal{E}_{i+1} :=$ `Exceptional`$(\mathcal{T}^*, \mathcal{E}_i)$;
  $\mathcal{D}_\infty^* := \mathcal{E}_i$;
  $\mathcal{T}^* := \mathcal{T}^* \cup \{C \sqsubseteq D \mid C \sqsubsim D \in \mathcal{D}_\infty^*\}$;
  $\mathcal{D}^* := \mathcal{D}^* \setminus \mathcal{D}_\infty^*$;
until $\mathcal{D}_\infty^* = \emptyset$;
$\mathcal{E} := (\mathcal{E}_0, \ldots, \mathcal{E}_{i-1})$;
return $(\mathcal{K}^* = (\mathcal{T}^*, \mathcal{D}^*), \mathcal{E})$;

The following example will illustrate the `WClosure` procedure in contrast to RC.

Example 2. *Consider a knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ that contains in $\mathcal{T}$ that penguins are birds ($P \sqsubseteq B$), birds are animals ($B \sqsubseteq A$), waterproof feathers are feathers ($Wf \sqsubseteq Fe$), and stuffed animals are animals ($Sta \sqsubseteq A$); in $\mathcal{D}$ we model the following defeasible inclusions: birds fly ($B \sqsubsim F$), penguins do not fly ($P \sqsubsim \neg F$), feathers are a part of the birds ($b \sqsubsim \exists part.Fe$), waterproof feathers are a part of the penguins ($P \sqsubsim \exists part.Wf$), stuffed animals do not have feathers ($Sta \sqsubsim \neg \exists part.Fe$), stuffed animals do not fly ($Sta \sqsubsim \neg F$), and animals are usually not stuffed ($A \sqsubsim \neg Sta$). Our ranked partition is*

- $\mathcal{D}_0^{\mathcal{K}} = \{B \sqsubsim F, B \sqsubsim \exists part.Fe, A \sqsubsim \neg Sta\}$;
- $\mathcal{D}_1^{\mathcal{K}} = \{P \sqsubsim \neg F, P \sqsubsim \exists part.Wf, Sta \sqsubsim \neg \exists part.Fe, Sta \sqsubsim \neg F\}$.

Since we do not have any defeasible inclusion of rank ∞, $\mathcal{T}^* = \mathcal{T}$ and $\mathcal{D}^* = \mathcal{D}$. We show that there is some defeasible inclusion, non-flying birds presumably have feathers*, that we can derive with System W and not with RC.*

RC. *We run the procedure* `RationalClosure`$(\mathcal{K}, B \sqcap \neg F \sqsubsim \exists part.Fe)$ *for RC from (Britz et al. 2021).*

- $\mathcal{T}^* \models \bigsqcap(\overline{D_1^{\mathcal{K}}} \cup \overline{D_0^{\mathcal{K}}}) \sqsubseteq \neg(B \sqcap \neg F)$: *No object can satisfy the antecedent of the query, remove the inclusions from $D_0^{\mathcal{K}}$ in the next step.*

Algorithm 3: $\texttt{WClosure}(\mathcal{K}, \alpha)$

Input: $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ and a query $\alpha = C \precsim D$.
Output: `true` if $\mathcal{K} \models_W$ C $\precsim$ D, else `False`

$(\mathcal{K}^*, \mathcal{E}) := \texttt{ComputeRanking}(\mathcal{K})$;
for $j := 0$ *to* $n-1$ **do**
  $\mathcal{D}_j := \mathcal{E}_j \setminus \mathcal{E}_{j+1}$;
   // $\mathcal{D}_j$ contains the DI's in $\mathcal{D}$ with rank j
$\mathcal{D}_n := \mathcal{E}_n$;
$\gamma_{n+1} := \{\emptyset\}$;
for $j := n$ *downto* 0 **do**
  $\gamma_j := \emptyset$;
  $\wp(\mathcal{D}_j) := \{p \mid p \subseteq \mathcal{D}_j\}$;
   // $\wp(\mathcal{D}_j)$ is the powerset of $\mathcal{D}_j$
  foreach $r \in \gamma_{j+1}$ **do**
    $\mathcal{P}_j := \emptyset$;
    foreach $p \in \wp(\mathcal{D}_j)$ **do**
      if $\mathcal{T}^* \not\models \bigsqcap\{\overline{r} \cup \overline{p}\} \sqsubseteq \neg C$ **then**
        $\mathcal{P}_j := \mathcal{P}_j \cup \{p\}$;
    $\mathcal{P}_j^{max} := \{p \mid p \in \mathcal{P}_j,$
      there is no $p' \in \mathcal{P}_j$ s.t. $p \subsetneq p'\}$;
    foreach $p \in \mathcal{P}_j^{max}$ **do**
      $\gamma_j := \gamma_j \cup \{\{r \cup p\}\}$;
if $\mathcal{T}^* \models \bigsqcap \overline{p} \sqcap C \sqsubseteq D$*, for all* $p \in \gamma_0$ **then**
  return `True` ;
else
  return `False` ;

- $\mathcal{T}^* \not\models \bigsqcap \overline{D_1^{\mathcal{K}}} \sqsubseteq \neg(B \sqcap \neg F)$*: The antecedent of the query is consistent with* $\overline{D_1^{\mathcal{K}}}$*; use it to check the entailment.*

We can see that $\mathcal{T}^* \not\models \bigsqcap \overline{D_1^{\mathcal{K}}} \sqcap \neg(B \sqcap \neg F) \sqsubseteq \exists part.Fe$*:* $\texttt{RCclosure}(\mathcal{K}, B \sqcap \neg F \precsim \exists part.Fe)$ *returns* `False`.

System W. *We run* $\texttt{Wclosure}(\mathcal{K}, B \sqcap \neg F \precsim \exists part.Fe)$*.*

- γ_1 *contains one set:* $p_1 = \mathcal{D}_1^{\mathcal{K}}$*.*
- γ_0 *contains one set that extends* p_1*:* $t_1 = p_1 \cup \{B \precsim \exists part.Fe, A \precsim \neg Sta\}$*.*

Now we verify that $\mathcal{T} \models \bigsqcap \overline{t_1} \sqcap (B \sqcap \neg F) \sqsubseteq \exists part.Fe$*. Hence* $\texttt{Wclosure}(\mathcal{K}, B \sqcap \neg F \precsim \exists part.Fe)$ *returns* `True`.

5 Lexicographic Closure for $\mathcal{DALC}$

A first proposal for modelling Lexicographic Closure (LC) for $\mathcal{DALC}$ was provided by Casini and Straccia (2012), but it lacks a semantic characterisation. Here we present both a semantic characterisation and a decision procedure, based on the original characterisation of LC given by Lehmann (1995) for propositional conditionals.

5.1 Semantic Definition: The Big L-Model

Lehmann (1995, Sect. 9.2) presents a semantic characterisation of LC based on a specific ranking of the worlds that is a refinement of the ordering $<_{\mathcal{K}}^{\mathtt{RC}}$ characterizing RC. As in the case of RC and System W, we propose a construction that orders the individuals. Since on the semantic side, like System W, LC is a refinement of the preference order of RC, we follow the same strategy used for System W: we start from the Big Ranked Model characterising RC for $\mathcal{DALC}$, and we refine the ranking of the individuals in a way that mirrors the way in which lexicographic closure refines the RC ordering in the propositional case.

As a first step we introduce the *seriousness ordering* among sets of defeasible inclusions. Following the propositional case, given a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ the seriousness ordering is based on the ranked partition $\langle \mathcal{D}_0^{\mathcal{K}}, \ldots, \mathcal{D}_n^{\mathcal{K}} \rangle$. Let $\mathcal{D}'$ be any subset of $\mathcal{D}$. We associate to $\mathcal{D}'$ a string of $n+1$ numbers $\langle k_0, \ldots, k_n \rangle_{\mathcal{D}'}$ such that for $i \in \{0, \ldots, n\}$ we have $k_i = |\mathcal{D}' \cap \mathcal{D}_{n-i}|$. That is, the first number indicates the number of the most exceptional axioms in $\mathcal{D}'$ (those with rank n and highest priority), and so on until we reach k_n, that indicates the number of the most generic axioms in $\mathcal{D}'$ (those with rank 0, lowest priority). Given any pair $\mathcal{D}', \mathcal{D}''$ of subsets of $\mathcal{D}$, with $\langle h_0, \ldots, h_n \rangle_{\mathcal{D}''}$ and $\langle k_0, \ldots, k_n \rangle_{\mathcal{D}'}$, we say that $\mathcal{D}' \prec_l \mathcal{D}''$ ($\mathcal{D}'$ is less serious than $\mathcal{D}''$) iff there is an $i \in \{0, \ldots, n\}$ such that

$$k_j = h_j \quad \text{for every } j \in \{0, \ldots, i-1\} \text{ and}$$
$$k_i < h_i.$$

That is, $\mathcal{D}' \prec_l \mathcal{D}''$ if $\mathcal{D}'$ includes a lower number of more specific axioms than $\mathcal{D}''$.

Following Proposition 3, we will refine the preference relation $\prec^{\mathcal{O}}$ over the objects in the Big Ranked Model in the following way.

Definition 12 (Big L-Model)**.** *Let* $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ *be any defeasible KB. The* Big L-Model *of* $\mathcal{K}$ *is the model* $L(\mathcal{O}) = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \prec^L \rangle$*, where* $\prec^L$ *is defined by, for any* $x, y \in \Delta^{\mathcal{O}}$*,* $x \prec^L y$ *iff*

$x \prec^{\mathcal{O}} y$*; or*
$h_{\mathcal{O}}(x) = h_{\mathcal{O}}(y)$ *and there is an* $m \in \{0, \ldots, n-1\}$
s.t. $|\xi^i(x)| = |\xi^i(y)| \quad \forall i \in \{m+1, \ldots, n-1\}$ *and*
$|\xi^m(x)| < |\xi^m(y)|$

where, for any $x \in \Delta^{\mathcal{O}}$*,* $\xi^m(x) = \{C \precsim D \in \mathcal{D}^j \mid x \in (C \sqcap \neg D)^{L(\mathcal{O})}\}$*.*

The definition of $\prec^L$ mirrors the LC-ordering for the propositional case: given any pair of objects o, o' in a $\Delta^{\mathcal{O}}$ we have that $o \prec^L o'$ iff $\xi(o) \prec_l \xi(o')$, that is, iff o violates a lower number of exceptional axioms than o'.

Proposition 11. *Given any defeasible KB* $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$*, let* $L(\mathcal{O}) = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \prec^L \rangle$ *be its Big L-Model. Then* $\prec^L$ *is equivalent to the relation* $\prec^*$*:* $\Delta^{\mathcal{O}} \times \Delta^{\mathcal{O}}$*, defined as follows. For any pair* $x, y \in \Delta^{\mathcal{O}}$*,* $x \prec^* y$ *iff there exists an* $m \in \{0, \ldots, k\}$ *such that*

$$|\xi^i(o)| = |\xi^i(o')| \quad \forall i \in \{m+1, \ldots, n-1\} \text{ and}$$
$$|\xi^m(o)| < |\xi^m(o')| \,.$$

where, for any $x \in \Delta^{\mathcal{O}}$*,* $\xi^m(x) = \{C \precsim D \in \mathcal{D}^j \mid x \in (C \sqcap \neg D)^{W(\mathcal{O})}\}$*.*

The model $L(\mathcal{O})$ is the characteristic model of the LC for $\mathcal{ALC}$, that is, we define lexicographic entailment for $\mathcal{DALC}$ as the inference induced by the Big L-model $L(\mathcal{O})$.

Definition 13 (Lexicographic entailment)**.** *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ be any $\mathcal{DALC}$ knowledge base and $C \tilde{\sqsubseteq} D$ a defeasible subsumption. We say that $\mathcal{K}$* L-entails *$C \tilde{\sqsubseteq} D$ (written $\mathcal{K} \models_L C \tilde{\sqsubseteq} D$) iff*

$$L(\mathcal{O}) \Vdash C \tilde{\sqsubseteq} D,$$

where $L(\mathcal{O})$ is the Big L-Model *for $\mathcal{K}$.*

Just as the Big Ranked Model and the Big W-model the Big L-Model is a preferential interpretation.

Proposition 12. *Let $L(\mathcal{O}) = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \prec^L \rangle$ be the Big L-Model of the knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$. Then $L(\mathcal{O})$ is a preferential model of $\mathcal{K}$.*

Actually, the Big L-Model $L(\mathcal{O})$, like the Big Ranked Model $\mathcal{O}$, is a *ranked interpretation* (see Section 3) implying additional properties, but we are not going to investigate that here. LC for $\mathcal{DALC}$ defines the most extensive entailment we consider here, since it extends System W, and, consequently, RC.

Proposition 13. *For any knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ and any defeasible subsumption $C \tilde{\sqsubseteq} D$, if $\mathcal{K} \models_W C \tilde{\sqsubseteq} D$, then $\mathcal{K} \models_L C \tilde{\sqsubseteq} D$.*

The reverse of Proposition 6 is not true in general, i.e., there are knowledge bases for which LC-entailment encompasses strictly more inclusions than W-entailment. Example 3 in the following section illustrates that.

Analogous to the case for Big W-Models, it holds that every consistent $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ has also a Big L-Model.

Proposition 14. *A knowledge base $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ is consistent iff it has a Big L-Model $L(\mathcal{O})$.*

Also for LC we can show that the definition for $\mathcal{DALC}$ properly generalizes the original LC, i.e., both definitions coincide for reasoning from propositional conditionals. For the correspondence between propositional logic and a fragment of $\mathcal{ALC}$ we refer to Definition 11, and again use the correspondence between the respective ranked partitions shown in Lemma 1. Moving from there we can prove that propositional LC coincides with LC for $\mathcal{DALC}$.

Proposition 15. *Let $\mathcal{K}$ be a set of (propositional) conditionals, and let $\mathcal{D} = \{ALC(C) \tilde{\sqsubseteq} ALC(D) \mid C \mid\!\sim D \in \mathcal{K}\}$. Then for formulas E, F we have $\mathcal{K} \models_L E \mid\!\sim F$ iff $\mathcal{D} \models_L ALC(E) \tilde{\sqsubseteq} ALC(F)$.*

5.2 An Algorithm for LC-entailment in $\mathcal{DALC}$

As for System W, we present a decision procedure for lexicographic closure, `LexClosure`$(\mathcal{K}, C \tilde{\sqsubseteq} D)$ (Algorithm 4). The main difference between the algorithms is the way the maximal sets are selected in each layer in Line 14: While `WClosure` is selecting inclusion-maximal sets, `LexClosure` selects sets with maximal arity.

The procedure `LexClosure` is correct and complete w.r.t. the entailment relation.

Theorem 2. *For a knowledge base $\mathcal{K}$ and concepts C, D,* `LexClosure`$(\mathcal{K}, C \tilde{\sqsubseteq} D)$ *returns* `True` *iff $\mathcal{K} \models_L C \tilde{\sqsubseteq} D$.*

Algorithm 4: `LexClosure`$(\mathcal{K}, \alpha)$

Input: $\mathcal{K} = (\mathcal{T}, \mathcal{D})$ and a query $\alpha = C \tilde{\sqsubseteq} D$.
Output: `true` if $\mathcal{K} \models_L$ C $\tilde{\sqsubseteq}$ D, else `False`

1 $(\mathcal{K}^*, \mathcal{E}) :=$ `ComputeRanking`$(\mathcal{K})$;
2 **for** $j := 0$ *to* $n-1$ **do**
3 $\quad \mathcal{D}_j := \mathcal{E}_j \setminus \mathcal{E}_{j+1}$; // $\mathcal{D}_j$ *contains the DI's in* $\mathcal{D}$ *with rank* j
4 $\mathcal{D}_n := \mathcal{E}_n$;
5 $\gamma_{n+1} := \{\emptyset\}$;
6 **for** $j := n$ *downto* 0 **do**
7 $\quad \gamma_j := \emptyset$;
8 $\quad \wp(\mathcal{D}_j) := \{p \mid p \subseteq \mathcal{D}_j\}$; // $\wp(\mathcal{D}_j)$ *is the powerset of* $\mathcal{D}_j$
9 $\quad$ **foreach** $r \in \gamma_{j+1}$ **do**
10 $\quad\quad \mathcal{P}_j := \emptyset$;
11 $\quad\quad$ **foreach** $p \in \wp(\mathcal{D}_j)$ **do**
12 $\quad\quad\quad$ **if** $\mathcal{T}^* \not\models \bigsqcap\{\overline{r} \cup \overline{p}\} \sqsubseteq \neg C$ **then**
13 $\quad\quad\quad\quad \mathcal{P}_j := \mathcal{P}_j \cup \{r \cup p\}$
14 $\quad \mathcal{P}_j^{max} := \{t \mid t \in \mathcal{P}_j,$ there is no $t' \in \mathcal{P}_j$ s.t. $|t'| > |t|\}$;
15 $\quad$ **foreach** $t \in \mathcal{P}_j^{max}$ **do**
16 $\quad\quad \gamma_j := \gamma_j \cup \{\{t\}\}$;
17 **if** $\mathcal{T}^* \models \bigsqcap \overline{t} \sqcap C \sqsubseteq D$ *for all* $t \in \gamma_0$ **then**
18 $\quad$ return `True`
19 **else**
20 $\quad$ return `False`

We conclude this section by illustrating `LexClosure` with an example that contrasts it with `WClosure`.

Example 3 (Example 2 cont'd)**.** *Let $\mathcal{K} = \langle \mathcal{T}, \mathcal{D} \rangle$ be the knowledge base from Example 2. Recall that the ranked partition of $\mathcal{K}$ is*

- $\mathcal{D}_0^{\mathcal{K}} = \{B \tilde{\sqsubseteq} F, B \tilde{\sqsubseteq} \exists part.Fe, A \tilde{\sqsubseteq} \neg Sta\}$;
- $\mathcal{D}_1^{\mathcal{K}} = \{P \tilde{\sqsubseteq} \neg F, P \tilde{\sqsubseteq} \exists part.Wf, Sta \tilde{\sqsubseteq} \neg\exists part.Fe, Sta \tilde{\sqsubseteq} \neg F\}$

and that $\mathcal{K}^* = \mathcal{K}$. We show that there is some defeasible inclusion, stuffed penguins presumably have feathers*, that we can derive with LC and not with System W.*

System W. *We run* `Wclosure`$(\mathcal{K}, P \sqcap Sta \tilde{\sqsubseteq} \exists part.Fe)$.

- *γ_1 contains two sets:*
  - $p_1 = \{P \tilde{\sqsubseteq} \neg F, P \tilde{\sqsubseteq} \exists part.Wf, Sta \tilde{\sqsubseteq} \neg F\}$;
  - $p_2 = \{P \tilde{\sqsubseteq} \neg F, Sta \tilde{\sqsubseteq} \neg\exists part.Fe, Sta \tilde{\sqsubseteq} \neg F\}$.
- *γ_0 still contains two sets, t_1 and t_2, that extend p_1 and p_2 in the following way:*
  - $t_1 = p_1 \cup \{B \tilde{\sqsubseteq} \exists part.Fe\}$;
  - *$t_2 = p_2$, since there is no axiom in $\mathcal{D}_0$ that can be added to p_2 without eliminating the possibility of stuffed penguins to exist.*

It is easy to check that $\mathcal{T} \models \bigsqcap \overline{t_1} \sqcap (P \sqcap Sta) \sqsubseteq \exists part.Fe$, but $\mathcal{T} \not\models \bigsqcap \overline{t_2} \sqcap (P \sqcap Sta) \sqsubseteq \exists part.Fe$. Hence, `Wclosure`$(\mathcal{K}, P \sqcap Sta \tilde{\sqsubseteq} \exists part.Fe)$ *returns* `False`.

Lexicographic Closure. *We run* LexClosure$(\mathcal{K}, P \sqcap Sta \mathrel{\sqsubset\!\!\!\sim} \exists part.Fe)$.

- γ_1 *contains, again, the two sets:*
  - $p_1 = \{P \mathrel{\sqsubset\!\!\!\sim} \neg F, P \mathrel{\sqsubset\!\!\!\sim} \exists part.Wf, Sta \mathrel{\sqsubset\!\!\!\sim} \neg F\}$;
  - $p_2 = \{P \mathrel{\sqsubset\!\!\!\sim} \neg F, Sta \mathrel{\sqsubset\!\!\!\sim} \neg\exists part.Fe, Sta \mathrel{\sqsubset\!\!\!\sim} \neg F\}$;

  since they both have the same cardinality of 3.
- *in* γ_0 *we end up instead only with the set* $t_1 = p_1 \cup \{B \mathrel{\sqsubset\!\!\!\sim} \exists part.Fe\}$, *since it contains one inclusion from* $\mathcal{D}_0^{\mathcal{K}}$, *while* t_2 *contains none.*

Now in γ_0 *we have only* t_1, *and, since* $\mathcal{T} \models \bigsqcap \overline{t_1} \sqcap (P \sqcap sta) \sqsubseteq \exists part.Fe$, LexClosure$(\mathcal{K}, (P \sqcap Sta) \mathrel{\sqsubset\!\!\!\sim} \exists part.Fe)$ *returns* True.

An immediate observation that can be done based on this example is that LC is more influenced by the syntax of the knowledge base in the following sense: in RC, conditional KBs that are satisfied by the same set of preferential interpretations also induce the same entailment relation, while in System W and LC, syntactic variation can result in different entailment relations. Consider $\mathcal{K}$ in the above example: by replacing $B \mathrel{\sqsubset\!\!\!\sim} F$ and $B \mathrel{\sqsubset\!\!\!\sim} \exists part.Fe$ by a single DCI $B \mathrel{\sqsubset\!\!\!\sim} F \sqcap \exists part.Fe$, we get an equivalent KB that induces the same entailment relation with respect to RC; but that does not entail $B \sqcap \neg F \mathrel{\sqsubset\!\!\!\sim} \exists part.Fe$ with System W and LC any more. This matches the observation by Casini et al. (2019) that once we look at entailment relations that are stronger than RC the syntax of the knowledge base becomes more relevant. Hence, both System W and LC depend on the syntactic form of the defeasible inclusions. In Example 3 we conclude with LC that *stuffed penguins presumably have feathers*, which seems a bit arbitrary looking at the knowledge base, but results from the form of the DCI and the use of cardinality to decide the preferred subsets of defeasible information to be used. System W is more cautious, and thus seems more justified in its derivations.

6 Related Work

To the best of our knowledge, Quantz and Royer (1992) were the first to consider lifting the preferential approach to defeasibility to a DL setting. As we have mentioned in Sections 4 and 5, the model-theoretic definitions of System W and LC for $\mathcal{DALC}$ rely on the definition of RC for $\mathcal{DALC}$ by Britz et al. (2021) and Giordano et al. (2015). More specifically, our definitions of the Big W-Model and the Big L-Model lean heavily on the Big Ranked Model of Britz et al. (2021).

Regarding System W for $\mathcal{DALC}$, *multipreference-closure* (*MP-closure* for short) is a form of entailment for the description logic with typicality $\mathcal{ALC} + \mathbf{T}_{\mathsf{R}}$ introduced by Giordano et al. (2018; 2020). MP-closure was adapted for reasoning with propositional conditionals by Giordano et al. (2021). While System W and MP-closure were developed independently in different contexts, and defined using distinct approaches, it has been shown that MP-closure for propositional conditionals coincides with System W (Haldimann and Beierle 2022a; Haldimann et al. 2023). We conjecture that MP-closure for $\mathcal{DALC}$ is the same as System W for $\mathcal{DALC}$, but proving whether or not this conjecture holds is left as future work.

For other extensions beyond RC in DLs, there is the work of Casini et al. (2014) on *relevant closure* for $\mathcal{DALC}$, and the work of Pensel and Turhan (2018) to extend both RC and relevant closure across role expressions in the description logic $\mathcal{EL}_{\bot}$. Casini and Straccia (2012) provided an initial proposal for LC for $\mathcal{DALC}$, but did not provide a model-theoretic characterisation or an algorithm for computing LC.

Finally, one of the motivations for investigating defeasible DLs, rather than full first-order logics, is that, being restricted versions of first-order logic, they are more manageable than the full first-order case. In the same spirit, Kern-Isberner and Thimm (2012), Beierle et al. (2017), and Casini et al. (2022) considered variants of RC for different, but related, restricted versions of first-order logic.

7 Conclusion and Future Work

The main contributions of this paper are (i) the model-based definitions of System W and LC for $\mathcal{DALC}$, (ii) the investigation of the basic properties of System W and LC for $\mathcal{DALC}$, and (iii) the presentation of algorithms for computing System W and LC entailments for $\mathcal{DALC}$.

The results presented here suggest a number of options for future work. Firstly, we plan to further evaluate the introduced inference relations with respect to their formal properties, and to evaluate the proposed algorithms with respect to their complexity. Furthermore, the reader familiar with DLs would have noticed that the current paper does not mention DL ABoxes at all. Casini and Straccia (2010) and others have pointed out that the introduction of ABoxes into $\mathcal{DALC}$ can complicate things. This applies to System W and LC for $\mathcal{DALC}$ as well, and a comprehensive investigation of introducing ABoxes into $\mathcal{DALC}$ is still lacking. It has also been pointed out by Pensel and Turhan (2018), amongst others, that defeasible forms of entailment for $\mathcal{DALC}$ (and indeed for most other defeasible description logics) lack a proper treatment of the quantifiers present in these logics. This applies to System W and LC for $\mathcal{DALC}$ as well. In a similar vein, there is a need to consider the impact of role-based defeasible constructors (Britz and Varzinczak 2019) into forms of defeasible entailment such as System W and LC for $\mathcal{DALC}$. Also, Bonatti (2019) has provided a semantic definition of RC that covers defeasible DL that are more expressive than $\mathcal{DALC}$. It would be interesting to see whether this form of RC can be employed to obtain versions of System W and LC for defeasible DLs beyond $\mathcal{DALC}$. Finally, work on the implementation of RC for $\mathcal{DALC}$ has yielded promising results (Casini et al. 2015), while recent works have shown that an implementation of System W for the propositional case can be quite efficient (Beierle, Spang, and Haldimann 2024). An obvious next step would be to see whether the insights gained from these two implementations can be applied to implementations for System W and LC for $\mathcal{DALC}$ as well.

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