

Finite Axiomatizability by Disjunctive Existential Rules

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Abstract

Rule-based languages lie at the core of several areas of central importance to databases and artificial intelligence such as deductive databases and knowledge representation and reasoning. Disjunctive existential rules (a.k.a. disjunctive tuple-generating dependencies in the database literature) form such a prominent rule-based language. The goal of this work is to pinpoint the expressive power of disjunctive existential rules in terms of insightful model-theoretic properties. More precisely, given a collection $\mathcal{C}$ of relational structures, we show that $\mathcal{C}$ is axiomatizable via a finite set Σ of disjunctive existential rules (i.e., $\mathcal{C}$ is precisely the set of models of Σ) iff $\mathcal{C}$ enjoys certain model-theoretic properties. This is achieved by using the well-known property of criticality, a refined version of closure under direct products, and a novel property called diagrammatic compatibility that relies on the method of diagrams. We further establish analogous characterizations for the well-behaved classes of linear and guarded disjunctive existential rules by adopting refined versions of diagrammatic compatibility that consider the syntactic restrictions imposed by linearity and guardedness; this illustrates the robustness of diagrammatic compatibility. We finally exploit diagrammatic compatibility to rewrite a set of guarded disjunctive existential rules into an equivalent set that falls in the weaker class of linear disjunctive existential rules, if one exists.

1 Introduction

Model theory is the study of the interaction between sentences in some logical formalism and their models, that is, structures that satisfy the sentences. There are two directions in this interaction: from syntax to semantics and from semantics to syntax. The first direction aims to identify properties that are satisfied by all models of sentences having common syntactic features, i.e., the syntax of a logical formalism is introduced first and then the properties of the mathematical structures that satisfy sentences of that formalism are explored. The second direction aims to characterize sentences in terms of their model-theoretic properties, i.e., given a set of properties, the goal is to determine whether or not these properties characterize some logical formalism. In general, establishing results in the second direction is much harder than establishing results in the first. In other words, obtaining model-theoretic characterizations of logical formalisms is a far greater challenge than identifying properties enjoyed by all models of sentences from a certain formalism.

In this work, we are concerned with the second direction of the interaction between logical sentences and their models. More precisely, we are interested in model-theoretic results of the following form. For a logical formalism $\mathcal{L}$, given a collection $\mathcal{C}$ of structures, the following are equivalent:

1. There exists a finite set Φ of sentences from $\mathcal{L}$ such that $\mathcal{C}$ is precisely the set of models of Φ , in which case we say that $\mathcal{C}$ is *finitely axiomatizable by $\mathcal{L}$* .¹
2. $\mathcal{C}$ enjoys certain model-theoretic properties.

Such results are welcome as they pinpoint the absolute expressive power of a logical formalism in terms of insightful model-theoretic properties. In other words, they equip us with model-theoretic tools that allow us to prove or disprove that a set of sentences Φ from some logical formalism can be equivalently rewritten as a finite set of sentences from $\mathcal{L}$. Indeed, by showing that the collection of models of Φ enjoy the properties listed in item (2), then Φ is logically equivalent to a finite set Φ' of sentences from $\mathcal{L}$; otherwise, we can safely conclude that such Φ' does not exist.

Two classical results in model theory of the above form are the characterizations of when a collection $\mathcal{C}$ of structures is (finitely) axiomatizable by first-order sentences via the property of closure under isomorphisms, and the more sophisticated closure properties under ultraproducts and ultrapower factors; the definitions of those notions are not essential for our discussion. In particular, we know that a collection $\mathcal{C}$ of structures over a schema (a.k.a. signature) $\mathbf{S}$ is axiomatizable by first-order sentences iff $\mathcal{C}$ is closed under isomorphisms, ultraproducts, and ultrapower factors. Moreover, we know that $\mathcal{C}$ is finitely axiomatizable by first-order sentences iff both $\mathcal{C}$ and the complement of $\mathcal{C}$, which collects all the structures over $\mathbf{S}$ that are not in $\mathcal{C}$, are closed under isomorphisms, ultraproducts, and ultrapower factors.

1.1 Characterizations for Rule-based Formalisms

Interesting characterizations have been also established for rule-based formalisms, expressed in suitable fragments of first-order logic, that have been used in numerous areas of central importance to databases and knowledge representation and reasoning. A prominent such formalism, which has

¹Note that one can also talk about the notion of axiomatizability by $\mathcal{L}$, where the set of sentences Φ from $\mathcal{L}$ might be infinite.

attracted considerable attention during the last decades, is that of existential rules (a.k.a. tuple-generating dependencies in the database literature) that are first-order sentences

$$\forall \bar{x} \forall \bar{y} (\phi(\bar{x}, \bar{y}) \rightarrow \exists \bar{z} \psi(\bar{x}, \bar{z})),$$

where ϕ and ψ are conjunctions of atoms. Existential rules have been originally introduced as a framework for database constraints (Abiteboul, Hull, and Vianu 1995). Later on, they have been deployed in the study of data exchange (Fagin et al. 2005) and data integration (Lenzerini 2002). More recently, they have been used for knowledge representation purposes, in fact, for modeling ontologies intended for data-intensive tasks such as ontology-based data access (Calì et al. 2010; Mugnier and Thomazo 2014).

Early characterizations of axiomatizability and finite axiomatizability by full existential rules (i.e., existential rules without existentially quantified variables) have been established by Makowsky and Vardi in the 1980s (Makowsky and Vardi 1986). Moreover, ten Cate and Kolaitis obtained analogous characterizations for source-to-target existential rules, which have been used to formalize data exchange between a source schema and a target schema (ten Cate and Kolaitis 2009). More recently, Console, Kolaitis and Pieris characterized finite axiomatizability by arbitrary existential rules, as well as existential rules from central guarded-based subclasses, that is, linear, guarded and frontier-guarded existential rules (Console, Kolaitis, and Pieris 2021).

To the best of our knowledge, the above results are the only known results in the literature concerning characterizations of axiomatizability and finite axiomatizability by rule-based formalisms. Having said that, let us stress that several preservation results can be found in the literature. When the target is a preservation result, one considers two formalisms $\mathcal{L}$ and $\mathcal{L}'$, where $\mathcal{L}'$ is typically a proper fragment of $\mathcal{L}$, and the goal is to obtain characterizations of the following form: a set Φ of sentences from $\mathcal{L}$ is logically equivalent to a set of sentences from $\mathcal{L}'$ iff the models of Φ enjoy certain model-theoretic properties. A prototypical example of such a result is the Łos-Tarski Theorem (Chang and Keisler 1992), which states that a set Φ of first-order sentences is equivalent to a set of universal first-order sentences iff the collection of models of Φ is closed under substructures. Such results for rule-based formalisms have been obtained in the context of description logics by Lutz et al. (Lutz, Piro, and Wolter 2011) and of existential rules by Zhang et al. (Zhang, Zhang, and Jiang 2020).

1.2 Disjunctive Existential Rules

The extension of existential rules with the feature of disjunction in the right-hand side of the implication, originally proposed in (Deutsch and Tannen 2003), leads to the central rule-based formalism of disjunctive existential rules that are first-order sentences of the form

$$\forall \bar{x} \forall \bar{y} \left(\phi(\bar{x}, \bar{y}) \rightarrow \bigvee_{i=1}^k \exists \bar{z}_i \psi_i(\bar{x}_i, \bar{z}_i) \right).$$

Disjunctive existential rules have also found numerous applications in several areas such as data exchange (Fagin et

al. 2008), expressive database query languages (Eiter, Gottlob, and Mannila 1997), and knowledge representation and reasoning (Alviano et al. 2012), to name a few.

Despite the characterizations concerning existential rules discussed above, similar characterizations for disjunctive existential rules have remained largely unexplored so far. The main objective of this work is to change this state of affairs by focusing on finite axiomatizability. Our results towards this end can be summarized as follows:

- We establish that a collection of structures $\mathcal{C}$ is finitely axiomatizable by disjunctive existential rules with at most n universally quantified variables, at most m existentially quantified variables, and at most ℓ disjuncts iff $\mathcal{C}$ is critical, closed under repairable direct products, and diagrammatically (n, m, ℓ) -compatible. Criticality is a standard property, which has been used in several works (see, e.g., (Console, Kolaitis, and Pieris 2021)), and states that, for each integer $\kappa > 0$, $\mathcal{C}$ contains a structure I_κ with κ domain elements, and each relation of I_κ contains all the tuples that can be formed using those κ elements. Closure under repairable direct products is a new property obtained by carefully refining the standard property of closure under direct products since it is easy to show that disjunctive existential rules violate closure under direct products. Finally, diagrammatic (n, m, ℓ) -compatibility, which is actually the main innovation of our characterizations, relies on the method of diagrams.
- Diagrammatic (n, m, ℓ) -compatibility turns out to be quite flexible. It can be tailored to other classes of disjunctive existential rules, so that it gives rise to the refined notions of linear-diagrammatic and guarded-diagrammatic (n, m, ℓ) -compatibility. By exploiting these refined properties, we obtain characterizations of finite axiomatizability by linear and guarded disjunctive existential rules.
- Finally, we study the problem of rewriting a finite set of guarded disjunctive existential rules into an equivalent set that falls in the weaker class of linear disjunctive existential rules, whenever one exists. We provide an algorithm for this non-trivial problem, which heavily exploits the notion of linear-diagrammatic compatibility, that generalizes existing rewritings for non-disjunctive guarded existential rules presented in (Console, Kolaitis, and Pieris 2021; Zhang, Zhang, and Jiang 2020).

2 Preliminaries

Let $\mathbf{C}$ and $\mathbf{V}$ be disjoint countably infinite sets of constants and variables, respectively. For an integer $n > 0$, we may write $[n]$ for the set of integers $\{1, \dots, n\}$.

Relational Structures. A *(relational) schema* $\mathbf{S}$ is a finite set of relation symbols (or predicates) with positive arity; we write $\text{ar}(R)$ for the arity of the relation symbol R . A *(relational) structure* I over a schema $\mathbf{S} = \{R_1, \dots, R_n\}$, or *$\mathbf{S}$ -structure*, is a tuple $(\text{dom}(I), R_1^I, \dots, R_n^I)$, where $\text{dom}(I) \subseteq \mathbf{C}$ is a (finite or infinite) domain and $R_1^I, \dots, R_n^I$ are relations over $\text{dom}(I)$, i.e., $R_i^I \subseteq \text{dom}(I)^{\text{ar}(R_i)}$ for each $i \in [n]$. We say that I is a *finite structure* if $\text{dom}(I)$ is finite. A *fact* of I is an expression of the form $R_i(\bar{c})$, where

$\bar{c} \in R_i^I$, and we denote by $\text{facts}(I)$ the set of facts of I . The *active domain* of a structure I , denoted $\text{adom}(I)$, is the set of elements of $\text{dom}(I)$ that occur in at least one fact of I . For an $\mathbf{S}$ -structure $J = (\text{dom}(J), R_1^J, \dots, R_n^J)$, we write $J \subseteq I$ if $\text{facts}(J) \subseteq \text{facts}(I)$. We say that J is a *substructure* of I , denoted $J \preceq I$, if $\text{dom}(J) \subseteq \text{dom}(I)$ and $R^J = R_{|\text{dom}(J)}^I$ for each $R \in \mathbf{S}$ with $R_{|\text{dom}(J)}^I$ being the *restriction of R^I over $\text{dom}(J)$* , i.e., the relation $\{\bar{c} \in R^I \mid \bar{c} \in \text{dom}(J)^{\text{ar}(R)}\}$. Note that $J \preceq I$ implies $J \subseteq I$, but the other direction is not necessarily hold. A *homomorphism* from I to J is a function $h : \text{dom}(I) \rightarrow \text{dom}(J)$ such that, for each $i \in [n]$, $\bar{c} = (c_1, \dots, c_m) \in R_i^I$ implies $h(\bar{c}) = (h(c_1), \dots, h(c_m)) \in R_i^J$. We write $h : I \rightarrow J$ for the fact that h is a homomorphism from I to J . Let $h(\text{facts}(I))$ be the set $\{R(h(\bar{c})) \mid R(\bar{c}) \in \text{facts}(I)\}$. Finally, we say that I and J are *isomorphic*, written $I \simeq J$, if there is a bijective homomorphism from I to J such that $h^{-1} : J \rightarrow I$.

Extended Structures. An *extended schema* is a pair $(\mathbf{S}, \mathbf{F})$, where $\mathbf{S} = \{R_1, \dots, R_n\}$ is a relational schema and $\mathbf{F} = \{f_1, \dots, f_m\}$ is a set of 0-ary function symbols. A *structure* I over $(\mathbf{S}, \mathbf{F})$ is a tuple $(\text{dom}(I), R_1^I, \dots, R_n^I, f_1^I, \dots, f_m^I)$, where $\text{dom}(I) \subseteq \mathbf{C}$, $R_1^I, \dots, R_n^I$ are relations over $\text{dom}(I)$, and $f_1^I, \dots, f_m^I$ are constants of $\text{dom}(I)$. Given a relational schema $\mathbf{S}$ and a finite non-empty set $C = \{c_1, \dots, c_m\}$ of constants from $\mathbf{C}$, we denote by $\mathbf{S}[C]$ the extended schema $(\mathbf{S}, \{f_{c_1}, \dots, f_{c_m}\})$. Furthermore, given an $\mathbf{S}$ -structure $I = (\text{dom}(I), R_1^I, \dots, R_n^I)$, we denote by $I[C]$ the structure over $\mathbf{S}[C]$ defined as $(\text{dom}(I) \cup C, R_1^I, \dots, R_n^I, f_{c_1}^I, \dots, f_{c_m}^I)$, where $f_{c_j}^I = c_j$ for each $j \in [m]$.²

By abuse of notation, we may write down first-order formulas that mention a constant $c \in \mathbf{C}$, which is actually the 0-ary function symbol f_c . For example, $\exists x(R(c, x, d) \wedge \neg(x = d))$, where $c, d \in \mathbf{C}$ and $x \in \mathbf{V}$, is essentially the sentence $\exists x(R(f_c, x, f_d) \wedge \neg(x = f_d))$ over the extended schema $(\{R\}, \{f_c, f_d\})$. Now, by abuse of terminology, given a structure I over a schema $\mathbf{S}$ and a first-order sentence Φ that mentions relation symbols of $\mathbf{S}$ and constants (not necessarily from $\text{dom}(I)$) of $\mathbf{C}$, we may say that I satisfies Φ , denoted $I \models \Phi$. With $C \subseteq \mathbf{C}$ being the set of constants occurring in Φ , $I \models \Phi$ essentially denotes the fact that $I[C] \models \Phi$, that is, the structure $I[C]$ over the extended schema $\mathbf{S}[C]$ satisfies Φ under the standard first-order semantics.

Disjunctive Existential Rules. An atom over a schema $\mathbf{S}$ is an expression of the form $R(\bar{v})$, where $R \in \mathbf{S}$ and $\bar{v}$ is an $\text{ar}(R)$ -tuple of variables from $\mathbf{V}$. A *disjunctive existential rule* (dextr) σ over a schema $\mathbf{S}$ is a constant-free sentence

$$\forall \bar{x} \forall \bar{y} \left(\phi(\bar{x}, \bar{y}) \rightarrow \bigvee_{i=1}^k \exists \bar{z}_i \psi_i(\bar{x}_i, \bar{z}_i) \right),$$

where $k > 0$, $\bar{x}, \bar{y}, \bar{x}_1, \dots, \bar{x}_k, \bar{z}_1, \dots, \bar{z}_k$ are tuples of variables of $\mathbf{V}$, the variables of $\bar{x}_i$ occur in $\bar{x}$ for $i \in [k]$, each variable of $\bar{x}$ occurs in $\bar{x}_i$ for some $i \in [k]$, $\phi(\bar{x}, \bar{y})$ is a (possibly empty) conjunction of atoms over $\mathbf{S}$, and $\psi_i(\bar{x}_i, \bar{z}_i)$ is a

non-empty conjunction of atoms over $\mathbf{S}$ for each $i \in [k]$. For brevity, we write σ as $\phi(\bar{x}, \bar{y}) \rightarrow \bigvee_{i=1}^k \exists \bar{z}_i \psi_i(\bar{x}_i, \bar{z}_i)$ and use comma instead of $\wedge$ for joining atoms. When $\phi(\bar{x}, \bar{y})$ is empty, σ is essentially the sentence $\bigvee_{i=1}^k \exists \bar{z}_i \psi_i(\bar{z}_i)$. We refer to $\phi(\bar{x}, \bar{y})$ and $\bigvee_{i=1}^k \psi_i(\bar{x}_i, \bar{z}_i)$ as the *body* and *head* of σ , denoted $\text{body}(\sigma)$ and $\text{head}(\sigma)$, respectively. By abuse of notation, we may treat a tuple of variables as a set of variables and a conjunction of atoms as a set of atoms. Assuming that $\phi(\bar{x}, \bar{y})$ is non-empty, an $\mathbf{S}$ -structure I satisfies σ if the following holds: whenever there exists a function $h : \bar{x} \cup \bar{y} \rightarrow \text{dom}(I)$ such that $h(\phi(\bar{x}, \bar{y})) \subseteq \text{facts}(I)$ (as usual, we write $h(\phi(\bar{x}, \bar{y}))$ for the set $\{R(h(\bar{v})) \mid R(\bar{v}) \in \phi(\bar{x}, \bar{y})\}$), then there exists an integer $i \in [k]$ and an extension h' of h such that $h'(\psi_i(\bar{x}_i, \bar{z}_i)) \subseteq \text{facts}(I)$. When $\phi(\bar{x}, \bar{y})$ is empty, an $\mathbf{S}$ -structure I satisfies σ if there exists $i \in [k]$ and a function $h : \bar{z}_i \rightarrow \text{dom}(I)$ such that $h(\psi_i(\bar{z}_i)) \subseteq \text{facts}(I)$. We write $I \models \sigma$ for the fact that I satisfies σ . The $\mathbf{S}$ -structure I satisfies a set Σ of dexrs over $\mathbf{S}$, written $I \models \Sigma$, in which case we say that I is a *model* of Σ , if $I \models \sigma$ for each $\sigma \in \Sigma$.

Finite Axiomatizability. Let $\mathcal{C}$ be a collection of structures over a schema $\mathbf{S}$. We say that $\mathcal{C}$ is *finitely axiomatizable* by dexrs if there exists a finite set Σ of dexrs over $\mathbf{S}$ such that $I \in \mathcal{C}$ iff $I \models \Sigma$, that is, $\mathcal{C}$ is precisely the set of models of Σ . Henceforth, since dexrs cannot distinguish isomorphic structures, we implicitly assume that collections $\mathcal{C}$ of structures are *closed under isomorphisms*, i.e., if $I \in \mathcal{C}$ and J is a structure such that $I \simeq J$, then $J \in \mathcal{C}$.

3 Model-theoretic Properties

We now introduce three model-theoretic properties of collections of structures that will play a crucial role in our characterizations. Fix an arbitrary schema $\mathbf{S} = \{R_1, \dots, R_r\}$.

3.1 Criticality

An $\mathbf{S}$ -structure $I = (\text{dom}(I), R_1^I, \dots, R_r^I)$ is κ -critical, for $\kappa > 0$, if $|\text{dom}(I)| = \kappa$ and $R_i^I = \text{dom}(I)^{\text{ar}(R_i)}$ for each $i \in [r]$. We can now lift criticality to collections of structures.

Definition 1. A collection $\mathcal{C}$ of $\mathbf{S}$ -structures is κ -critical, for $\kappa > 0$, if it contains a κ -critical $\mathbf{S}$ -structure. We further say that $\mathcal{C}$ is critical if it is κ -critical for each $\kappa > 0$. ■

It is easy to show the following lemma:

Lemma 2. A collection $\mathcal{C}$ of structures that is finitely axiomatizable by dexrs is critical.

3.2 Closure Under Repairable Direct Products

Consider two $\mathbf{S}$ -structures $I = (\text{dom}(I), R_1^I, \dots, R_r^I)$ and $J = (\text{dom}(J), R_1^J, \dots, R_r^J)$. Their *direct product*, denoted $I \otimes J$, is the $\mathbf{S}$ -structure $K = (\text{dom}(K), R_1^K, \dots, R_r^K)$, where $\text{dom}(K) = \text{dom}(I) \times \text{dom}(J)$ and

$$R_i^K = \{((a_1, b_1), \dots, (a_{\text{ar}(R_i)}, b_{\text{ar}(R_i)})) \mid (a_1, \dots, a_{\text{ar}(R_i)}) \in R_i^I \text{ and } (b_1, \dots, b_{\text{ar}(R_i)}) \in R_i^J\},$$

for each $i \in [r]$. A collection of structures $\mathcal{C}$ is closed under direct products if, for every $I, J \in \mathcal{C}$, the direct product of I and J belongs to $\mathcal{C}$. Closure under direct products

²The additional notions defined above for relational structures (such as subset, substructure, homomorphism, and isomorphism) are not needed in the paper for structures over extended schemas.

is a well-known property that has been extensively used in model theory. However, it is not appropriate towards a characterization of finite axiomatizability by dexrs.

Example 1. Consider the set Σ consisting of the single dextr

$$R(x) \rightarrow S(x) \vee T(x)$$

and the structures I_1 and I_2 with $\text{facts}(I_1) = \{R(a), S(a)\}$ and $\text{facts}(I_2) = \{R(a), T(a)\}$, and thus, $\text{facts}(I_1 \otimes I_2) = \{R(aa)\}$. Clearly, $I_1 \models \Sigma$ and $I_2 \models \Sigma$, but $I_1 \otimes I_2 \not\models \Sigma$. ■

The above example essentially tells us that a collection of structures that is finitely axiomatizable by dexrs is not necessarily closed under direct products, and thus, we need to revisit the standard notion of direct product. To this end, we define the function $\pi_K : \text{dom}(K) \rightarrow \text{dom}(I)$ as follows: for every $(a_I, a_J) \in \text{dom}(K)$, $\pi_K((a_I, a_J)) = a_I$. It is clear that π_K is a homomorphism from K to I , which we call the *projective homomorphism* for K . Now, a *repairable direct product* of the $\mathbf{S}$ -structures I and J is an $\mathbf{S}$ -structure L such that (1) $I \otimes J \subseteq L$, and (2) there exists an extension h of π_K that is a homomorphism from L to I . We now define the following novel model-theoretic property:

Definition 3. A collection $\mathcal{C}$ of $\mathbf{S}$ -structures is closed under repairable direct products if, for every two $\mathbf{S}$ -structures $I, J \in \mathcal{C}$, there exists an $\mathbf{S}$ -structure in $\mathcal{C}$ that is a repairable direct product of I and J . ■

Coming back to Example 1, it is easy to verify that the structure L with $\text{facts}(L) = \{R(aa), S(aa)\}$ is a repairable direct product of I_1 and I_2 ; indeed, $I_1 \otimes I_2 \subseteq L$ and $\pi_{I_1 \otimes I_2}$ is a homomorphism from L to I_1 . We can show the following:

Lemma 4. A collection $\mathcal{C}$ of structures that is finitely axiomatizable by dexrs is closed under repairable direct products.

3.3 Diagrammatic Compatibility

We now introduce our new property of collections of structures, which in turn relies on the notion of relative diagram of a finite structure. We first introduce relative diagrams and then define the new model-theoretic property of interest.

Relative Diagrams. Consider a structure I over a schema $\mathbf{S}$ and a finite structure $K \subseteq I$ with $\text{dom}(K) = \text{adom}(K)$. We are interested in the so-called (m, ℓ) -diagrams of K relative to I for integers $m, \ell \geq 0$, which we define below. To this end, let $A_{K,m}$ be the set of all atomic formulas of the form $R(\bar{u})$ that can be formed using predicates from $\mathbf{S}$, constants from $\text{dom}(K)$, and m distinct variables $y_1, \dots, y_m$ from $\mathbf{V}$, i.e., $R \in \mathbf{S}$ and $\bar{u} \in (\text{dom}(K) \cup \{y_1, \dots, y_m\})^{\text{ar}(R)}$. Furthermore, let $C_{K,m}$ be the set of all conjunctions of atomic formulas from $A_{K,m}$. Observe that both $A_{K,m}$ and $C_{K,m}$ are finite sets since $\text{dom}(K)$ is finite. Let

$$N_{K,m}^I = \{\gamma(\bar{y}) \in C_{K,m} \mid I \not\models \exists \bar{y} \gamma(\bar{y})\},$$

which is clearly finite since $C_{K,m}$ is finite. For a set of formulas $G \subseteq N_{K,m}^I$, the G -diagram of K relative to I , denoted $\Delta_{K,G}^I$, is defined as the first-order sentence

$$\underbrace{\bigwedge_{\alpha \in \text{facts}(K)} \alpha}_{\Psi_1} \wedge \underbrace{\bigwedge_{\substack{c,d \in \text{dom}(K), \\ c \neq d}} \neg(c=d) \wedge \bigwedge_{\gamma(\bar{y}) \in G} \neg(\exists \bar{y} \gamma(\bar{y}))}_{\Psi_2}.$$

Note that Ψ_1 and Ψ_2 might be empty (i.e., they have no conjuncts). In particular, Ψ_1 is empty if K is empty, whereas Ψ_2 is empty if I is 1-critical, which means that $|\text{dom}(K)| = 1$ and $I \models \exists \bar{y} \gamma(\bar{y})$ for each $\gamma(\bar{y}) \in C_{K,m}$, and thus, $G = \emptyset$. When Ψ_1 and Ψ_2 are empty, $\Delta_{K,G}^I$ is the truth value true, i.e., a tautology. Intuitively, $\Delta_{K,G}^I$ provides a witness for the fact that a dextr, whose head-disjuncts belong to G , is violated by the structure I due to the structure $K \subseteq I$.

Definition 5. Consider an $\mathbf{S}$ -structure I and a finite structure $K \subseteq I$ with $\text{dom}(K) = \text{adom}(K)$. An (m, ℓ) -diagram of K relative to I , for $m, \ell \geq 0$, is a G -diagram of K relative to I , where $G \subseteq N_{K,m}^I$ and $|G| \leq \ell$. ■

Let $\Phi_{K,G}^I(\bar{x})$ be the formula obtained from $\Delta_{K,G}^I$ by replacing each constant $c \in \text{dom}(K)$ with a new variable $x_c \in \mathbf{V} \setminus \{y_1, \dots, y_m\}$. If $\Delta_{K,G}^I$ is empty, then $\Phi_{K,G}^I$ is the truth value true, i.e., a tautology. It is easy to verify the following lemma, which will be used in our proofs:

Lemma 6. It holds that $I \models \exists \bar{x} \Phi_{K,G}^I(\bar{x})$.

The Property. Consider a collection $\mathcal{C}$ of structures over a schema $\mathbf{S}$ and an $\mathbf{S}$ -structure I . For $n, m, \ell \geq 0$, we say that $\mathcal{C}$ is *diagrammatically (n, m, ℓ) -compatible* with I if, for every $K \preceq I$ with $\text{dom}(K) = \text{adom}(K)$ and $|\text{dom}(K)| \leq n$, and every (m, ℓ) -diagram $\Delta_{K,G}^I$ of K relative to I , where $G \subseteq N_{K,m}^I$, there is $J \in \mathcal{C}$ such that $J \models \Delta_{K,G}^I$. Roughly, this states that no matter how a dextr σ with a bounded number of variables (at most n universally quantified and at most m existentially quantified variables) and a bounded number of disjuncts in its head (at most ℓ disjuncts) is violated by the structure I , there is always a structure $J \in \mathcal{C}$ that violates σ for the same reason. The new property of interest follows.

Definition 7. A collection $\mathcal{C}$ of $\mathbf{S}$ -structures is diagrammatically (n, m, ℓ) -compatible, for $n, m, \ell \geq 0$, if, for every $\mathbf{S}$ -structure I , the following holds: $\mathcal{C}$ is diagrammatically (n, m, ℓ) -compatible with I implies $I \in \mathcal{C}$. ■

The next lemma establishes that collections of structures that are finitely axiomatizable by dexrs are diagrammatically compatible. Actually, it shows a stronger claim since it relates the integers n, m and ℓ that witness diagrammatic (n, m, ℓ) -compatibility with the number of universally quantified variables, existentially quantified variables, and head-disjuncts, respectively, occurring in the dexrs. A dextr is called (n, m, ℓ) -dextr, for $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$, if it mentions at most n universally quantified variables in its body, at most m existentially quantified variables in each disjunct of its head, and at most ℓ disjuncts in its head. Note that we require $n + m > 0$ since, by definition, a dextr has at least one variable that is either universally or existentially quantified. Furthermore, we require $\ell > 0$ since, by definition, a dextr has at least one disjunct in its head.

Lemma 8. For $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$, every collection $\mathcal{C}$ of structures that is finitely axiomatizable by (n, m, ℓ) -dexrs is diagrammatically (n, m, ℓ) -compatible.

Proof. Let Σ be a finite set of (n, m, ℓ) -dexrs with $I \in \mathcal{C}$ iff $I \models \Sigma$. Consider a structure I and assume that $\mathcal{C}$ is diagrammatically (n, m, ℓ) -compatible with I . We proceed to show

that $I \in \mathcal{C}$, or, equivalently, $I \models \Sigma$. Consider a dextr $\sigma \in \Sigma$ of the form $\phi(\bar{x}, \bar{y}) \rightarrow \bigvee_{i=1}^k \exists \bar{z}_i \psi_i(\bar{x}_i, \bar{z}_i)$. We assume that $\phi(\bar{x}, \bar{y})$ is non-empty; the case where $\phi(\bar{x}, \bar{y})$ is empty is treated similarly. Assume that there exists a function $h : \bar{x} \cup \bar{y} \rightarrow \text{dom}(I)$ such that $h(\phi(\bar{x}, \bar{y})) \subseteq \text{facts}(I)$. We need to show that there exists $i \in [k]$ such that $\lambda(\psi_i(\bar{x}_i, \bar{z}_i)) \subseteq \text{facts}(I)$ with λ being an extension of h . By contradiction, assume that such $i \in [k]$ does not exist. This means that, for every $i \in [k]$, $I \models \neg \exists \bar{z}_i \psi'_i(\bar{z}_i)$, where $\psi'_i(\bar{z}_i)$ is obtained from $\psi_i(\bar{x}_i, \bar{z}_i)$ by replacing each variable x of $\bar{x}_i$ with $h(x)$. Let K be the structure $(\text{dom}(K), R_1^K, \dots, R_r^K)$, where $\text{dom}(K)$ is the set of constants occurring in $h(\phi(\bar{x}, \bar{y}))$, and, for each $i \in [r]$, $R_i^K = R_i^I|_K$. It is clear that $K \preceq I$ with $\text{dom}(K) = \text{adom}(K)$ and $|\text{adom}(K)| \leq n$ since $\phi(\bar{x}, \bar{y})$ mentions at most n variables. Let $G = \{\psi'_i(\bar{z}_i)\}_{i \in [k]}$. Since, for each $i \in [k]$, $\psi_i(\bar{x}_i, \bar{z}_i)$ mentions at most m existentially quantified variables and $k \leq \ell$, it is easy to see that $G \subseteq N_{K,m}^I$ and $|G| \leq \ell$. Therefore, $\Delta_{K,G}^I$, that is, the G -diagram of K relative to I is an (m, ℓ) -diagram of K relative to I . Since $\mathcal{C}$ is diagrammatically (n, m, ℓ) -compatible with I , we get that there exists $J \in \mathcal{C}$ that satisfies $\Delta_{K,G}^I$. The latter implies that $h(\phi(\bar{x}, \bar{y})) \subseteq \text{facts}(J)$, but there is no extension λ of h such that $\lambda(\psi_i(\bar{x}_i, \bar{z}_i)) \subseteq \text{facts}(J)$ for some $i \in [k]$. Consequently, $J \not\models \sigma$, and thus, $J \not\models \Sigma$. But this contradicts the fact that $J \in \mathcal{C}$, which is equivalent to say that $J \models \Sigma$, and the claim follows. $\square$

4 Finite Axiomatizability by Disjunctive Existential Rules

We proceed to characterize when a collection $\mathcal{C}$ of structures is finitely axiomatizable by dexrs. More precisely, the goal of this section is to establish the following result:

Theorem 9. *For a collection $\mathcal{C}$ of structures and $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$, the following are equivalent:*

1. $\mathcal{C}$ is finitely axiomatizable by (n, m, ℓ) -dexrs.
2. $\mathcal{C}$ is critical, closed under repairable direct products, and diagrammatically (n, m, ℓ) -compatible.

It is clear that the direction (1) $\Rightarrow$ (2) immediately follows from Lemmas 2, 4 and 8. The rest of this section is devoted to discussing the proof of (2) $\Rightarrow$ (1). To this end, we need to introduce disjunctive dependencies.

Disjunctive Dependencies. A *disjunctive dependency* (dd) δ over a schema $\mathbf{S}$ is a constant-free sentence of the form

$$\forall \bar{x} \left(\phi(\bar{x}) \rightarrow \bigvee_{i=1}^k \psi_i(\bar{x}_i) \right),$$

where $k \geq 0$, $\bar{x}$ is a (possibly empty) tuple of variables of $\mathbf{V}$, the expression $\phi(\bar{x})$ is a (possibly empty) conjunction of atoms over $\mathbf{S}$, and, assuming $k > 0$, for each $i \in [k]$, $\bar{x}_i \subseteq \bar{x}$ and the expression $\psi_i(\bar{x}_i)$ is either an equality formula $y = z$ with $\bar{x}_i = \{y, z\}$, or a formula $\exists \bar{y}_i \chi_i(\bar{x}_i, \bar{y}_i)$ with $\bar{y}_i$ being a tuple of variables from $\mathbf{V} \setminus \bar{x}$ and $\chi_i(\bar{x}_i, \bar{y}_i)$ a (non-empty) conjunction of atoms over $\mathbf{S}$. When $k = 0$, there are no disjuncts in the conclusion of δ . In this case, if

$\phi(\bar{x})$ is empty, then δ is interpreted as the truth value false, i.e., a contradiction; otherwise, δ is essentially the sentence $\forall \bar{x} (\phi(\bar{x}) \rightarrow \text{false}) \equiv \neg(\exists \bar{x} \phi(\bar{x}))$. Now, when $k > 0$ and $\phi(\bar{x})$ is empty, δ is essentially the sentence $\bigvee_{i=1}^k \psi_i$. If $k > 0$ and, for each $i \in [k]$, $\psi_i(\bar{x}_i)$ is an equality formula, then δ is called a *disjunctive equality rule* (deqr).

Assuming that $k = 0$ and $\phi(\bar{x})$ is non-empty, an $\mathbf{S}$ -structure I satisfies δ if there is no function $h : \bar{x} \rightarrow \text{dom}(I)$ such that $h(\phi(\bar{x})) \subseteq \text{facts}(I)$. Assume now that $k > 0$. If $\phi(\bar{x})$ is non-empty, then δ is satisfied by an $\mathbf{S}$ -structure I if, whenever there exists a function $h : \bar{x} \rightarrow \text{dom}(I)$ such that $h(\phi(\bar{x})) \subseteq \text{facts}(I)$, then there is $i \in [k]$ such that, if $\psi_i(\bar{y}_i)$ is $y = z$, then $h(y) = h(z)$; otherwise, if $\psi_i(\bar{x}_i)$ is $\exists \bar{y}_i \chi_i(\bar{x}_i, \bar{y}_i)$, then there is an extension h' of h such that $h'(\chi_i(\bar{x}_i, \bar{y}_i)) \subseteq \text{facts}(I)$. In case $\phi(\bar{x})$ is empty, then $\delta = \bigvee_{i \in [k]} \exists \bar{y}_i \chi_i(\bar{y}_i)$ is satisfied by I if there is $i \in [k]$ and a function $h : \bar{y}_i \rightarrow \text{dom}(I)$ such that $h(\chi_i(\bar{y}_i)) \subseteq \text{facts}(I)$. We write $I \models \delta$ for the fact that I satisfies δ . The structure I satisfies a set Σ of dds, written $I \models \Sigma$, in which case we say that I is a *model* of Σ , if $I \models \delta$ for each $\delta \in \Sigma$.

For a schema $\mathbf{S}$, let $\text{DD}_{n,m,\ell}^{\mathbf{S}}$, for integers $n, m, \ell \geq 0$, be the set of dds over $\mathbf{S}$ of the form

$$\forall \bar{x} \left(\phi(\bar{x}) \rightarrow \bigvee_{i=1}^k \psi_i(\bar{x}_i) \right),$$

where $k \geq 0$, such that

1. $\bar{x}$ consists of at most n distinct variables,
2. if $k \geq 1$, then, for each $i \in [k]$, if $\psi_i(\bar{x}_i)$ is a formula of the form $\exists \bar{y}_i \chi_i(\bar{x}_i, \bar{y}_i)$ with $\chi_i(\bar{x}_i, \bar{y}_i)$ being a non-empty conjunction of atoms, then $\bar{y}_i$ consists of at most m distinct variables, and
3. if $k \geq 1$, then it holds that

$$|\{i \in [k] \mid \psi_i(\bar{x}_i) \text{ is not an equality formula}\}| \leq \ell,$$

i.e., at most ℓ disjuncts are a non-empty conjunction of atoms (i.e., not an equality formula).

Note that $\text{DD}_{n,m,\ell}^{\mathbf{S}}$ is a finite set (up to variable renaming) since $\mathbf{S}$ is finite, and the number of variables and number of disjuncts in each element of $\text{DD}_{n,m,\ell}^{\mathbf{S}}$ is finite.

4.1 Proving the Direction (2) $\Rightarrow$ (1)

We now have all the ingredients needed for discussing the proof of the direction (2) $\Rightarrow$ (1) of Theorem 9. Consider a collection $\mathcal{C}$ of structures over a schema $\mathbf{S}$ that is critical, closed under repairable direct products, and diagrammatically (n, m, ℓ) -compatible for integers $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$. We proceed to show that $\mathcal{C}$ is finitely axiomatizable by (n, m, ℓ) -dexrs in three steps:

1. We first define a finite set $\Sigma^\vee$ of dds from $\text{DD}_{n,m,\ell}^{\mathbf{S}}$ such that, for every $\mathbf{S}$ -structure I , $I \in \mathcal{C}$ iff $I \models \Sigma^\vee$. This exploits the fact that $\mathcal{C}$ is 1-critical (since it is critical) and diagrammatically (n, m, ℓ) -compatible.
2. We then show that there is a finite set $\Sigma^{\exists,=}$ of (n, m, ℓ) -dexrs and n -deqrs (i.e., deqrs with at most n universally quantified variables with the corner case of 0-deqr being

the value true) over $\mathbf{S}$ such that $\Sigma^\vee \equiv \Sigma^{\exists,=}$; in fact, $\Sigma^{\exists,=}$ is the set of dexrs and deqrs occurring in $\Sigma^\vee$. This exploits the fact that $\mathcal{C}$ is closed under repairable direct products.

3. We finally argue that $\Sigma^{\exists,=}$ consists only of dexrs, which implies that $\mathcal{C}$ is finitely axiomatizable by (n, m, ℓ) -dexrs. This exploits the fact that $\mathcal{C}$ is critical.

We proceed to give further details for the above three steps.

Step 1: The finite set $\Sigma^\vee$ of dds

Let $\Sigma^\vee$ be the set of all dds from $\text{DD}_{n,m,\ell}^{\mathbf{S}}$ that are satisfied by every structure of $\mathcal{C}$, that is,

$$\Sigma^\vee = \left\{ \delta \in \text{DD}_{n,m,\ell}^{\mathbf{S}} \mid \text{for each } I \in \mathcal{C}, \text{ we have } I \models \delta \right\}.$$

Clearly, the set $\Sigma^\vee$ is finite (up to variable renaming) since $\Sigma^\vee \subseteq \text{DD}_{n,m,\ell}^{\mathbf{S}}$. We proceed to show that $\mathcal{C}$ is precisely the set of $\mathbf{S}$ -structures that satisfy $\Sigma^\vee$.

Lemma 10. *For every $\mathbf{S}$ -structure I , $I \in \mathcal{C}$ iff $I \models \Sigma^\vee$.*

Proof. The $(\Rightarrow)$ direction holds by construction. We proceed with the non-trivial direction $(\Leftarrow)$. Consider an $\mathbf{S}$ -structure I such that $I \models \Sigma^\vee$. We are going to show that $\mathcal{C}$ is diagrammatically (n, m, ℓ) -compatible with I , which in turn implies that $I \in \mathcal{C}$ since $\mathcal{C}$ is diagrammatically (n, m, ℓ) -compatible. Consider an arbitrary substructure K of I with $\text{dom}(K) = \text{adom}(K)$ and $|\text{dom}(K)| \leq n$, and an arbitrary (m, ℓ) -diagram $\Delta_{K,G}^I$ of K relative to I , where $G \subseteq N_{K,m}^I$. We need to show that there exists $J \in \mathcal{C}$ that satisfies $\Delta_{K,G}^I$. To this end, we first establish the following auxiliary claim:

Claim 11. *There is $\delta \in \text{DD}_{n,m,\ell}^{\mathbf{S}}$ with $\delta \equiv \neg \exists \bar{x} \Phi_{K,G}^I(\bar{x})$.*

Let $\delta \in \text{DD}_{n,m,\ell}^{\mathbf{S}}$ be the dd provided by Claim 11 such that $\delta \equiv \neg \exists \bar{x} \Phi_{K,G}^I(\bar{x})$. We claim that $\delta \notin \Sigma^\vee$. By contradiction, assume that $\delta \in \Sigma^\vee$. This implies that $I \models \delta$, which cannot be the case since, by Lemma 6, $I \models \exists \bar{x} \Phi_{K,G}^I(\bar{x})$. The fact that $\delta \notin \Sigma^\vee$ implies that there exists an $\mathbf{S}$ -structure $L \in \mathcal{C}$ such that $L \not\models \delta$, which means that $L \models \exists \bar{x} \Phi_{K,G}^I(\bar{x})$. Therefore, there is an $\mathbf{S}$ -structure J such that $J \simeq L$ and $J \models \Delta_{K,G}^I$. Since $\mathcal{C}$ is closed under isomorphisms, we can conclude that $J \in \mathcal{C}$, and the claim follows. $\square$

Step 2: The finite set $\Sigma^{\exists,=}$ of dexrs and deqrs

It is clear that $\Sigma^\vee \models \Sigma^{\exists,=}$, that is, each model of $\Sigma^\vee$ is a model of $\Sigma^{\exists,=}$, since $\Sigma^{\exists,=} \subseteq \Sigma^\vee$. It remains to show that $\Sigma^{\exists,=} \models \Sigma^\vee$. By contradiction, assume that $\Sigma^{\exists,=} \not\models \Sigma^\vee$. This implies that there exists a dd $\delta \in \Sigma^\vee$ such that $\Sigma^{\exists,=} \not\models \delta$. Clearly, δ is neither a dextr nor a deqr. Thus, δ can be written as a sentence of the form

$$\forall \bar{x} \left(\phi(\bar{x}) \rightarrow \bigvee_{i=1}^{k_1} (z_i = w_i) \vee \bigvee_{i=1}^{k_2} \exists \bar{y}_i \chi_i(\bar{x}_i, \bar{y}_i) \right),$$

where $k_1, k_2 \geq 1$. Let δ_- be the deqr

$$\forall \bar{x} \left(\phi(\bar{x}) \rightarrow \bigvee_{i=1}^{k_1} (z_i = w_i) \right).$$

Since $\Sigma^{\exists,=} \not\models \delta$, we can conclude that $\Sigma^\vee \not\models \delta_-$; otherwise, $\delta_- \in \Sigma^{\exists,=}$ which cannot be the case. Therefore, we get that there exists an $\mathbf{S}$ -structure $I_- \in \mathcal{C}$ such that $I_- \not\models \delta_-$. Analogously, with δ_+ being the dextr

$$\forall \bar{x} \left(\phi(\bar{x}) \rightarrow \bigvee_{i=1}^{k_2} \exists \bar{y}_i \chi_i(\bar{x}_i, \bar{y}_i) \right),$$

we can show that there exists an $\mathbf{S}$ -structure $I_+ \in \mathcal{C}$ such that $I_+ \not\models \delta_+$. We now proceed, by exploiting the structures I_- and I_+ , to show that there is an $\mathbf{S}$ -structure J such that $J \in \mathcal{C}$ and $J \not\models \delta$, which leads to a contradiction. Since $\mathcal{C}$ is closed under repairable direct products, we get that there exists an $\mathbf{S}$ -structure $J \in \mathcal{C}$ that is a repairable direct product of I_+ and I_- , which means that (i) $I_+ \otimes I_- \subseteq J$, and (ii) there exists an extension $h_{I_+ \otimes I_-}$ of $\pi_{I_+ \otimes I_-}$ that is a homomorphism from J to I_+ . It remains to show that $J \not\models \delta$.

Since $I_- \not\models \delta_-$, we get that there exists a function $h_- : \bar{x} \rightarrow \text{dom}(I_-)$ such that $h_-(\phi(\bar{x})) \subseteq \text{facts}(I_-)$ and $h_-(z_i) \neq h_-(w_i)$, for each $i \in [k_1]$. Similarly, since $I_+ \not\models \delta_+$, we get that there exists a function $h_+ : \bar{x} \rightarrow \text{dom}(I_+)$ such that $h_+(\phi(\bar{x})) \subseteq \text{facts}(I_+)$ and there is no extension h'_+ of h_+ such that $h'_+(\chi_i(\bar{x}_i, \bar{y}_i)) \subseteq \text{facts}(I_+)$, for each $i \in [k_2]$. Let $L_\phi^\pm$ be the structure with $\text{dom}(L_\phi^\pm) = h_\pm(\bar{x})$ and $\text{facts}(L_\phi^\pm) = h_\pm(\phi(\bar{x}))$. Analogously, let $L_\chi^\pm$ be the structure with $\text{dom}(L_\chi^\pm) = h_\pm(\bar{x})$ and $\text{facts}(L_\chi^\pm) = h_\pm(\chi_i(\bar{x}_i, \bar{y}_i))$. It is easy to verify that there exists a function $h : \bar{x} \rightarrow \text{dom}(L_\phi^\pm \otimes L_\chi^\pm)$ such that $h(\phi(\bar{x})) \subseteq \text{facts}(L_\phi^\pm \otimes L_\chi^\pm)$; in particular, for each $x \in \bar{x}$, $h(x) = (h_+(x), h_-(x))$. We show that h witnesses the fact that $J \not\models \delta$.

Clearly, $L_\phi^\pm \otimes L_\chi^\pm \subseteq I_+ \otimes I_- \subseteq J$, which implies that $h(\phi(\bar{x})) \subseteq \text{facts}(J)$. It remains to show that (i) for each $i \in [k_1]$, $h(z_i) \neq h(w_i)$, and (ii) for each $i \in [k_2]$, there is no extension h' of h such that $h'(\chi_i(\bar{x}_i, \bar{y}_i)) \subseteq \text{facts}(J)$. Concerning item (i), we observe that $h_-(z_i) \neq h_-(w_i)$ since h_- witnesses the fact that $I_- \not\models \delta_-$. Therefore, by definition of h , we get that $h(z_i) \neq h(w_i)$, as needed. Concerning item (2), by contradiction, assume that there exists $i \in [k_2]$ and an extension h' of h such that $h'(\chi_i(\bar{x}_i, \bar{y}_i)) \subseteq \text{facts}(J)$. Let $h'' = h_{I_+ \otimes I_-} \circ h'$. By definition of h , we get that $h''(x) = h_+(x)$ for each $x \in \bar{x}$. Moreover, by composition, we get that $h''(\chi_i(\bar{x}_i, \bar{y}_i)) \subseteq \text{facts}(I_+)$. The latter implies that h'' is an extension of h_+ such that $h''(\chi_i(\bar{x}_i, \bar{y}_i)) \subseteq \text{facts}(I_+)$, which contradicts the fact that h_+ witnesses $I_+ \not\models \delta_+$.

Step 3: The set $\Sigma^{\exists,=}$ consists only of dexrs

By contradiction, assume that a deqr δ of the form

$$\forall \bar{x} \left(\phi(\bar{x}) \rightarrow \bigvee_{i=1}^k (z_i = w_i) \right),$$

where $k \geq 1$, occurs in $\Sigma^{\exists,=}$. Since $\mathcal{C}$ is critical, there exists a $|\bar{x}|$ -critical structure $I \in \mathcal{C}$, where $|\bar{x}|$ is the number of distinct variables in $\bar{x}$, and a function $h : \bar{x} \rightarrow \text{dom}(I)$ such that $h(\phi(\bar{x})) \subseteq \text{facts}(I)$ and $h(z_i) \neq h(w_i)$ for each $i \in [k]$. This implies that $I \not\models \delta$, which contradicts the fact that every dd of $\Sigma^{\exists,=}$ is satisfied by every structure of $\mathcal{C}$.

4.2 Diagrammatic Compatibility vs Locality

As discussed in Section 1, a characterization similar to Theorem 9 for existential rules (i.e., the special case of disjunctive existential rules where the head consists of exactly one disjunct) has been established in (Console, Kolaitis, and Pieris 2021) by exploiting a model-theoretic property called locality; for our discussion, the formal definition of locality is not crucial and is omitted. More precisely, it was shown that, for a collection $\mathcal{C}$ of structures and $n, m \geq 0$, with $n + m > 0$, the following are equivalent:

1. $\mathcal{C}$ is finitely axiomatizable by existential rules with at most n universally and m existentially quantified variables.
2. $\mathcal{C}$ is critical, closed under direct products, and (n, m) -local.

The question that comes up is whether we could use the notion of locality, instead of introducing the new property of diagrammatic compatibility, for achieving the characterization for dexrs stated in Theorem 9. It turns out, as shown below, that this is not the case, which justifies the introduction of diagrammatic compatibility that can be understood as a refined notion of locality that explicitly takes into account the number of disjuncts in the head of a rule.

Proposition 12. *Consider a collection $\mathcal{C}$ of structures. For every $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$, $\mathcal{C}$ is diagrammatically (n, m, ℓ) -compatible implies $\mathcal{C}$ is (n, m) -local.*

The above proposition implies that every collection of structures that is finitely axiomatizable by (n, m, ℓ) -dexrs is (n, m) -local. This means that (n, m) -locality is not powerful enough to distinguish between classes of structures $\mathcal{C}$ and $\mathcal{C}'$ such that $\mathcal{C}$ is finitely axiomatizable by (n, m, ℓ) -dexrs and $\mathcal{C}'$ is finitely axiomatizable by (n, m, ℓ') -dexrs, for $\ell \neq \ell'$. To further illustrate this fact, let us consider again the set Σ of dexrs from Example 1 consisting of the dextr $R(x) \rightarrow S(x) \vee T(x)$ and let $\mathcal{C}_\Sigma$ be the set of models of Σ . By definition, $\mathcal{C}_\Sigma$ is finitely axiomatizable by $(1, 0, 2)$ -dexrs. Therefore, $\mathcal{C}_\Sigma$ is diagrammatically $(1, 0, 2)$ -compatible (by Theorem 9) and $(1, 0)$ -local (by Proposition 12). Moreover, by Proposition 12, we get that every collection of structures that is finitely axiomatizable by $(1, 0, 1)$ -dexrs is also $(1, 0)$ -local. However, by exploiting our characterization, we can show that $\mathcal{C}_\Sigma$ is not finitely axiomatizable by $(1, 0, 1)$ -dexrs.

5 Finite Axiomatizability by Guarded-based Disjunctive Existential Rules

The goal here is to establish a result analogous to Theorem 9 for the two main members of the guarded family of dexrs:

- A dextr is *linear* if it has at most one atom in its body.
- A dextr is *guarded* if its body is empty or has an atom that mentions all the universally quantified variables.

Interestingly, to achieve the desired characterizations for the above classes of dexrs, we simply need to replace the diagrammatic compatibility property in Theorem 9 with a refined version of it that takes into account the syntactic property underlying linear and guarded dexrs. We start our analysis with linear dexrs, and then proceed with guarded dexrs.

5.1 Linear Disjunctive Existential Rules

A structure J is called *linear* if $|\text{facts}(J)| \leq 1$. Consider a collection $\mathcal{C}$ of $\mathbf{S}$ -structures and an $\mathbf{S}$ -structure I . For integers $n, m, \ell \geq 0$, we say that $\mathcal{C}$ is *linear-diagrammatically (n, m, ℓ) -compatible* with I if, for every linear structure $K \subseteq I$ with $\text{dom}(K) = \text{adom}(K)$ and $|\text{dom}(K)| \leq n$, and every (m, ℓ) -diagram $\Delta_{K,G}^I$ of K relative to I , where $G \subseteq N_{K,m}^I$, there exists $J \in \mathcal{C}$ such that $J \models \Delta_{K,G}^I$.

Definition 13. *A collection $\mathcal{C}$ of $\mathbf{S}$ -structures is linear-diagrammatically (n, m, ℓ) -compatible, for $n, m, \ell \geq 0$, if, for every $\mathbf{S}$ -structure I , $\mathcal{C}$ is linear-diagrammatically (n, m, ℓ) -compatible with I implies $I \in \mathcal{C}$.* ■

It is important to observe that linear-diagrammatic compatibility implies diagrammatic compatibility as this will be crucial for obtaining our main characterization.

Lemma 14. *Consider a collection $\mathcal{C}$ of $\mathbf{S}$ -structures that is linear-diagrammatically (n, m, ℓ) -compatible, for integers $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$. It holds that $\mathcal{C}$ is diagrammatically (n, m, ℓ) -compatible.*

The characterization of interest for linear dexrs follows:

Theorem 15. *For a collection $\mathcal{C}$ of structures and $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$, the following are equivalent:*

1. $\mathcal{C}$ is finitely axiomatizable by linear (n, m, ℓ) -dexrs.
2. $\mathcal{C}$ is critical, closed under repairable direct products, and linear-diagrammatically (n, m, ℓ) -compatible.

To establish the above characterization, we first show a technical lemma, called *Linearization Lemma*, which is interesting in its own right as it characterizes when a collection of structures that is finitely axiomatizable by dexrs is finitely axiomatizable by linear dexrs. To this end, the property of linear-diagrammatic compatibility plays a central role.

Lemma 16. *Consider a collection $\mathcal{C}$ of structures that is finitely axiomatizable by (n, m, ℓ) -dexrs, for integers $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$. For every integer $\ell' > 0$, the following are equivalent:*

1. $\mathcal{C}$ is finitely axiomatizable by linear (n, m, ℓ') -dexrs.
2. $\mathcal{C}$ is linear-diagrammatically (n, m, ℓ') -compatible.

The proof of the direction $(1) \Rightarrow (2)$ of the Linearization Lemma is analogous to the proof of Lemma 8, whereas the direction $(2) \Rightarrow (1)$ is shown by following the same strategy as in the proof of the direction $(2) \Rightarrow (1)$ of Theorem 9. Having the Linearization Lemma in place, it is now not difficult to prove the characterization provided by Theorem 15:

Proof of Theorem 15. The direction $(1) \Rightarrow (2)$ follows from Lemmas 2, 4, and 16 (direction $(1) \Rightarrow (2)$). For $(2) \Rightarrow (1)$, since $\mathcal{C}$ is linear-diagrammatically (n, m, ℓ) -compatible, by Lemma 14 we get that $\mathcal{C}$ is also diagrammatically (n, m, ℓ) -compatible. Since $\mathcal{C}$ is critical and closed under repairable direct products, we get from Theorem 9 that $\mathcal{C}$ is finitely axiomatizable by (n, m, ℓ) -dexrs. This allows us to apply Lemma 16 (direction $(2) \Rightarrow (1)$), and get that $\mathcal{C}$ is finitely axiomatizable by linear (n, m, ℓ) -dexrs, as needed. □

5.2 Guarded Disjunctive Existential Rules

Let us now proceed with guarded dexrs and perform a similar analysis as for linear dexrs. As one might suspect, the refined notion of diagrammatic compatibility with a structure I is defined as diagrammatic compatibility with I with the key difference that only guarded substructures K of I are considered. Formally, a structure J is *guarded* if either $\text{facts}(J) = \emptyset$, or there is $R(c_1, \dots, c_r) \in \text{facts}(J)$ such that $\text{adom}(J) = \{c_1, \dots, c_r\}$. Consider now a collection $\mathcal{C}$ of structures over a schema $\mathbf{S}$ and an $\mathbf{S}$ -structure I . For integers $n, m, \ell \geq 0$, we say that $\mathcal{C}$ is *guarded-diagrammatically* (n, m, ℓ) -compatible with I if, for every guarded substructure K of I with $\text{dom}(K) = \text{adom}(K)$ and $|\text{dom}(K)| \leq n$, and every (m, ℓ) -diagram $\Delta_{K,G}^I$ of K relative to I , where $G \subseteq N_{K,m}^I$, there exists $J \in \mathcal{C}$ such that $J \models \Delta_{K,G}^I$. The refined model-theoretic property follows.

Definition 17. A collection $\mathcal{C}$ of $\mathbf{S}$ -structures is guarded-diagrammatically (n, m, ℓ) -compatible, for $n, m, \ell \geq 0$, if, for every $\mathbf{S}$ -structure I , $\mathcal{C}$ is guarded-diagrammatically (n, m, ℓ) -compatible with I implies $I \in \mathcal{C}$. ■

As for linear-diagrammatic compatibility, it is straightforward to show the following, which will be used later:

Lemma 18. Consider a collection $\mathcal{C}$ of $\mathbf{S}$ -structures that is guarded-diagrammatically (n, m, ℓ) -compatible, for integers $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$. It holds that $\mathcal{C}$ is diagrammatically (n, m, ℓ) -compatible.

The characterization of interest for guarded dexrs follows:

Theorem 19. For a collection $\mathcal{C}$ of structures and $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$, the following are equivalent:

1. $\mathcal{C}$ is finitely axiomatizable by guarded (n, m, ℓ) -dexrs.
2. $\mathcal{C}$ is critical, closed under repairable direct products, and guarded-diagrammatically (n, m, ℓ) -compatible.

To establish the above characterization, we first show a technical lemma in the spirit of the Linearization Lemma, called *Guardedization Lemma*, which characterizes when a collection of structures that is finitely axiomatizable by dexrs is finitely axiomatizable by guarded dexrs. To this end, guarded-diagrammatic compatibility plays a crucial role.

Lemma 20. Consider a collection $\mathcal{C}$ of structures that is finitely axiomatizable by (n, m, ℓ) -dexrs, for integers $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$. For every integer $\ell' > 0$, the following are equivalent:

1. $\mathcal{C}$ is finitely axiomatizable by guarded (n, m, ℓ') -dexrs.
2. $\mathcal{C}$ is guarded-diagrammatically (n, m, ℓ') -compatible.

The proof of the direction $(1) \Rightarrow (2)$ of the Guardedization Lemma is analogous to the proof of Lemma 8, whereas the direction $(2) \Rightarrow (1)$ is shown by following the same strategy as in the proof of the direction $(2) \Rightarrow (1)$ of Theorem 9. The Guardedization Lemma allows us to prove the characterization provided by Theorem 19:

Proof of Theorem 19. The direction $(1) \Rightarrow (2)$ follows from Lemmas 2, 4, and 20 (direction $(1) \Rightarrow (2)$). For

$(2) \Rightarrow (1)$, since $\mathcal{C}$ is guarded-diagrammatically (n, m, ℓ) -compatible, by Lemma 18 we get that $\mathcal{C}$ is also diagrammatically (n, m, ℓ) -compatible. Since $\mathcal{C}$ is critical and closed under repairable direct products, we get from Theorem 9 that $\mathcal{C}$ is finitely axiomatizable by (n, m, ℓ) -dexrs. Thus, we can apply Lemma 20 (direction $(2) \Rightarrow (1)$), and get that $\mathcal{C}$ is finitely axiomatizable by guarded (n, m, ℓ) -dexrs. □

6 From Guarded to Linear Disjunctive Existential Rules

The goal of this last section is to understand whether our new diagrammatic compatibility property can be used to solve the non-trivial problem of rewriting a set of guarded dexrs into an equivalent set of dexrs that falls in the weaker class of linear dexrs rules. Formally, we are interested in the following algorithmic problem:

PROBLEM :	G-to-L
INPUT :	A finite set Σ of guarded dexrs.
OUTPUT :	A finite set Σ' of linear dexrs such that $\Sigma \equiv \Sigma'$, if one exists; otherwise, fail.

Our goal is to show that:

Theorem 21. G-to-L is computable in elementary time.

6.1 Bounded Linearization Lemma

To obtain an algorithm that solves our rewritability problem, we need a stronger version of the Linearization Lemma, established in the previous section (see Lemma 16), that allows us to bound the number of variables and the number of head-disjuncts occurring in the linear dexrs of the equivalent set, which in turn allows us to focus on finitely many linear dexrs. More precisely, we need a result that allows us to conclude the following: given a set Σ of dexrs, there is a set Σ' of linear dexrs that is equivalent to Σ iff there is one consisting of linear dexrs with a bounded number of variables and a bounded number of head-disjuncts. This is achieved by the following result, dubbed *Bounded Linearization Lemma*. For a schema $\mathbf{S}$, we write $\text{ar}(\mathbf{S})$ for the maximum arity over all predicates of $\mathbf{S}$, that is, the integer $\max_{R \in \mathbf{S}} \text{ar}(R)$.

Lemma 22. Consider a collection $\mathcal{C}$ of structures over a schema $\mathbf{S}$ that is finitely axiomatizable by (n, m, ℓ) -dexrs, for integers $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$. Let $\ell' = \ell \cdot |\mathbf{S}| \cdot (n + m + 1)^{m \cdot \text{ar}(\mathbf{S})}$. The following are equivalent:

1. $\mathcal{C}$ is finitely axiomatizable by linear dexrs.
2. $\mathcal{C}$ is linear-diagrammatically (n, m, ℓ') -compatible.

Remark. To establish a Bounded Linearization Lemma, one could simply let $\ell' = 2^{|\mathbf{S}| \cdot (n + m)^{\text{ar}(\mathbf{S})}}$. Indeed, with $n + m$ variables, we can construct at most $|\mathbf{S}| \cdot (n + m)^{\text{ar}(\mathbf{S})}$ atoms that mention a predicate of $\mathbf{S}$ and any subset of those atoms may give rise to a conjunction of atoms that can appear in a head-disjunct. However, we would like to obtain an optimal bound on the number of head-disjuncts, whereas the naive double-exponential bound discussed above is clearly suboptimal. The Bounded Linearization Lemma established

Algorithm 1: Rewrite

Input: A set Σ of guarded (n, m, ℓ) -dexrs over $\mathbf{S}$, for $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$.

Output: A set Σ' of linear (n, m, ℓ') -dexrs, where $\ell' = \ell \cdot |\mathbf{S}| \cdot (n + m)^{\text{ar}(\mathbf{S}) \cdot (n+1)}$, such that $\Sigma \equiv \Sigma'$, if one exists; otherwise, fail.

$\ell' := \ell \cdot |\mathbf{S}| \cdot (n + m)^{\text{ar}(\mathbf{S}) \cdot (n+1)}$

$\Sigma' := \{ \sigma \mid$

σ is a linear (n, m, ℓ') -dextr over $\mathbf{S}$ and $\Sigma \models \sigma \}$

if $\Sigma' \neq \emptyset$ and $\Sigma' \models \Sigma$ **then**

return Σ'

else

return fail

above provides an exponential bound, which significantly improves the naive bound, by exploiting the fact that the collection $\mathcal{C}$ of $\mathbf{S}$ -structures is finitely axiomatizable by dexrs with at most ℓ head-disjuncts. The question whether we can establish a stronger version of the Bounded Linearization Lemma, which provides a polynomial bound on the number of disjuncts, remains an interesting open problem.

6.2 The Rewriting Algorithm

Let Σ be a finite set of guarded dexrs over $\mathbf{S}$. Clearly, there are integers $n, m \geq 0$, with $n + m > 0$, and $\ell > 0$, such that Σ consists only of (n, m, ℓ) -dexrs. By the Bounded Linearization Lemma, we get that the following are equivalent:

- There is a finite set Σ' of linear dexrs over $\mathbf{S}$ with $\Sigma \equiv \Sigma'$.
- There is a finite set Σ' of linear (n, m, ℓ') -dexrs, where $\ell' = \ell \cdot |\mathbf{S}| \cdot (n + m + 1)^{m \cdot \text{ar}(\mathbf{S})}$, over $\mathbf{S}$ with $\Sigma \equiv \Sigma'$.

This means that, even though there are infinitely many finite sets of linear dexrs over $\mathbf{S}$, it suffices to search only for linear dexrs over $\mathbf{S}$ that mention at most n universally quantified variables, at most m existentially quantified variables, and at most ℓ' head-disjuncts, which are finitely many, to find a set Σ' that is equivalent to Σ . This leads to the simple algorithm depicted in Algorithm 1. It first collects in Σ' all the linear (n, m, ℓ') -dexrs over $\mathbf{S}$ that are entailed by the input set Σ of guarded dexrs, and then checks whether Σ' is non-empty and entails Σ ; the latter is actually done by checking whether $\Sigma' \models \sigma$, for each $\sigma \in \Sigma$. We proceed to show that Rewrite runs in elementary time, which will imply Theorem 21.

We first observe that the number of linear (n, m, ℓ') -dexrs over the schema $\mathbf{S}$ is bounded by the integer

$$\underbrace{|\mathbf{S}| \cdot n^{\text{ar}(\mathbf{S})}}_{\geq \# \text{ of linear bodies}} \cdot \underbrace{\sum_{i=1}^{\ell'} \binom{H}{i}}_{\geq \# \text{ of heads}},$$

where $H = 2^{|\mathbf{S}| \cdot (n+m)^{\text{ar}(\mathbf{S})}}$. Therefore, one can enumerate all the linear (n, m, ℓ') -dexrs over $\mathbf{S}$ in elementary time (in fact, in triple-exponential time). It remains to analyze the complexity of deciding whether a set of guarded dexrs entails a linear dextr (needed in the construction of Σ'), and

the complexity of deciding whether a set of linear dexrs entails a guarded dextr (needed for checking whether $\Sigma' \models \Sigma$). Given a set Σ of dexrs and a single dextr σ of the usual form $\phi(\bar{x}, \bar{y}) \rightarrow \bigvee_{i=1}^k \exists \bar{z}_i \psi_i(\bar{x}_i, \bar{z}_i)$, it is not difficult to show that the following statements are equivalent:

1. $\Sigma \models \sigma$.
2. $\phi(\rho(\bar{x}), \rho(\bar{y})) \wedge \Sigma \models \exists \bar{z}_i \psi_i(\rho(\bar{x}_i), \bar{z}_i)$, for some $i \in [k]$, where ρ is a renaming function that replaces each variable u in $\phi(\bar{x}, \bar{y})$ with a new constant $\rho(u)$.

The problem of deciding whether $\phi(\rho(\bar{x}), \rho(\bar{y})) \wedge \Sigma \models \exists \bar{z}_i \psi_i(\rho(\bar{x}_i), \bar{z}_i)$, for some $i \in [k]$, if Σ is a set of guarded (and thus, linear) dexrs is decidable in elementary time (in particular, in double-exponential time); the latter is immediately inherited from (Bourhis et al. 2016) that analyzes the problem of conjunctive query answering under guarded dexrs. Putting everything together, we get that the algorithm Rewrite runs in elementary time and Theorem 21 follows.

At this point, let us stress that a more refined complexity analysis allows us to conclude that G-to-L is computable in triple-exponential time, whereas the best that we can hope for is double-exponential time since already the problem of deciding whether a set of guarded existential rules (i.e., dexrs with only one head-disjunct) can be rewritten as an equivalent set of linear existential rules is 2EXPTIME-hard (Console, Kolaitis, and Pieris 2021). More precisely, from the proof of the Bounded Linearization Lemma, we can conclude that whenever the collection $\mathcal{C}$ of $\mathbf{S}$ -structures is finitely axiomatizable by linear dexrs where each head-disjunct mentions at most $p > 0$ atoms, then it is finitely axiomatizable by linear (n, m, ℓ') -dexrs with the same bound on the number of atoms in each head-disjunct. Therefore, the integer H in the above analysis can be set to

$$\sum_{i=1}^p \binom{|\mathbf{S}| \cdot (n + m)^{\text{ar}(\mathbf{S})}}{i} \leq \sum_{i=1}^p \left(|\mathbf{S}| \cdot (n + m)^{\text{ar}(\mathbf{S})} \right)^i,$$

which implies that we need to consider double-exponentially many linear dexrs. This in turn allows us to argue, by using results on the complexity of the satisfiability problem for the guarded fragment of first-order logic (Grädel 1999), that our rewriting algorithm runs in triple-exponential time.

Remark. As said above, it remains open whether a stronger version of the Bounded Linearization Lemma that provides a polynomial bound on the number of head-disjuncts can be established. Let us remark that having such a stronger version of the Bounded Linearization Lemma in place, we can show that our problem G-to-L is computable in double-exponential time, which is the best that we can hope for.

7 Future Work

We would like to perform a similar analysis that goes beyond finite axiomatizability by dexrs. In particular, we are planning to consider finite axiomatizability by dexrs, disjunctive equality rules, and denial constraints. Moreover, it is interesting to prove a stronger version of the Bounded Linearization Lemma (see the last remark of Section 6), which will allow us to establish that G-to-L is computable in double-exponential time, which is worst-case optimal.

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