Non-Rigid Designators in Modal and Temporal Free Description Logics

Alessandro Artale[1] , **Roman Kontchakov**[2] , **Andrea Mazzullo**[1,3] , **Frank Wolter**[4]
[1]Free University of Bozen-Bolzano
[2]Birkbeck, University of London
[3]University of Trento
[4]University of Liverpool
artale@inf.unibz.it, roman@dcs.bbk.ac.uk, andrea.mazzullo@unitn.it, wolter@liverpool.ac.uk

Abstract

Definite descriptions, such as 'the General Chair of KR 2024,' are a semantically transparent device for object identification in knowledge representation. In first-order modal logic, definite descriptions have been widely investigated for their non-rigidity, which allows them to designate different objects (or none at all) at different states. We propose expressive modal description logics with non-rigid definite descriptions and names, and investigate decidability and complexity of the satisfiability problem. We first systematically link satisfiability for the one-variable fragment of first-order modal logic with counting to our modal description logics. Then, we prove a promising NExpTime-completeness result for concept satisfiability for the fundamental epistemic multi-agent logic $\mathbf{S5}^n$ and its neighbours, and show that some expressive logics that are undecidable with constant domain become decidable (but Ackermann-hard) with expanding domains. Finally, we conduct a fine-grained analysis of decidability of temporal logics.

1 Introduction

Definite descriptions, like 'the General Chair of KR 2024,' are expressions of the form 'the unique x such that φ.' Together with *individual names* such as 'Pierre,' they are used as *referring expressions* to identify objects in a given domain (Reiter and Dale 2000; Borgida, Toman, and Weddell 2016; Borgida, Toman, and Weddell 2017).

Definite descriptions and individual names can also *fail to designate* any object at all, as in the case of the definite description 'the KR Conference held after KR 2018 and before KR 2020,' or the individual name 'KR 2019.' In order to admit these as genuine terms of the language, while allowing for their possible lack of referents, formalisms based on *free logic* semantics have been developed (Bencivenga 2002; Lehmann 2002; Indrzejczak 2021; Indrzejczak and Zawidzki 2021). In contrast, classical logic approaches assume that individual names always designate, and that definite descriptions can be paraphrased in terms of existence and uniqueness conditions, an approach dating back to Russell (1905). Recently, definite descriptions have been introduced to enrich standard description logics (DLs) with nominals, $\mathcal{ALCO}$ and $\mathcal{ELO}$ (Neuhaus, Kutz, and Righetti 2020; Artale et al. 2020; Artale et al. 2021).

In particular, DLs $\mathcal{ALCO}^\iota_u$ and $\mathcal{ELO}^\iota_u$ (Artale et al. 2020; Artale et al. 2021) include the universal role, u, as well as nominals and definite descriptions of the form $\{\iota C\}$ ('the unique object in C') as basic concept constructs, while also employing a free logic semantics that allows non-designating terms. Hence, for instance, using nominal $\{\mathsf{kr24}\}$ to designate KR 2024, we can refer to the General Chair of KR 2024 by means of the definite description $\{\iota \exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\}$. Then, in $\mathcal{ALCO}^\iota_u$, we can say that Pierre is the General Chair of KR 2024 with the following concept:

$$\exists u.(\{\mathsf{pierre}\} \sqcap \{\iota \exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\}).$$

Indeed, if this concept is satisfiable (in other words, has a non-empty extension), then there are objects p and k designated by pierre and kr24, respectively, and the pair (p, k) belongs to the interpretation of role isGenChair, which contains no other (o, k), thus making p the only object connected to k by isGenChair. The free DLs $\mathcal{ALCO}^\iota_u$ and $\mathcal{ELO}^\iota_u$ have, respectively, ExpTime-complete ontology satisfiability and PTime-complete entailment problems, thus matching the complexity of the classical DL counterparts.

Names and descriptions display interesting behaviours also in *modal* (epistemic, temporal) settings. These are indeed *referentially opaque* contexts, where the *intension* (i.e., the meaning) of a term might not coincide with its *extension* (that is, its referent) (Fitting 2004). Here, referring expressions can behave as *non-rigid designators*, meaning that they can designate different individuals across different states (epistemic alternatives, instants of time, etc.).

For example, in an epistemic scenario, even if everybody is aware that Pierre is the General Chair of KR 2024, not everyone thereby knows that he is also the General Chair of the (only, so far, and excluding virtual location) KR Conference held in Southeast Asia, despite the fact that 'the KR Conference held in Southeast Asia' and 'KR 2024' refer (to this day) to the same object. Indeed, 'the KR Conference held in Southeast Asia' can be conceived to designate another event by someone unaware of its actual reference to KR 2024. Similarly, in a temporal setting, 'the General Chair of KR' refers to Pierre in 2024, but will designate someone else over the years. So, for instance, if we assume that Pierre works and will continue working in Europe, we can conclude that the General Chair of KR currently works in Europe, but we should not infer that it will always remain the case.

Due to this fundamental and challenging interplay between designation and modalities, non-rigid descriptions and names have been widely investigated in first-order modal and temporal logics as individual concepts or flexible terms capable of taking different values across states (Cocchiarella 1984; Garson 2001; Braüner and Ghilardi 2007; Kröger and Merz 2008; Fitting and Mendelsohn 2012; Corsi and Orlandelli 2013; Indrzejczak 2020; Orlandelli 2021; Indrzejczak and Zawidzki 2023).

The aim of this contribution is to introduce modal DLs that have sufficient expressive power to represent the phenomena discussed above, explore their relationship to standard modal DLs without definite descriptions and non-designating names, and investigate decidability and complexity of reasoning in these formalisms.

In detail, we propose the language $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, which is a modalised extension of the free DL $\mathcal{ALCO}^\iota_u$. For instance, under an epistemic reading of the modal operator $\Box$, we can express with an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ concept that *of* Pierre it is *known* that he is the General Chair of KR 2024:

$$\exists u.(\{\mathsf{pierre}\} \sqcap \Box\{\iota\exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\}),$$

while *of* Pierre it is *not* known that he is the General Chair of the KR Conference held in Southeast Asia:

$$\exists u.(\{\mathsf{pierre}\} \sqcap \neg\Box\{\iota\exists \mathsf{isGenChair}. \{\iota(\mathsf{KRConf} \sqcap \exists \mathsf{hasLoc}.\mathsf{SEAsiaLoc})\}\}).$$

In a temporal setting, where $\Box$ is read as 'always' and its dual $\Diamond$ as 'at some point in the future,' KR 2024 can be made a rigid designator, which refers to the same object at all time instants. This can be achieved, for example, by means of an ontology that holds globally, at all time instants, and consists of the following concept inclusion (CI):

$$\{\mathsf{kr24}\} \sqsubseteq \Box\{\mathsf{kr24}\}.$$

In contrast, we can use the nominal $\{\mathsf{kr}\}$ to refer to the current edition of the KR Conference, for instance stating that KR 2024 is the current KR Conference with the concept $\exists u.(\{\mathsf{kr24}\} \sqcap \{\mathsf{kr}\})$. Moreover, by reading $\bigcirc$ as 'next year', we can exploit its lack of rigidity to express, e.g., that there will be *other* KR Conferences in the future, with the CIs:

$$\top \sqsubseteq \Diamond\exists u.\{\mathsf{kr}\}, \qquad \{\mathsf{kr}\} \sqsubseteq \neg\bigcirc\{\mathsf{kr}\}.$$

Compared to first-order modal logics with non-rigid designators (Stalnaker and Thomason 1968; Fitting and Mendelsohn 2012), both the definition of the language and the scope distinctions for modal operators are simplified in modalised DLs, as first-order variables are replaced by a class-based DL syntax leaving quantification and predicate abstraction implicit. We, however, show that our modal DLs can be translated to a natural fragment of first-order modal logic with definite descriptions and predicate abstraction.

In this work, we consider interpretations with both constant domains (in which the first-order domain is fixed across all worlds) and expanding domains (in which it can grow when moving to accessible worlds). We first establish, for any class of Kripke frames and both constant and expanding domains, polytime satisfiability-preserving reductions (with and without ontology) to the language $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ without definite descriptions. We will show that, in addition, we can assume that each nominal designates in every world, but importantly is still non-rigid.

We then study the satisfiability problem for various fundamental modal logics with epistemic and temporal interpretations. While for first-order modal logics with rigid designators and no counting the restriction to *monodic* formulas (in which modal operators are applied only to formulas with at most one free variable) very often ensures decidability, this is no longer the case if non-rigid designators and/or some counting are admitted (Gabbay et al. 2003). For our modal DLs, this implies that the standard recipe for designing decidable languages — apply modal operators only to concepts — does not always work anymore. Here, we explore in detail when this recipe still works, and when it does not.

First, we closely link the two main sources of bad computational behaviour, non-rigid designators and counting, enabling us to use the results and machinery introduced for first-order modal logics with counting (Hampson and Kurucz 2012; Hampson and Kurucz 2015; Hampson 2016).

Next, we prove that, rather surprisingly, for some fundamental modal epistemic logic, non-rigid designators come for free: concept satisfiability for the modal logics of all Kripke frames with n accessibility relation, $\mathbf{K}^n$, and of all Kripke frames with n equivalence relations, $\mathbf{S5}^n$, is in NExpTime and thus no harder than without names at all. This holds under both constant and expanding domains, and the proof is by showing the exponential finite model property. This answers an open problem discussed in (Hampson 2016). With ontologies, however, concept satisfiability becomes undecidable under constant domains. While for $\mathbf{S5}^n$ (because of symmetric accessibility relations) constant domains coincide with expanding domains, for $\mathbf{K}^n$ decidability under expanding domains remains open with ontologies. As a fundamental example of an expressive modal logic, we investigate the extension $\mathbf{K}^{*n}$ of $\mathbf{K}^n$ with a modal operator interpreted by the transitive closure of the union of the n accessibility relations, which can be interpreted as common knowledge (Fagin et al. 1995) but also as a fragment of propositional dynamic logic, PDL (Harel, Kozen, and Tiuryn 2001). In this case concept satisfiability is undecidable under expanding and constant domains, but becomes decidable, though Ackermann-hard, for the corresponding logic, $\mathbf{K}\boldsymbol{f}^{*n}$, on finite acyclic models with expanding domains. This answers an open problem posed in (Wolter and Zakharyaschev 2001). Note that Ackermann-hardness means that the time required to establish (un)satisfiability is not bounded by any primitive recursive (or computable) function. We refer the reader to Table 1 for an overview of our results. Recall that with rigid designators all these problems are known to be in 2NExpTime (Wolter and Zakharyaschev 2001; Gabbay et al. 2003).

Finally, in the temporal setting, we show that undecidability is a widespread phenomenon: concept satisfiability under global ontology with constant domain is undecidable for all our $\mathcal{ALCO}$-based fragments; the same applies to concept satisfiability in the languages with the universal modality and the temporal $\Diamond$ operator. Reasoning becomes decidable only when considering concept satisfiability (under global ontology) with expanding domains over finite flows of time

modal logic L	domain	concept sat.	concept sat. under global ont.
$\mathbf{K}^n$, $n \geq 1$	const.	NEXP-c. [T 11]	undecidable [T 14]
	exp.	NEXP-c. [T 11]	?
$\mathbf{S5}$	—	NEXP-complete [T 11]	
$\mathbf{S5}^n$, $n \geq 2$	—	NEXP-c. [T 11]	undecidable [T 14]
$\mathbf{K}^{*n}$, $n \geq 1$	const.	Σ^1_1-complete [L 13 + T 15]	
	exp.	undecidable [L 13 + T 16.1]	
$\mathbf{K}\boldsymbol{f}^{*n}$, $n \geq 1$	const.	undecidable [L 13 + T 15]	
	exp.	decidable [T 12], Ackermann-hard [L 13 + T 16.2]	

Table 1: Concept satisfiability (under global ontology) for $L_{\mathcal{ALCO}^\iota_u}$

(though the problem is Ackermann-hard), or in fragments with the $\bigcirc$ operator only, for which we prove EXPTIME-membership of concept satisfiability (without ontologies).

It is to be emphasised that the non-rigidity of symbol interpretation by itself is *not* the cause for the satisfiability problem to become harder. For instance, rigid roles are known to often cause an increase in the hardness of the satisfiability problem compared with the case of non-rigid roles only (Gabbay et al. 2003). What makes non-rigid designators computationally much harder than rigid designators is their ability to count in an unbounded way across worlds.

The full version of the paper with proofs and additional material is available on ArXiv (Artale et al. 2024).

Related Work Other than in non-modal DLs (Neuhaus, Kutz, and Righetti 2020; Artale et al. 2020; Artale et al. 2021), and in the already mentioned *first-order* modal and temporal settings, definite descriptions have been recently investigated in the context of *propositional* hybrid logics with nominals and the @ operator (Walega and Zawidzki 2023). Here, the additional ι operator allows one to refer to the (one and only) state of a model that satisfies a certain condition.

Non-rigid designators have received, to the best of our knowledge, little attention in modal DLs, despite the extensive body of research both on temporal (Wolter and Zakharyaschev 1998; Artale and Franconi 2005; Lutz, Wolter, and Zakharyaschev 2008; Artale et al. 2014) and epistemic (Donini et al. 1998; Artale, Lutz, and Toman 2007; Calvanese et al. 2008; Console and Lenzerini 2020) extensions. As a notable exception, Mehdi and Rudolph (2011) investigate non-rigid individual names in an epistemic DL context, where abstract individual names are interpreted on an infinite common domain, but without definite descriptions.

2 Preliminaries

The $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ language is a modalised extension of the free description logic (DL) $\mathcal{ALCO}^\iota_u$ (Artale et al. 2020; Artale et al. 2021). Let $\mathsf{N_C}$, $\mathsf{N_R}$ and $\mathsf{N_I}$ be countably infinite and pairwise disjoint sets of *concept names*, *role names* and *individual names*, respectively, and let $I = \{1, \dots, n\}$ be a finite set of *modalities*. $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ *terms* and *concepts* are defined by the following grammar:

$$\tau ::= a \mid \iota C,$$
$$C ::= A \mid \{\tau\} \mid \neg C \mid (C \sqcap C) \mid \exists r.C \mid \exists u.C \mid \Diamond_i C,$$

where $a \in \mathsf{N_I}$, $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, $u \notin \mathsf{N_R}$ is the *universal role*, and $\Diamond_i$, with $i \in I$, is a *diamond* operator. A term of the form ιC is called a *definite description* and C its *body*; a concept $\{\tau\}$ is called a (*term*) *nominal*. All the usual syntactic abbreviations are assumed: $\bot = A \sqcap \neg A$, $\top = \neg\bot$, $C \sqcup D = \neg(\neg C \sqcap \neg D)$, $C \Rightarrow D = \neg C \sqcup D$, $C \Leftrightarrow D = (C \Rightarrow D) \sqcap (D \Rightarrow C)$, $\forall s.C = \neg\exists s.\neg C$, for $s \in \mathsf{N_R} \cup \{u\}$, and *box* operator $\Box_i C = \neg\Diamond_i\neg C$. A *concept inclusion* (*CI*) is of the form $C \sqsubseteq D$, for concepts C, D. We use $C \equiv D$ to abbreviate $C \sqsubseteq D$ and $D \sqsubseteq C$. An *ontology* $\mathcal{O}$ is a finite set of CIs.

Fragments $\mathcal{ML}^n_{\mathcal{ALCO}_u}$, $\mathcal{ML}^n_{\mathcal{ALCO}^\iota}$, and $\mathcal{ML}^n_{\mathcal{ALCO}}$ of the full language are defined by restricting the available DL constructors: they do not contain, respectively, definite descriptions, the universal role, and both definite descriptions and the universal role.

Given a concept C, the set of *subconcepts* of C, denoted by $\mathsf{sub}(C)$, is defined as usual (see Appendix A in the full version): we only note that $\mathsf{sub}(\{\iota C\})$ contains C along with its own subconcepts. The *signature* of C, denoted by Σ_C, is the set of concept, role and individual names in C. The signature of a CI or an ontology is defined similarly. The set of *connectives* of an ontology $\mathcal{O}$ is the constructors from the following list that occur in $\mathcal{O}$: ι, $\{\cdot\}$, $\neg$, $\sqcap$, $\exists$ with roles in $\mathsf{N_R}$, $\exists u$, and $\Diamond_i$ with $i \in I$. The *modal depth* of terms and concepts is the maximum number of nested modal operators: $md(A) = 0$, $md(\iota C) = md(C)$ and $md(\Diamond_i C) = md(C) + 1$, for example. The *modal depth* of a CI or an ontology is the maximum modal depth of their concepts.

A *frame* is a pair $\mathfrak{F} = (W, \{R_i\}_{i \in I})$, where W is a non-empty set of *worlds* (or *states*) and each $R_i \subseteq W \times W$, for $i \in I$, is a binary *accessibility relation* on W. A *partial interpretation with expanding domains* based on a frame $\mathfrak{F} = (W, \{R_i\}_{i \in I})$ is a triple $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$, where Δ is a function associating with every $w \in W$ a non-empty set, Δ^w, called the *domain of* w *in* $\mathfrak{M}$, such that $\Delta^w \subseteq \Delta^v$, whenever wR_iv, for some $i \in I$; and $\mathcal{I}$ is a function associating with every $w \in W$ a *partial* DL interpretation $\mathcal{I}_w = (\Delta^w, \cdot^{\mathcal{I}_w})$ that maps every $A \in \mathsf{N_C}$ to a subset $A^{\mathcal{I}_w}$ of Δ^w, every $r \in \mathsf{N_R}$ to a subset $r^{\mathcal{I}_w}$ of $\Delta^w \times \Delta^w$, the universal role u to the set $\Delta^w \times \Delta^w$, and every a in *some subset* of $\mathsf{N_I}$ to an element $a^{\mathcal{I}_w}$ in Δ^w. Hence, every $\cdot^{\mathcal{I}_w}$ is a total function on $\mathsf{N_C} \cup \mathsf{N_R}$ but a *partial* function on $\mathsf{N_I}$. If $\mathcal{I}_w$ is defined on $a \in \mathsf{N_I}$, then we say that *a designates at w*. If every $a \in \mathsf{N_I}$ designates at $w \in W$, then $\mathcal{I}_w$ is called *total*. We say that $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$ is a *total* interpretation if every $\mathcal{I}_w$, $w \in W$, is a *total* interpretation. In the sequel, we refer to partial interpretations as interpretations, and add the adjective 'total' explicitly whenever this is the case.

An *interpretation with constant domains* is defined as a special case, where the function Δ is such that $\Delta^w = \Delta^v$, for every $w, v \in W$. With an abuse of notation, we denote the common domain by Δ and call it the *domain of* $\mathfrak{M}$.

Given $\mathfrak{M} = (\mathfrak{F}, \Delta, \mathcal{I})$, with $\mathfrak{F} = (W, \{R_i\}_{i\in I})$, we define the *value* $\tau^{\mathcal{I}_w}$ of a term τ in world $w \in W$ as $a^{\mathcal{I}_w}$, for $\tau = a$, and as follows, for $\tau = \iota C$:

$$(\iota C)^{\mathcal{I}_w} = \begin{cases} d, & \text{if } C^{\mathcal{I}_w} = \{d\}, \text{ for some } d \in \Delta^w; \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

A term τ is said to *designate at* w if $\tau^{\mathcal{I}_w}$ is defined. The *extension* $C^{\mathcal{I}_w}$ of a concept C in $w \in W$ is defined as follows, where $s \in \mathsf{N_R} \cup \{u\}$:

$$\begin{aligned}
(\neg C)^{\mathcal{I}_w} &= \Delta^w \setminus C^{\mathcal{I}_w}, \\
(C \sqcap D)^{\mathcal{I}_w} &= C^{\mathcal{I}_w} \cap D^{\mathcal{I}_w}, \\
(\exists s.C)^{\mathcal{I}_w} &= \{d \in \Delta^w \mid \exists e \in C^{\mathcal{I}_w} : (d, e) \in s^{\mathcal{I}_w}\}, \\
(\Diamond_i C)^{\mathcal{I}_w} &= \{d \in \Delta^w \mid \exists v \in W : wR_iv \text{ and } d \in C^{\mathcal{I}_v}\}, \\
\{\tau\}^{\mathcal{I}_w} &= \begin{cases} \{\tau^{\mathcal{I}_w}\}, & \text{if } \tau \text{ designates at } w, \\ \emptyset, & \text{otherwise.} \end{cases}
\end{aligned}$$

A concept C is *satisfied at* $w \in W$ *in* $\mathfrak{M}$ if $C^{\mathcal{I}_w} \neq \emptyset$; C is *satisfied in* $\mathfrak{M}$ if it is satisfied at some $w \in W$ in $\mathfrak{M}$. A CI $C \sqsubseteq D$ is *satisfied in* $\mathfrak{M}$, written $\mathfrak{M} \models C \sqsubseteq D$, if $C^{\mathcal{I}_w} \subseteq D^{\mathcal{I}_w}$, for every $w \in W$. An ontology $\mathcal{O}$ is *satisfied in* $\mathfrak{M}$, written $\mathfrak{M} \models \mathcal{O}$, if every CI in $\mathcal{O}$ is satisfied in $\mathfrak{M}$; we also say a concept C is *satisfied in* $\mathfrak{M}$ *under an ontology* $\mathcal{O}$ if $\mathfrak{M} \models \mathcal{O}$ and $C^{\mathcal{I}_w} \neq \emptyset$, for some $w \in W$.

An ontology $\mathcal{O}'$ is called a *model conservative extension* of an ontology $\mathcal{O}$ if every interpretation that satisfies $\mathcal{O}'$ also satisfies $\mathcal{O}$, and every interpretation that satisfies $\mathcal{O}$ can be turned to satisfy $\mathcal{O}'$ by modifying the interpretation of symbols in $\Sigma_{\mathcal{O}'} \setminus \Sigma_{\mathcal{O}}$, while keeping fixed the interpretation of symbols in $\Sigma_{\mathcal{O}}$. Similarly, a concept C' is said to be a *model conservative extension* of a concept C if every interpretation that satisfies C' also satisfies C, and every interpretation satisfying C can be turned into an interpretation that satisfies C', by modifying the interpretation of symbols in $\Sigma_{C'} \setminus \Sigma_C$, while keeping fixed the interpretation of symbols in Σ_C.

Remark 1 (Encoding of assertions)**.** Assertions can be introduced as syntactic sugar using the universal role, with $C(\tau)$ and $r(\tau_1, \tau_2)$ abbreviations for, respectively, concepts

$$\exists u.(\{\tau\} \sqcap C) \text{ and } \exists u.(\{\tau_1\} \sqcap \exists r.\{\tau_2\}).$$

The first example in Sec. 1 is thus $\mathsf{isGenChair}(\mathsf{pierre}, \mathsf{kr24})$.

To avoid ambiguities, we need to use square brackets when applying negation and modal operators to assertions, as in $\neg[C(\tau)]$ and $\Diamond_i[C(\tau)]$. Observe that, in an assertion of the form $\Diamond_i C(\tau)$, the diamond acts as a *de re* operator, since the concept $\Diamond_i C$ applies to the object, if any, designated by the term τ at the current world w, and the assertion is false at a world whenever τ fails to designate at w. On the other hand, in an expression of the form $\Diamond_i[C(\tau)]$, the diamond plays the role of a *de dicto* modality, as it refers to the world of evaluation for the whole assertion $C(\tau)$. Using the lambda abstraction notation for first-order modal logic (Fitting and Mendelsohn 2012), assertion $\Diamond_i A(a)$ corresponds to $\exists x.(\langle \lambda y.x = y\rangle(a) \wedge \Diamond_i A(x))$, whereas $\Diamond_i[A(a)]$ stands for $\Diamond_i \exists x.(\langle \lambda y.x = y\rangle(a) \wedge A(x))$; see Appendix A of the full version for details on the standard translation.

3 Reasoning Problems and Reductions

Let $\mathcal{C}$ be a class of frames (e.g., frames with n equivalence relations) and $\mathcal{ML}^n_{\mathcal{DL}}$ a language. We consider the following two main reasoning problems.

Concept $\mathcal{C}$-Satisfiability: Given an $\mathcal{ML}^n_{\mathcal{DL}}$-concept C, is there an interpretation $\mathfrak{M}$ based on a frame in $\mathcal{C}$ such that C is satisfied in $\mathfrak{M}$?

Concept $\mathcal{C}$-Satisfiability under Global Ontology: Given an $\mathcal{ML}^n_{\mathcal{DL}}$-concept C and an $\mathcal{ML}^n_{\mathcal{DL}}$-ontology $\mathcal{O}$, is there an interpretation $\mathfrak{M}$ based on a frame in $\mathcal{C}$ such that C is satisfied in $\mathfrak{M}$ under $\mathcal{O}$?

In the sequel, for the case of concept $\mathcal{C}$-satisfiability under global ontology, we will assume without loss of generality that C is a concept name. Indeed, we can extend $\mathcal{O}$ with CI $A \equiv C$, for a fresh concept name A, and consider satisfiability of A under the extended ontology, which is a model conservative extension of $\mathcal{O}$.

We begin with a few observations on polytime reductions between the concept satisfiability problems (under global ontology) for two main languages, $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, and various semantic conditions, including (non-)rigid designators, total and partial interpretations, and expanding and constant domains. We also show how to eliminate the universal role using the global ontology and how to replace definite descriptions with nominals, and the other way round, which in particular means that satisfiability in the full $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ and its fragment without ι has the same computational properties. These observations will be useful in our constructions below, where we also apply them to smaller fragments, if the results carry over. Proofs are available in Appendix B of the full version.

No-RDA Subsumes RDA An interpretation $\mathfrak{M}$ satisfies the *rigid designator assumption* (*RDA*) if every individual name $a \in \mathsf{N_I}$ is a *rigid designator* in $\mathfrak{M}$, in the sense that, for every $w, v \in W$ such that wR_iv, if a designates at w, then it designates at v and $a^{\mathcal{I}_w} = a^{\mathcal{I}_v}$. For instance, concept

$$\exists u.(\{a\} \sqcap \Box C) \sqcap \Diamond \exists u.(\{a\} \sqcap \neg C),$$

is unsatisfiable in interpretations *with the RDA*, but is satisfiable otherwise, as a can designate differently at different worlds. Note that an individual that fails to designate at all worlds is vacuously rigid. The following proposition shows the RDA can be enforced in interpretations by an ontology.

Proposition 1. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, concept $\mathcal{C}$-satisfiability under global ontology with the RDA is polytime-reducible to concept $\mathcal{C}$-satisfiability under global ontology, with both constant and expanding domains.*

The proof is based on the observation that CIs of the form $\{a\} \sqsubseteq \Box_i\{a\}$, for $i \in I$, ensure that a can be made a rigid designator in any interpretation. This provides a reduction for the case of global ontology, where a given $\mathcal{O}$ is extended with such CIs for every a. A similar reduction is provided in Appendix B for concept satisfiability in total interpretations.

In the sequel, we assume implicitly that interpretations do *not* satisfy the RDA, and explicitly write 'with the RDA' where necessary.

From Total to Partial Satisfiability and Back Partial interpretations are a generalisation of the classical, total, interpretations, where all nominals designate at all possible worlds. It turns out that satisfiability in partial and total interpretations are polytime-reducible to each other.

Proposition 2. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, total concept $\mathcal{C}$-satisfiability (under global ontology) is polytime-reducible to concept $\mathcal{C}$-satisfiability (under global ontology, respectively), with both constant and expanding domains.*

We sketch the proof for the case of global ontology. Let $\mathcal{O}$ be an ontology. Consider the extension $\mathcal{O}'$ of $\mathcal{O}$ with CIs $\top \sqsubseteq \exists u.\{a\}$, for all individual names a in $\mathcal{O}$. Clearly, these CIs ensure that each a designates in every accessible world. The case of concept satisfiability is shown in Appendix B.

Next, we provide the converse reduction.

Proposition 3. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, concept $\mathcal{C}$-satisfiability (under global ontology) is polytime-reducible to total concept $\mathcal{C}$-satisfiability (under global ontology, respectively), with both constant and expanding domains.*

We again sketch the proof for the case of global ontology. Let $\mathcal{O}$ be an ontology. Consider $\mathcal{O}'$ obtained from $\mathcal{O}$ by replacing every nominal $\{a\}$ in $\mathcal{O}$ with a fresh concept name N_a and by extending the result with all CIs $N_a \sqsubseteq \{a\}$. It follows that every a in $\mathcal{O}'$ can designate in all worlds, but the corresponding concept N_a may still be interpreted by the empty set in some worlds, thus reflecting the fact that a in $\mathcal{O}$ could have failed to designate in those worlds. The case of concept satisfiability is dealt with in Appendix B.

Normal Form for Ontologies and Concepts Next, we define normal form that will help us prove further polytime reductions, e.g., Lemma 6 and Proposition 8, and then complexity upper bounds. Let $\mathcal{O}$ be an ontology and C a subconcept in $\mathcal{O}$. Denote by $\mathcal{O}[C/A]$ the result of replacing every occurrence of C in $\mathcal{O}$ with a fresh concept name A, called the *surrogate* of C. Clearly, $\mathcal{O}[C/A] \cup \{C \equiv A\}$ is a model conservative extension of $\mathcal{O}$. We can systematically apply this procedure to obtain an ontology in *normal form* where connectives are applied only to concept names: e.g., definite descriptions occur only in the form of ιB, for a concept name B. If surrogates are introduced for innermost connectives first, then the transformation runs in polytime.

Lemma 4. *For any $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ ontology $\mathcal{O}$, we can construct in polytime an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ ontology $\mathcal{O}'$ in normal form that is a model conservative extension of $\mathcal{O}$. Moreover, $\mathcal{O}'$ uses the same set of connectives as $\mathcal{O}$.*

If the language contains the universal role, then a similar construction transforms concepts D into normal form. For a single modality ($n = 1$), we can use $\Box^k_1 \forall u.(C \Leftrightarrow A)$, for all $k \leq md(D)$. If $n > 1$, then we need to carefully select sequences of boxes to avoid an exponential blowup. So, for an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ concept D and its subconcept C, we define the set of *C-relevant paths in D*, denoted by $\mathsf{rp}(D, C)$, as the sequences $(i_1, \dots, i_n)$ of the $\Diamond_{i_j}$ operators under which C occurs in D. For example, for $D = \Diamond_1 \neg A \sqcap \Diamond_2 \Diamond_3 A$, we have $\mathsf{rp}(D, A) = \{(1), (2, 3)\}$ and $\mathsf{rp}(D, \neg A) = \{(1)\}$. Note that the maximum length of a path in $\mathsf{rp}(D, C)$ is $md(D)$.

We also define the '$\Box$-modality' for each path: for a concept E, we recursively define

$$\Box^\epsilon E = E \text{ and } \Box^{i \cdot \pi} E = \Box_i \Box^\pi E, \text{ for any path } \pi.$$

As before, the *surrogate* of C is a fresh concept name A, and $D[C/A]$ denotes the result of replacing C with A in D.

Lemma 5. *Let D be an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ concept and C its subconcept. Denote by D' the conjunction of $D[C/A]$ and*

$$\Box^\pi \forall u.(C \Leftrightarrow A), \text{ for all } \pi \in \mathsf{rp}(D, C). \tag{1}$$

Then D' is a model conservative extension of D. Moreover, $\mathsf{rp}(D', A) = \mathsf{rp}(D, C)$ and $\mathsf{rp}(D', E) = \mathsf{rp}(D, E)$, for any subconcept E of C.

For any concept D, by repeatedly replacing non-atomic subconcepts with their surrogates, one can obtain a concept D^* in *normal form*, which is a conjunction of a concept name and concepts of the form (1). By Lemma 5, D^* is a model conservative extension of D. Moreover, if surrogates are introduced for innermost non-atomic concepts first, the procedure runs in polytime in the size of D.

Spy Points: Eliminating the Universal Role Our next observation allows us to eliminate occurrences of the universal role from ontologies.

Lemma 6. *Let $\mathcal{O}$ be an $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ ontology in normal form. Denote by $\mathcal{O}'$ the $\mathcal{ML}^n_{\mathcal{ALCO}^\iota}$ ontology obtained from $\mathcal{O}'$ by replacing*

- *each CI of the form $B \sqsubseteq \exists u.B'$ with $B \sqsubseteq \exists r.B'$, and*
- *each CI of the form $\exists u.B \sqsubseteq B'$ with the following:*

$$\top \sqsubseteq \exists r.\{e\}, \;\; A \sqsubseteq \{e\}, \;\; \neg B' \sqsubseteq \exists r.A, \;\; \exists r.A \sqsubseteq \neg B,$$

where r, e and A are fresh role, nominal and concept names, respectively. Then $\mathcal{O}'$ is a model conservative extension of $\mathcal{O}$, and the size of $\mathcal{O}'$ is linear in the size of $\mathcal{O}$.

Intuitively, positive occurrences of $\exists u.B'$ ensure that B' is non-empty, which can also be achieved with a fresh role r. For negative occurrences of $\exists u.B'$, we use a spy-point e, which is accessible, via a fresh r, from every domain element and belongs to A whenever $\neg B'$ is non-empty (that is, whenever B' does not coincide with the domain). If this is the case, then no domain element can be in B, which, by contraposition, implies $\exists u.B \sqsubseteq B'$. Note that e can be rigid.

From Nominals to Definite Descriptions and Back We first observe that nominals can be easily encoded with definite descriptions. Indeed, given an ontology $\mathcal{O}$, take a fresh concept name N_a for each individual name a in $\mathcal{O}$, and let $\mathcal{O}'$ be the result of replacing every occurrence of $\{a\}$ in $\mathcal{O}$ with $\{\iota N_a\}$. Clearly, $\mathcal{O}'$ is a model conservative extension of $\mathcal{O}$, and vice versa. Note that $\mathcal{O}'$ is in the fragment $\mathcal{ML}^n_{\mathcal{ALC}^\iota_u}$ without nominals, which has no distinction between partial and total interpretations, and between the RDA and no-RDA cases. Thus, we have the following result.

Proposition 7. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, concept $\mathcal{C}$-satisfiability (under global ontology) is polytime-reducible to $\mathcal{ML}^n_{\mathcal{ALC}^\iota_u}$ concept $\mathcal{C}$-satisfiability (under global ontology, respectively), with both constant and expanding domains.*

Conversely, we now show how to replace definite descriptions ιC with nominals using the universal role.

Proposition 8. *$\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ concept $\mathcal{C}$-satisfiability (under global ontology) is polytime-reducible to $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ concept $\mathcal{C}$-satisfiability (under global ontology, respectively), with both constant and expanding domains.*

The proof reduces the total $\mathcal{C}$-satisfiability problems in $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ to total $\mathcal{C}$-satisfiability in $\mathcal{ML}^n_{\mathcal{ALCO}_u}$, which, by Propositions 3 and 2 gives us the required result. We sketch the case of the global ontology. By Lemma 4, we can assume that the given $\mathcal{O}$ is in normal form. Let $\mathcal{O}^*$ be the result of replacing each $A_{\{\iota B\}} \equiv \{\iota B\}$ in $\mathcal{O}$ with CIs

$$A_{\{\iota B\}} \sqsubseteq B \sqcap \{a_B\} \text{ and } B \sqcap \forall u.(B \Rightarrow \{a_B\}) \sqsubseteq A_{\{\iota B\}},$$

where a_B is a fresh individual name. Intuitively, the first CI ensures that the surrogate for $\{\iota B\}$ belongs to B and is never interpreted by more than one domain element. The second CI ensures that if B is a singleton, then that element belongs to the surrogate for $\{\iota B\}$. Formally, we show that $\mathcal{O}^*$ is a model conservative extension of $\mathcal{O}$. The case of concept satisfiability relies on normal form of concepts (Lemma 5) and is treated in Appendix B.

Expanding to Constant Domains It is known that, for the satisfiability problems, the interpretations with expanding domains can be simulated by constant domain interpretations, where a fresh concept name representing the domain is used to relativise concepts and CIs; see e.g., (Gabbay et al. 2003, Proposition 3.32 (ii), (iv)). We restate this standard result in our setting for completeness:

Proposition 9. *In $\mathcal{ML}^n_{\mathcal{ALCO}_u}$ and $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$, concept $\mathcal{C}$-satisfiability (under global ontology) with expanding domains is polytime-reducible to concept $\mathcal{C}$-satisfiability (under global ontology, respectively) with constant domain.*

In the sequel, we implicitly adopt the *constant domain assumption* (Gabbay et al. 2003), and explicitly write when we consider interpretations with expanding domains instead.

4 Non-Rigid Designators and Counting

We prove a strong link between non-rigid designators and the first-order one-variable modal logic enriched with the 'elsewhere' quantifier, $\mathcal{ML}^n_{\text{Diff}}$, introduced and investigated in (Hampson and Kurucz 2012; Hampson and Kurucz 2015; Hampson 2016). We define $\mathcal{ML}^n_{\text{Diff}}$ using DL-style syntax: concepts in $\mathcal{ML}^n_{\text{Diff}}$ are of the form

$$C ::= A \mid \neg C \mid (C \sqcap C) \mid \exists u.C \mid \exists^{\neq} u.C \mid \Diamond_i C,$$

where $i \in I$. Observe that the language has no terms and no roles apart from the universal role u. All constructs are interpreted as before, with the addition of

$$(\exists^{\neq} u.C)^{\mathcal{I}_w} = \big\{d \in \Delta^w \mid C^{\mathcal{I}_w} \setminus \{d\} \neq \emptyset\big\}.$$

Note that our language contains existential quantification, which in (Hampson and Kurucz 2015) is introduced as an abbreviation for $C \sqcup \exists^{\neq} u.C$. In fact, $\mathcal{ML}^n_{\text{Diff}}$ can be regarded as a basic first-order modal logic with counting because the counting quantifier $\exists^{=1} u.C$ ('there is exactly one C') with

$$(\exists^{=1} u.C)^{\mathcal{I}_w} = \big\{d \in \Delta^w \mid |C^{\mathcal{I}_w}| = 1\big\}$$

is clearly logically equivalent to the $\mathcal{ML}^n_{\text{Diff}}$-concept $\exists u.(C \sqcap \neg\exists^{\neq} u.C)$. Conversely, $\exists^{\neq} u.C$ is logically equivalent to $\exists u.C \sqcap (C \Rightarrow \neg\exists^{=1} u.C)$. So, one could replace $\exists^{\neq}$ by $\exists^{=1}$ in the definition $\mathcal{ML}^n_{\text{Diff}}$ and obtain a logic with exactly the same expressive power.

Theorem 10. (1) *$\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concepts (under global ontology) can be reduced in double exponential time to $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\text{Diff}}$-concepts (under global ontology, respectively), with both constant and expanding domains.*

(2) *Conversely, $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\text{Diff}}$-concepts (under global ontology) is polytime-reducible to $\mathcal{C}$-satisfiability of $\mathcal{ML}^n_{\mathcal{ALCO}_u}$-concepts (under global ontology, respectively), with both constant and expanding domains.*

We first give the main ingredients for the proof of Item (1) with global ontology $\mathcal{O}$. If $\mathcal{O}$ contains no roles apart from u, then we introduce a surrogate concept name $\{a\}^\sharp$ for each individual name a in $\mathcal{O}$ and denote by $\mathcal{O}^\sharp$ the result of replacing each $\{a\}$ with $\{a\}^\sharp$ and then extending the ontology with CIs of the form $\top \sqsubseteq \exists^{=1} u.\{a\}^\sharp$. The resulting $\mathcal{ML}^n_{\text{Diff}}$ ontology $\mathcal{O}^\sharp$ is clearly a model conservative extension of $\mathcal{O}$, which completes the reduction. If, however, $\mathcal{O}$ contains role names, then we apply the quasimodel technique (Gabbay et al. 2003) to deal with binary relations (roles), whose interpretations in different worlds are independent from each other. This technique is also used in Sections 5 and 6. In quasimodels, each DL interpretation is represented as a *quasistate*, which is a non-empty set of *types*, maximal consistent sets of subconcepts of $\mathcal{O}$ and their negations: each type $\boldsymbol{t}$ represents all domain elements satisfying the concepts in $\boldsymbol{t}$. In this proof, we can characterise possible quasistates for $\mathcal{O}$ using concepts: the *description* $\Xi_{\boldsymbol{T}}$ of a quasistate $\boldsymbol{T}$ is

$$\forall u. \bigsqcup_{\boldsymbol{t} \in \boldsymbol{T}} \boldsymbol{t} \;\sqcap\; \bigsqcap_{\boldsymbol{t} \in \boldsymbol{T}} \exists u.\boldsymbol{t},$$

where $\boldsymbol{t}$ is the conjunction ($\sqcap$) of all concepts $C \in \boldsymbol{t}$. The set $\mathcal{S}_{\mathcal{O}}$ of quasistates that can possibly occur in interpretations satisfying $\mathcal{O}$ can be obtained by checking whether the *modal abstraction* $\Xi^*_{\boldsymbol{T}}$ of its description $\Xi_{\boldsymbol{T}}$ is satisfiable, where $\cdot^*$ is the result of replacing subconcepts starting with a modal operator by fresh concept names. Since satisfiability of $\mathcal{ALCO}_u$-concepts under $\mathcal{ALCO}_u$ ontologies is in ExpTime, the set $\mathcal{S}_{\mathcal{O}}$ can be computed in double exponential time. In order to ensure that quasistates fit together to form a representation of an interpretation satisfying $\mathcal{O}$, we use the *DL-abstraction* $\cdot^\sharp$ described above, except that now we replace not only nominals but also existential restrictions with fresh concept names. We construct $\mathcal{ML}^n_{\text{Diff}}$-ontology $\mathcal{O}^\sharp$ by replacing CIs $C \sqsubseteq D$ in $\mathcal{O}$ with $C^\sharp \sqsubseteq D^\sharp$ and extending the result with CIs $\top \sqsubseteq \bigsqcup_{\boldsymbol{T} \in \mathcal{S}_{\mathcal{O}}} \Xi^\sharp_{\boldsymbol{T}}$ and $\top \sqsubseteq \exists^{=1} u.\{a\}^\sharp$, for all individual names a. Then we show that A is $\mathcal{C}$-satisfiable under $\mathcal{O}$ iff $\boldsymbol{t}^\sharp$ is $\mathcal{C}$-satisfiable under $\mathcal{O}^\sharp$, for some type $\boldsymbol{t}$ containing A. See Appendix C for full details and the case of concept satisfiability (without ontology).

Item (2) is proved by lifting to the modal description logic setting the observation of Gargov and Goranko (1993) that the difference modality and nominals are mutually interpretable by each other; see the equivalences that show the same expressive power of $\exists^{\neq}$ and $\exists^{=1}$.

Using Theorem 10, one can transfer a large number of (un)decidability results and lower complexity bounds from first-order modal logics with 'elsewhere' to modal DLs with non-rigid designators; see, e.g., Theorem 14 in Section 5. Conversely, the results proved below entail new results for first-order modal logics with 'elsewhere' and/or counting.

5 Reasoning in Modal Free Description Logics

The aim of this section is to show the results presented in Table 1 and also discuss a few related logics. For some basic frame classes, it will be convenient to use standard modal logic notation when discussing the satisfiability problem. So, given a propositional modal logic L with n operators and a DL fragment $\mathcal{DL}$, the problem of $L_{\mathcal{DL}}$ *concept satisfiability* (*under global ontology*) is the problem of deciding $\mathcal{C}_L$-satisfiability of $\mathcal{ML}^n_{\mathcal{DL}}$-concepts (under global ontology, respectively), where $\mathcal{C}_L$ is the class of all frames validating L. We focus on the modal logics L characterised by the following classes $\mathcal{C}_L$ of frames, with $n \geq 1$:

$\mathbf{K}^n$: $\mathcal{C}_L$ is the class of all frames $(W, R_1, \dots, R_n)$;

$\mathbf{S5}^n$: $\mathcal{C}_L$ is the class of all frames $(W, R_1, \dots, R_n)$ such that the R_i are equivalence relations;

$\mathbf{K}^{*n}$: $\mathcal{C}_L$ is the class of all frames $(W, R_1, \dots, R_n, R)$ such that R is the transitive closure of $R_1 \cup \dots \cup R_n$;

$\mathbf{K}\boldsymbol{f}^{*n}$: $\mathcal{C}_L$ is as for $\mathbf{K}^{*n}$ and, in addition, W is finite and R is irreflexive (in other words, there are no chains $w_0 R_{i_1} w_1 \cdots R_{i_n} w_n$ with $w_0 = w_n$).

To illustrate the language $\mathbf{S5}^2_{\mathcal{ALCO}^\iota_u}$, we express that Agent 2 *knows that* Agent 1 does *not know of* the General Chair of KR 2024 that they are the General Chair of the KR Conference held in Southeast Asia:

$$\Box_2 \exists u.(\{\iota \exists \mathsf{isGenChair}.\{\mathsf{kr24}\}\} \sqcap \neg \Box_1 \{\iota \exists \mathsf{isGenChair}. \{\iota(\mathsf{KRConf} \sqcap \exists \mathsf{hasLoc}.\mathsf{SEAsiaLoc})\}\}).$$

The two main new results in Table 1 are the NExpTime upper bound for $\mathbf{K}^n$ and $\mathbf{S5}^n$ and decidability of $\mathbf{K}\boldsymbol{f}^{*n}$. The remaining results are by (sometimes non-trivial) reductions to known results.

Theorem 11. *For* $L \in \{\mathbf{K}^n, \mathbf{S5}^n\}$ *with* $n \geq 1$, $L_{\mathcal{ALCO}^\iota_u}$ *concept satisfiability is in* NExpTime *with both constant and expanding domains.*

We provide a sketch of the main ideas of the proof for $\mathbf{S5}^n$. First, observe (using an unfolding argument) that any satisfiable concept C is satisfied in world w_0 in a model based on a frame $\mathfrak{F} = (W, R_1, \dots, R_n)$ such that the domain W of $\mathfrak{F}$ is a prefix-closed set of words of the form

$$\boldsymbol{w} = w_0 i_0 w_1 \cdots i_{m-1} w_m,$$

where $1 \leq i_j \leq n$, $i_j \neq i_{j+1}$, and each R_i is the smallest equivalence relation containing all pairs of the form $(\boldsymbol{w}, \boldsymbol{w}iw) \in W \times W$. One can assume that m is smaller than the modal depth of C. Next, one can substitute the first-order domain by quasistates (as introduced in Section 4) and work with *quasimodels* $\mathfrak{Q} = (\mathfrak{F}, \boldsymbol{q}, \mathfrak{R})$, in which $\boldsymbol{q}$ associates a quasistate with any world, and a set of *runs* $\mathfrak{R}$ represents first-order domain elements as functions mapping every world w to a type in $\boldsymbol{q}(w)$; note that runs were implicit in the proof of Theorem 10 and could be defined as the domain elements in the interpretations for $\mathcal{ML}^n_{\mathsf{Diff}}$. If $\boldsymbol{t} \in \boldsymbol{q}(w)$ contains a nominal, there is only one $r \in \mathfrak{R}$ with $r(w) = \boldsymbol{t}$; this condition was expressed by the CIs of the form $\top \sqsubseteq \exists^{=1} u.\{a\}^\sharp$. Now, one can apply selective filtration and some surgery to such a quasimodel in order to obtain a quasimodel with at most exponential outdegree (and so of at most exponential size), from which one can then extract a model of exponential size.

Theorem 12. $\mathbf{K}\boldsymbol{f}^{*n}_{\mathcal{ALCO}^\iota_u}$ *concept satisfiability under global ontology is decidable with expanding domains, for* $n \geq 1$.

The proof is again based on appropriate quasimodels, which are now based on expanding domain models using finite frames $\mathfrak{F} = (W, R_1, \dots, R_n, R)$ such that the transitive closure R of $R_1 \cup \dots \cup R_n$ contains no cycles. Unfolding shows that we may assume that (W, R) is a forest. We show decidability by proving a recursive bound on the size of these models. Let $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$, where we assume $m \leq \infty$ and $m + \infty = \infty$, for all $m \in \mathbb{N}_\infty$. We fix an ordering $\boldsymbol{t}_1, \dots, \boldsymbol{t}_k$ of the types and represent a quasistate as a vector $(x_1, \dots, x_k) \in \mathbb{N}^k_\infty$, where x_i is the number of domain elements that satisfy type $\boldsymbol{t}_i$ in a world (equivalently, the number of runs through $\boldsymbol{t}_i$). Let $|\boldsymbol{x}| = x_1 + \dots + x_k$ for $\boldsymbol{x} = (x_1, \dots, x_k) \in \mathbb{N}^k_\infty$. Observe that expanding domains correspond to the condition that wR_iv implies $|\boldsymbol{x}| \leq |\boldsymbol{y}|$, where $\boldsymbol{q}(w), \boldsymbol{q}(v)$ are represented by $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{N}^k_\infty$, respectively. To obtain a recursive bound on the size of a finite model satisfying a concept we then apply Dickson's Lemma to the quasistates. Define the product ordering $\leq$ on $\mathbb{N}^k_\infty$ by setting $\boldsymbol{x} \leq \boldsymbol{y}$ if $x_i \leq y_i$ for all $1 \leq i \leq k$, where $\boldsymbol{x} = (x_1, \dots, x_k)$ and $\boldsymbol{y} = (y_1, \dots, y_k)$. A pair $\boldsymbol{x}, \boldsymbol{y}$ with $\boldsymbol{x} \leq \boldsymbol{y}$ is called an *increasing pair*. Dickson's Lemma states every infinite sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots \in \mathbb{N}^k_\infty$ contains an increasing pair $\boldsymbol{x}_{i_1}, \boldsymbol{x}_{i_2}$ with $i_1 < i_2$. In fact, assuming $|\boldsymbol{x}_i| \leq |\boldsymbol{x}_{i+1}|$ for all $i \geq 0$ and given recursive bounds on $|\boldsymbol{x}_1|$ and $|\boldsymbol{x}_{i+1}| - |\boldsymbol{x}_i|$, one can compute a recursive bound on the length of the longest sequence without any increasing pair (Figueira et al. 2011). Now, the proof of a recursive bound on the size of a satisfying model consists in manipulating the quasimodel so that the outdegree of the forest is recursively bounded and Dickson's Lemma becomes applicable. The expanding domain assumption is crucial for this.

We comment on the remaining results in Table 1. The NExpTime-hardness results already hold without nominals (Gabbay et al. 2003, Theorem 14.14) (the proof goes through also with expanding domain). The lower bounds for $\mathbf{K}\boldsymbol{f}^{*n}$ and $\mathbf{K}^{*n}$ follow from the following lemma and the corresponding lower bounds proved in the next section for temporal DLs (Table 2).

Lemma 13. *Concept satisfiability for* $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}_u}$ *and* $\mathbf{LTL}_{\mathcal{ALCO}_u}$ *are polytime-reducible to concept satisfiability* *for* $\mathbf{K}\boldsymbol{f}^{*n}_{\mathcal{ALCO}_u}$ *and* $\mathbf{K}^{*n}_{\mathcal{ALCO}_u}$*, respectively, with and without ontology and with both constant and expanding domains.*

The proof of this reduction from logics of linear frames with transitive closure to logics of branching frames with transitive closure is not trivial but can be done by adapting the reduction given in the proof of Theorem 6.24 in (Gabbay et al. 2003) for product modal logics.

Finally, undecidability of concept satisfiability under global ontology for $\mathbf{K}^n$ with $n \geq 1$ follows from undecidability of $\mathcal{ML}^n_{\text{Diff}}$ (Hampson and Kurucz 2012) using the reduction of Theorem 10. The result for $\mathbf{S5}^n$ with $n \geq 2$ can be obtained in a similar way, see, e.g., (Gabbay et al. 2003).

Theorem 14. *Concept satisfiability under global ontology is undecidable with constant domains for* $\mathbf{S5}^n_{\mathcal{ALCO}_u}$*, with* $n \geq 2$*, and for* $\mathbf{K}^n_{\mathcal{ALCO}_u}$ *with* $n \geq 1$*.*

For many important modal logics the decidability status of modal DLs with non-rigid designators remains open. Most prominently, for the modal logics of (reflexive) transitive frames $\mathbf{K4}$ (and $\mathbf{S4}$, respectively), decidability of concept satisfiability with or without ontologies and with expanding or constant domains is open. As a "finitary" approximation and a first step to understand $\mathbf{K4}$ and $\mathbf{S4}$, we prove, as a specialisation of the proof of Theorem 12, decidability of concept satisfiability for the Gödel-Löb logic $\mathbf{GL}_{\mathcal{ALCO}^\iota_u}$ ($\mathbf{GL}$ is the logic of all transitive and Noetherian[1] frames (Boolos 1995)) and Grzegorczyk's logic $\mathbf{Grz}_{\mathcal{ALCO}^\iota_u}$ ($\mathbf{Grz}$ is the logic of all reflexive and transitive Noetherian frames (Grzegorczyk 1967)) in expanding domain models with and without ontologies. Alternatively, the decidability of concept satisfiability for $\mathbf{GL}_{\mathcal{ALCO}^\iota_u}$ and $\mathbf{Grz}_{\mathcal{ALCO}^\iota_u}$ can be proved by a double-exponential-time reduction (similar to Theorem 10) to satisfiability in expanding domain products of transitive Noetherian frames, which is known to be decidable (Gabelaia et al. 2006).

6 Reasoning in Temporal Free Description Logics

In this section, we consider the temporal setting. For the temporal DL language, we build $\mathcal{TL}_{\mathcal{ALCO}^\iota_u}$ *terms*, *concepts*, *concept inclusions* and *ontologies* similarly to the $\mathcal{ML}^n_{\mathcal{ALCO}^\iota_u}$ case, with $n = 2$: the language has two modalities — temporal operators 'sometime in the future', $\Diamond$, and 'at the next moment', $\bigcirc$. In particular, the $\mathcal{TL}_{\mathcal{ALCO}^\iota_u}$ *concepts* are defined by the grammar

$$C ::= A \mid \{\tau\} \mid \neg C \mid (C \sqcap C) \mid \exists r.C \mid \exists u.C \mid \Diamond C \mid \bigcirc C,$$

where τ is a $\mathcal{TL}_{\mathcal{ALCO}^\iota_u}$ *term*, defined as in Section 2.

A *flow of time* $\mathfrak{F}$ is a pair $(T, <)$, where T is either the set $\mathbb{N}$ of non-negative integers or a subset of $\mathbb{N}$ of the form $[0, n]$, for $n \in \mathbb{N}$, and $<$ is the strict linear order on T. Elements of T are called *instants* (rather than worlds). A flow of time $(T, <)$ naturally gives rise to a frame $(T, <, \mathit{succ})$ for $\mathcal{TL}_{\mathcal{ALCO}^\iota_u}$, where *succ* is the successor relation: $\mathit{succ} = \{(t, t+1) \mid t, t+1 \in T\}$. So, we will often say that an interpretation $\mathfrak{M}$ is based on a flow $\mathfrak{F}$ if its frame is induced by $\mathfrak{F}$. We denote it (with an abuse of notation) by $\mathfrak{M} = (\Delta, (\mathcal{I}_t)_{t\in T})$. If $\mathfrak{M}$ is based on $(\mathbb{N}, <)$, then we call it an $\mathbf{LTL}_{\mathcal{ALCO}^\iota_u}$ *interpretation*; if it is based on $([0, n], <)$, for some $n \in \mathbb{N}$, it is called an $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}^\iota_u}$ *interpretation*.

Given $\mathfrak{M} = (\Delta, (\mathcal{I}_t)_{t\in T})$, the *value* of a $\mathcal{TL}_{\mathcal{ALCO}^\iota_u}$ term τ at $t \in T$ and the *extension* of a $\mathcal{TL}_{\mathcal{ALCO}^\iota_u}$ concept C at $t \in T$, are defined as in the modal case for $n = 2$: in particular, we have

$$(\bigcirc D)^{\mathcal{I}_t} = \begin{cases} D^{\mathcal{I}_{t+1}}, & \text{if } t+1 \in T, \\ \emptyset, & \text{otherwise,} \end{cases} \quad \text{and } (\Diamond D)^{\mathcal{I}_t} = \bigcup_{t' \in T \text{ with } t < t'} D^{\mathcal{I}_{t'}}.$$

Note that $\Diamond$ is interpreted by $<$ and so does not include the current instant.

We will consider restrictions of the base language $\mathcal{TL}_{\mathcal{ALCO}^\iota_u}$ along both the DL and temporal dimensions. First, $\mathcal{TL}_{\mathcal{ALCO}^\iota}$, $\mathcal{TL}_{\mathcal{ALCO}_u}$ and $\mathcal{TL}_{\mathcal{ALCO}}$ stand for the fragments of $\mathcal{TL}_{\mathcal{ALCO}^\iota_u}$ without the universal role, definite descriptions and both constructs, respectively. In addition to the basic free description logic $\mathcal{ALCO}^\iota_u$, we define temporal extensions of the light-weight free DL $\mathcal{ELO}^\iota_u$, which does not contain negation (and so disjunction). More precisely, the language $\mathcal{TL}_{\mathcal{ELO}^\iota_u}$ is obtained from $\mathcal{TL}_{\mathcal{ALCO}^\iota_u}$ by allowing only $\top$ (considered as a primitive logical symbol), concept names, term nominals, conjunctions and existential restrictions in the construction of concepts. Then, by removing the universal role or/and definite descriptions, we define $\mathcal{TL}_{\mathcal{ELO}^\iota}$, $\mathcal{TL}_{\mathcal{ELO}_u}$ and $\mathcal{TL}_{\mathcal{ELO}}$ in the obvious way.

In the temporal dimension, given a DL $\mathcal{DL}$, the$\Diamond\Box$-*fragment*, $\mathcal{TL}^{\Diamond}_{\mathcal{DL}}$, and the $\bigcirc$-*fragment*, $\mathcal{TL}^{\bigcirc}_{\mathcal{DL}}$, are obtained from $\mathcal{TL}_{\mathcal{DL}}$ by disallowing the $\bigcirc$ and the $\Diamond/\Box$ operators, respectively. Both fragments correspond to the unimodal language $\mathcal{ML}^1_{\mathcal{DL}}$, but with different accessibility relations.

In the following we will combine the syntactic restrictions (fragments) with the semantic restrictions on interpretations and refer, for example, to the satisfiability problem for $\mathcal{TL}^{\Diamond}_{\mathcal{ALCO}_u}$ concepts in $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}^\iota_u}$ interpretations simply as $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_u}$ concept satisfiability.

As an example, in $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}^\iota_u}$, we express that whoever is a Program Chair of KR will not be Program Chair of KR again, but is always appointed as either the General Chair or a PC member of next year's KR, by means of the CI:

$$\exists \mathsf{isProgChair}.\{\mathsf{kr}\} \sqsubseteq \neg\Diamond\exists \mathsf{isProgChair}.\{\mathsf{kr}\} \sqcap$$
$$(\{\iota\exists \mathsf{isGenChair}.\bigcirc\{\mathsf{kr}\}\} \sqcup \exists \mathsf{isPCMember}.\bigcirc\{\mathsf{kr}\}).$$

We begin the study of the satisfiability problems for temporal DLs based on $\mathcal{ALCO}$ by showing that concept satisfiability in constant domains is undecidable or even Σ^1_1-complete (highly undecidable in the analytical hierarchy) over the infinite flow of time $(\mathbb{N}, <)$. This is very different from the classical case with RDA, which was shown to be decidable in the absence of definite descriptions (Gabbay et al. 2003, Theorem 14.12).

Theorem 15. *With constant domains, concept satisfiability is* Σ^1_1*-complete for* $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}_u}$ *and* $\mathbf{LTL}_{\mathcal{ALCO}_u}$*, and undecidable for* $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_u}$*; also, concept satisfiability under global ontology is* Σ^1_1*-complete for* $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}}$ *and* $\mathbf{LTL}_{\mathcal{ALCO}}$*, and undecidable for* $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}}$*.*

[1] (W, R) is *Noetherian* if there is no infinite chain $w_0 R w_1 R \cdots$ with $w_i \neq w_{i+1}$.

temporal logic	concept satisfiability: const. domain	concept satisfiability: exp. domains	concept sat. under global ontology: const. domain	concept sat. under global ontology: exp. domains
$\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}_u}$, $\mathbf{LTL}_{\mathcal{ALCO}_u}$	Σ^1_1-complete [T 15]	undecidable [T 16.1]	Σ^1_1-complete [T 15]	undecidable [T 16.1]
$\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_u}$, $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}_u}$	undecidable [T 15]	decidable, Ackermann-hard [T 16.2]	undecidable [T 15]	decidable, Ackermann-hard [T 16.2]
$\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}}$, $\mathbf{LTL}_{\mathcal{ALCO}}$	?	?	Σ^1_1-complete [T 15]	undecidable [T 16.1]
$\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}}$, $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}}$	?	?	undecidable [T 15]	decidable, Ackermann-hard [T 16.2]
$\mathbf{LTL}^{\bigcirc}_{\mathcal{ALCO}_u}$ / $\mathbf{LTL}^{\bigcirc}_{\mathcal{ALCO}}$	ExP-c. / in ExP [T 18]	ExP-c. / in ExP [T 18]	undecidable [T 19]	?
$\mathbf{LTL}\boldsymbol{f}^{\bigcirc}_{\mathcal{ALCO}_u}$ / $\mathbf{LTL}\boldsymbol{f}^{\bigcirc}_{\mathcal{ALCO}}$	ExP-c. / in ExP [T 18]	ExP-c. / in ExP [T 18]	undecidable [T 19]	decidable

Table 2: Concept satisfiability (under global ontology) for temporal DLs

The lower bounds follow, using Theorem 10 (2), from the respective undecidability results for the first-order one-variable temporal logic with counting (Hampson and Kurucz 2015). Observe that, for $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}}$, we apply the spy-point universal role elimination of Lemma 6. Moreover, Proposition 3 also applies to $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}}$ under global ontology, and so the results hold in both total and partial interpretations.

They could also be proven more directly by encoding the Σ^1_1-complete recurrence and undecidable reachability problems for the Minsky counter machines (Minsky 1961; Alur and Henzinger 1994). Intuitively, the value of a counter of the Minsky machine can be represented as the cardinality of a certain concept. Then non-rigid nominals (or indeed the counting quantifier) can be used to ensure that the value of the counter is incremented/decremented (depending on the command) by the transition: for instance, a CI of the form

$$Q_i \sqsubseteq \bigcirc R_k \Leftrightarrow (R_k \sqcup \{a_k\})$$

could be used to say that from state Q_i, the value of counter k is increased by one. Note, however, that this CI uses the $\bigcirc$ operator. Without it, the proof is considerably more elaborate and represents a counter as a pair of concepts: R_k is used to increment the counter, while S_k to decrement it, so that the counter value is the cardinality of $R_k \sqcap \neg S_k$. Both concepts are made 'monotone': $R_k \sqsubseteq \Box R_k$ and $S_k \sqsubseteq \Box S_k$, and for each transition of the Minsky machine, the non-rigid nominals pick an element that, for example, has never been in R_k before but will remain in R_k from the next instant on: $\neg R_k \sqcap \Box R_k$. A sequence of these elements allows us to linearly order the domain and construct a 'diagonal' in the two-dimensional interpretation necessary for the encoding of the computation using only the $\Diamond$ operator.

Reasoning in expanding domains turns out to be less complex, and we obtain the following:

Theorem 16. (1) *With expanding domains, concept satisfiability is undecidable for* $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}_u}$*, and concept satisfiability under global ontology is undecidable for* $\mathbf{LTL}^{\Diamond}_{\mathcal{ALCO}}$.

(2) *With expanding domains, concept satisfiability (under global ontology) is decidable for* $\mathbf{LTL}\boldsymbol{f}_{\mathcal{ALCO}_u}$*. However, both problems are Ackermann-hard for* $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}_u}$*; moreover, concept satisfiability under global ontology is Ackermann-hard for* $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ALCO}}$.

Undecidability and Ackermann-hardness are proven similarly to Theorem 15. In this case, however, the master problems are, respectively, the ω-reachability and reachability problems for lossy Minsky machines (Konev, Wolter, and Zakharyaschev 2005; Schnoebelen 2010), which in addition to normal transitions can also arbitrarily decrease the counter values. Such computations can be naturally encoded *backwards* in interpretations with expanding domains: the arbitrary decreases of counter values correspond to the extension of the interpretation domain with fresh elements.

The positive decidability results over the finite flows of time follows from Theorem 12 by Lemma 13 (together with the reduction in Proposition 8).

Temporal Free DLs Based on $\mathcal{ELO}$. Next, we transfer the above results to the $\mathcal{ELO}$ family. As $\mathcal{TL}_{\mathcal{ELO}}$ concepts do not contain negation and the empty concept ($\bot$), they are trivially satisfiable. Thus, our main reasoning problem is based on the notion of entailment (rather than satisfiability).

CI Entailment (over Finite Flows): Given a $\mathcal{TL}_{\mathcal{DL}}$-CI $C_1 \sqsubseteq C_2$ and a $\mathcal{TL}_{\mathcal{DL}}$-ontology $\mathcal{O}$, is it the case that $C_1^{\mathcal{I}_t} \subseteq C_2^{\mathcal{I}_t}$, for every $t \in T$ in every interpretation $\mathfrak{M}$ satisfying $\mathcal{O}$ and based on $(\mathbb{N}, <)$ (every finite flow, respectively)?

It turns out that disjunction can be modelled in the temporal extension of $\mathcal{ELO}$ with the help of the $\Diamond$ modality (Artale et al. 2007): intuitively, any CI of the form $\top \sqsubseteq B_1 \sqcup B_2$ is replaced with $\top \sqsubseteq \exists q.(\Diamond X_1 \sqcap \Diamond X_2)$, which says that X_1 and X_2 occur in some order in the future (possibly on another domain element). It then remains to check the order of X_1 and X_2 and, if, say, X_1 precedes X_2, then B_1 is chosen, otherwise B_2 is chosen. So, this reduction shows that the entailment problem for the fragments of $\mathcal{TL}^{\Diamond}_{\mathcal{ELO}}$ essentially has the same complexity as the complement of the satisfiability problem for the corresponding $\mathcal{TL}^{\Diamond}_{\mathcal{ALCO}}$ fragment:

Theorem 17. (1) *CI entailment with constant domains is* Π^1_1*-complete for* $\mathbf{LTL}^{\Diamond}_{\mathcal{ELO}}$ *and undecidable for* $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ELO}}$.

(2) *CI entailment with expanding domains is undecidable for* $\mathbf{LTL}^{\Diamond}_{\mathcal{ELO}}$.

(3) *CI entailment with expanding domains is decidable but Ackermann-hard for* $\mathbf{LTL}\boldsymbol{f}^{\Diamond}_{\mathcal{ELO}}$.

Next-Only Temporal Free DLs. As we have seen above, the $\Diamond$-only fragments normally exhibit the same bad computational behaviour as the full logics with both $\Diamond$ and $\bigcirc$.

We now provide some results for the fragments that contain only $\bigcirc$. We begin with some positive results for the satisfiability problem (without global ontology):

Theorem 18. *With constant and with expanding domains, concept satisfiability is* ExpTime*-complete for* $\mathbf{LTL}^{\bigcirc}_{\mathcal{ALCO}_u}$ *and* $\mathbf{LTL}\boldsymbol{f}^{\bigcirc}_{\mathcal{ALCO}_u}$ *and in* ExpTime *for* $\mathbf{LTL}^{\bigcirc}_{\mathcal{ALCO}}$ *and* $\mathbf{LTL}\boldsymbol{f}^{\bigcirc}_{\mathcal{ALCO}}$.

The ExpTime upper complexity bound can be shown by a type elimination procedure, similarly to the case of the product $\mathbf{Alt} \times \mathbf{K}_n$ of modal logics $\mathbf{Alt}$, whose accessibility relation is a partial function, and multi-modal $\mathbf{K}_n$, which is a notational variant of $\mathcal{ALC}$; see (Gabbay et al. 2003, Theorem 6.6). One has to, in addition, take care of nominals and the universal role, but that can be done in exponential time. The matching lower bound is inherited from $\mathcal{ALCO}_u$, but for the fragment without the universal role the exact complexity remains an open problem.

Our final result indicates that with the global ontology, the $\bigcirc$-fragments behaves nearly as badly as the full language:

Theorem 19. *With constant domains, concept satisfiability under global ontology is undecidable for* $\mathbf{LTL}^{\bigcirc}_{\mathcal{ALCO}}$ *and* $\mathbf{LTL}\boldsymbol{f}^{\bigcirc}_{\mathcal{ALCO}}$.

The proof is by a direct reduction of the reachability problem for Minsky machines, similarly to the simplified sketch for Theorem 15; note the proof makes use of the spy-point universal role elimination in Lemma 6.

7 Discussion and Future Work

We have introduced and investigated novel fragments of first-order modal logic with non-rigid (and possibly non-referring) individual names and definite descriptions. Potential applications that remain to be explored include business process management, where formalisms for representing the dynamic behaviour of data and information are crucial (Delgrande et al. 2023; Deutsch et al. 2018), and context, knowledge or standpoint-dependent reasoning for which possible worlds semantics is needed (Ghidini and Serafini 2017; Gómez Álvarez, Rudolph, and Strass 2023).

Besides the open decidability problems discussed above, future research directions include the extension of our results to more expressive monodic fragments (Gabbay et al. 2003; Hodkinson, Wolter, and Zakharyaschev 2002), automated support for the construction of definite descriptions and referring expressions (Artale et al. 2021; Kurucz, Wolter, and Zakharyaschev 2023), the design of 'practical' reasoning algorithms for the languages considered here, and the extension of our results to modal DLs with hybrid (Braüner 2014; Indrzejczak and Zawidzki 2023), branching-time (Hodkinson, Wolter, and Zakharyaschev 2002; Gutiérrez-Basulto, Jung, and Lutz 2012), dynamic (Harel 1979), or non-normal operators (Dalmonte et al. 2023).

Acknowledgements

Andrea Mazzullo acknowledges the support of the MUR PNRR project FAIR - Future AI Research (PE00000013) funded by the NextGenerationEU.

References

Alur, R., and Henzinger, T. A. 1994. A really temporal logic. *J. ACM* 41(1):181–204.

Artale, A., and Franconi, E. 2005. Temporal description logics. In *Handbook of Temporal Reasoning in Artificial Intelligence*, volume 1 of *Foundations of Artificial Intelligence*. Elsevier. 375–388.

Artale, A.; Kontchakov, R.; Lutz, C.; Wolter, F.; and Zakharyaschev, M. 2007. Temporalising tractable description logics. In *Proc. of the 14th International Symposium on Temporal Representation and Reasoning (TIME-07)*, 11–22. IEEE Computer Society. ISBN: 0-7695-2836-8.

Artale, A.; Kontchakov, R.; Ryzhikov, V.; and Zakharyaschev, M. 2014. A cookbook for temporal conceptual data modelling with description logics. *ACM Trans. Comput. Log.* 15(3):25:1–25:50.

Artale, A.; Mazzullo, A.; Ozaki, A.; and Wolter, F. 2020. On free description logics with definite descriptions. In *Proceedings of the 33rd International Workshop on Description Logics (DL-20)*, volume 2663 of *CEUR Workshop Proceedings*. CEUR-WS.org.

Artale, A.; Mazzullo, A.; Ozaki, A.; and Wolter, F. 2021. On free description logics with definite descriptions. In *Proceedings of the 18th International Conference on Principles of Knowledge Representation and Reasoning (KR-21)*, 63–73.

Artale, A.; Kontchakov, R.; Mazzullo, A.; and Wolter, F. 2024. Non-Rigid Designators in Modal and Temporal Free Description Logics (Extended Version). *CoRR* abs/2405.07656.

Artale, A.; Lutz, C.; and Toman, D. 2007. A description logic of change. In *IJCAI*, 218–223.

Bencivenga, E. 2002. Free logics. In *Handbook of Philosophical Logic*. Springer. 147–196.

Boolos, G. 1995. *The logic of provability*. Cambridge university press.

Borgida, A.; Toman, D.; and Weddell, G. E. 2016. On referring expressions in query answering over first order knowledge bases. In *Proceedings of the 15th International Conference on Principles of Knowledge Representation and Reasoning (KR-16)*, 319–328. AAAI Press.

Borgida, A.; Toman, D.; and Weddell, G. E. 2017. Concerning referring expressions in query answers. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence, (IJCAI-17)*, 4791–4795. ijcai.org.

Braüner, T., and Ghilardi, S. 2007. First-order Modal Logic. In *Handbook of Modal Logic*. Elsevier. 549–620.

Braüner, T. 2014. First-order hybrid logic: introduction and survey. *Log. J. IGPL* 22(1):155–165.

Calvanese, D.; De Giacomo, G.; Lembo, D.; Lenzerini, M.; and Rosati, R. 2008. Inconsistency tolerance in P2P data integration: An epistemic logic approach. *Inf. Syst.* 33(4-5):360–384.

Cocchiarella, N. B. 1984. Philosophical perspectives on quantification in tense and modal logic. II: Extensions of Classical Logic:309–353.

Console, M., and Lenzerini, M. 2020. Epistemic integrity constraints for ontology-based data management. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI-20)*, 2790–2797. AAAI Press.

Corsi, G., and Orlandelli, E. 2013. Free quantified epistemic logics. *Studia Logica* 101(6):1159–1183.

Dalmonte, T.; Mazzullo, A.; Ozaki, A.; and Troquard, N. 2023. Non-normal modal description logics. In Gaggl, S. A.; Martinez, M. V.; and Ortiz, M., eds., *Logics in Artificial Intelligence - 18th European Conference, JELIA 2023, Dresden, Germany, September 20-22, 2023, Proceedings*, volume 14281 of *Lecture Notes in Computer Science*, 306–321. Springer.

Delgrande, J. P.; Glimm, B.; Meyer, T. A.; Truszczynski, M.; and Wolter, F. 2023. Current and future challenges in knowledge representation and reasoning. *CoRR* abs/2308.04161.

Deutsch, A.; Hull, R.; Li, Y.; and Vianu, V. 2018. Automatic verification of database-centric systems. *ACM SIGLOG News* 5(2):37–56.

Donini, F. M.; Lenzerini, M.; Nardi, D.; Nutt, W.; and Schaerf, A. 1998. An epistemic operator for description logics. *Artif. Intell.* 100(1-2):225–274.

Fagin, R.; Halpern, J. Y.; Moses, Y.; and Vardi, M. Y. 1995. *Reasoning About Knowledge*. MIT Press.

Figueira, D.; Figueira, S.; Schmitz, S.; and Schnoebelen, P. 2011. Ackermannian and primitive-recursive bounds with Dickson's lemma. In *Proceedings of the 26th Annual IEEE Symposium on Logic in Computer Science, LICS 2011, June 21-24, 2011, Toronto, Ontario, Canada*, 269–278. IEEE Computer Society.

Fitting, M., and Mendelsohn, R. L. 2012. *First-order Modal Logic*. Springer Science & Business Media.

Fitting, M. 2004. First-order intensional logic. *Ann. Pure Appl. Log.* 127(1-3):171–193.

Gabbay, D. M.; Kurucz, A.; Wolter, F.; and Zakharyaschev, M. 2003. *Many-dimensional Modal Logics: Theory and Applications*. North Holland Publishing Company.

Gabelaia, D.; Kurucz, A.; Wolter, F.; and Zakharyaschev, M. 2006. Non-primitive recursive decidability of products of modal logics with expanding domains. *Ann. Pure Appl. Log.* 142(1-3):245–268.

Gargov, G., and Goranko, V. 1993. Modal logic with names. *J. Philos. Log.* 22(6):607–636.

Garson, J. W. 2001. Quantification in modal logic. In *Handbook of philosophical logic*, volume II: Extensions of Classical Logic. Springer. 267–323.

Ghidini, C., and Serafini, L. 2017. Distributed first order logic. *Artif. Intell.* 253:1–39.

Grzegorczyk, A. 1967. Some relational systems and the associated topological spaces. *Fundamenta Mathematicae* 3(60):223–231.

Gutiérrez-Basulto, V.; Jung, J. C.; and Lutz, C. 2012. Complexity of branching temporal description logics. In *Proceedings of the 20th European Conference on Artificial Intelligence (ECAI-12)*, volume 242 of *Frontiers in Artificial Intelligence and Applications*, 390–395. IOS Press.

Gómez Álvarez, L.; Rudolph, S.; and Strass, H. 2023. Tractable diversity: Scalable multiperspective ontology management via standpoint EL. In *Proceedings of the Thirty-Second International Joint Conference on Artificial Intelligence, IJCAI 2023, 19th-25th August 2023, Macao, SAR, China*, 3258–3267. ijcai.org.

Hampson, C., and Kurucz, A. 2012. On modal products with the logic of 'elsewhere'. In *Advances in Modal Logic 9, papers from the ninth conference on "Advances in Modal Logic," held in Copenhagen, Denmark, 22-25 August 2012*, 339–347.

Hampson, C., and Kurucz, A. 2015. Undecidable propositional bimodal logics and one-variable first-order linear temporal logics with counting. *ACM Trans. Comput. Log.* 16(3):27:1–27:36.

Hampson, C. 2016. Decidable first-order modal logics with counting quantifiers. In *Advances in Modal Logic 11, proceedings of the 11th conference on "Advances in Modal Logic," held in Budapest, Hungary, August 30 - September 2, 2016*, 382–400.

Harel, D.; Kozen, D.; and Tiuryn, J. 2001. Dynamic logic. *SIGACT News* 32(1):66–69.

Harel, D. 1979. *First-Order Dynamic Logic*, volume 68 of *Lecture Notes in Computer Science*. Springer.

Hodkinson, I. M.; Wolter, F.; and Zakharyaschev, M. 2002. Decidable and undecidable fragments of first-order branching temporal logics. In *Proceedings of the 17th IEEE Symposium on Logic in Computer Science (LICS-02)*, 393–402. IEEE Computer Society.

Indrzejczak, A., and Zawidzki, M. 2021. Tableaux for free logics with descriptions. In Das, A., and Negri, S., eds., *Proceedings of the 30th International Conference on Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX-21)*, volume 12842 of *Lecture Notes in Computer Science*, 56–73. Springer.

Indrzejczak, A., and Zawidzki, M. 2023. Definite descriptions and hybrid tense logic. *Synthese* 202(3):98.

Indrzejczak, A. 2020. Existence, definedness and definite descriptions in hybrid modal logic. In *Proceedings of the 13th Conference on Advances in Modal Logic (AiML-20)*, 349–368. College Publications.

Indrzejczak, A. 2021. Free logics are cut-free. *Stud Logica* 109(4):859–886.

Konev, B.; Wolter, F.; and Zakharyaschev, M. 2005. Temporal logics over transitive states. In Nieuwenhuis, R., ed., *Automated Deduction - CADE-20, 20th International Conference on Automated Deduction, Tallinn, Estonia, July 22-27, 2005, Proceedings*, volume 3632 of *Lecture Notes in Computer Science*, 182–203. Springer.

Kröger, F., and Merz, S. 2008. *Temporal Logic and State Systems*. Texts in Theoretical Computer Science. An EATCS Series. Springer.

Kurucz, A.; Wolter, F.; and Zakharyaschev, M. 2023. Definitions and (uniform) interpolants in first-order modal logic. *CoRR* abs/2303.04598.

Lehmann, S. 2002. More free logic. In *Handbook of Philosophical Logic*. Springer. 197–259.

Lutz, C.; Wolter, F.; and Zakharyaschev, M. 2008. Temporal description logics: A survey. In *Proceedings of the 15th International Symposium on Temporal Representation and Reasoning (TIME-08)*, 3–14. IEEE Computer Society.

Mehdi, A., and Rudolph, S. 2011. Revisiting semantics for epistemic extensions of description logics. In *Proceedings of the 25th AAAI Conference on Artificial Intelligence (AAAI-11)*. AAAI Press.

Minsky, M. L. 1961. Recursive unsolvability of Post's problem of "tag" and other topics in theory of Turing machines. *Annals of Mathematics* 74(3):437–455.

Neuhaus, F.; Kutz, O.; and Righetti, G. 2020. Free description logic for ontologists. In *Proceedings of the Joint Ontology Workshops (JOWO-20)*, volume 2708 of *CEUR Workshop Proceedings*. CEUR-WS.org.

Orlandelli, E. 2021. Labelled calculi for quantified modal logics with definite descriptions. *J. Log. Comput.* 31(3):923–946.

Reiter, E., and Dale, R., eds. 2000. *Building Natural Language Generation Systems*. Cambridge University Press.

Russell, B. 1905. On Denoting. *Mind* 14(56):479–493.

Schnoebelen, P. 2010. Revisiting Ackermann-hardness for lossy counter machines and reset Petri nets. In *Mathematical Foundations of Computer Science 2010, 35th International Symposium, MFCS 2010, Brno, Czech Republic, August 23-27, 2010. Proceedings*, volume 6281 of *Lecture Notes in Computer Science*, 616–628. Springer.

Stalnaker, R. C., and Thomason, R. H. 1968. Abstraction in first-order modal logic. *Theoria* 34(3):203–207.

Walega, P. A., and Zawidzki, M. 2023. Hybrid modal operators for definite descriptions. In Gaggl, S. A.; Martinez, M. V.; and Ortiz, M., eds., *Logics in Artificial Intelligence - 18th European Conference, JELIA 2023, Dresden, Germany, September 20-22, 2023, Proceedings*, volume 14281 of *Lecture Notes in Computer Science*, 712–726. Springer.

Wolter, F., and Zakharyaschev, M. 1998. Temporalizing description logics. In *Proceedings of the 2nd International Symposium on Frontiers of Combining Systems (FroCoS-98)*, 104–109. Research Studies Press/Wiley.

Wolter, F., and Zakharyaschev, M. 2001. Decidable Fragments of First-Order Modal Logics. *J. Symb. Log.* 66(3):1415–1438.