Expressive Power of Definite Descriptions in Modal Logics

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Abstract

Motivated by applications in knowledge representation and reasoning, modal and description logics have been recently extended with definite description operators. Such operators provide us with a tool for referring to a particular element of a model by stating a property satisfied only by this element. This mechanism resembles the way we refer to objects in natural language, which makes it an attractive component of ontology and query languages. In this paper, we aim to provide a tool for analysing the expressive power of logics with definite descriptions. In particular, we introduce an adequate bisimulation notion for the basic modal logic extended with definite descriptions. We exploit the introduced bisimulation to relate expressive power of definite descriptions to other operators and we develop an algorithm for computing the maximal bisimulation between a pair of models. Furthermore, we consider a simplified setting, where expressions used in definite descriptions do not mention modal operators. We show how this restriction impacts our results.

1 Introduction

Definite descriptions are expressions aiming to refer to a single element by stating its unique property, as in the famous example 'the present king of France' (Russell 1905). Such expressions provide us with a natural way of referring to objects, which has been intensively studied by logicians, linguists, and philosophers and gave rise to a number of alternatives theories (Pelletier and Linsky 2005; Hilbert and Bernays 1968; Rosser 1978; Lambert 2001).

In recent year, there is a renewed interest in formal aspects of definite descriptions. In particular, a number of methods for automated reasoning with definite descriptions have been proposed, including tableau systems, sequent calculi, and natural deduction (Fitting and Mendelsohn 2023; Indrzejczak and Zawidzki 2021; Indrzejczak and Zawidzki 2023; Indrzejczak 2019; Indrzejczak 2023; Orlandelli 2021; Kürbis 2021a; Kürbis 2021b). There are also several successful implementation, for example, in systems *KeYamera* X (Bohrer, Fernández, and Platzer 2019), *PROVER9* (Oppenheimer and Zalta 2011), and *Isabelle/HOL* (Benzmüller and Scott 2020; Blumson 2020).

Furthermore, definite descriptions gained interest in the area of Knowledge Representation (Borgida, Toman, and Weddell 2016a; Toman and Weddell 2018; Toman and Weddell 2019b), as they introduce a natural way of identifying objects (Borgida, Toman, and Weddell 2016b; Artale et al. 2021), they can be used instead of object identifiers (Borgida, Toman, and Weddell 2016b; Borgida, Toman, and Weddell 2017), and as more informative answers to queries (Toman and Weddell 2019a). In particular, definite descriptions have been studied in the setting of description logics, by introducing to the language a generalised form of nominals of the form $\{\iota C\}$, for a complex concept C (Mazzullo 2022). The intended extension of $\{\iota C\}$ is a singleton containing a unique element of the model of which C holds. A similar approach has been considered in modal logics by extending the (hybrid) satisfaction operators $@_i$. The standard satisfaction operator $@_i$ allows us to refer to the unique modal world satisfying the nominal i. To allow for complex definite descriptions, modal logics have been recently extended with a generalisation of $@_i$ to the form $@_\varphi$, for arbitrarily complex formulas φ (Wałęga and Zawidzki 2023). Such operators allow us to express interesting properties, for example, $@_{(\neg\Diamond\top)}\top$ states that 'there exists exactly one modal world which has no outgoing accessibility relation', $@_{(\Diamond\Diamond\Diamond\top)}\top$ states that 'the longest path (via accessibility relation) in the model is of length 3', and $@_{(p\vee\neg\varphi)}\varphi$ states that 'formula φ holds in every world (and p holds in exactly one world)'. In both modal and description logic setting, recent research tries to determine what is the exact impact of adding definite descriptions on the expressive power of the logic (Mazzullo 2022; Wałęga and Zawidzki 2023).

A key tool for analysing and characterising expressive power is a bisimulation (van Benthem 2014; Milner 1971; Park 1981)—a relation between elements of two models, which relates elements that are indistinguishable from the perspective of a given formal language L. Two main properties, which are usually required from an adequate notion of an L-bisimulation are the bisimulation invariance and Hennessy-Milner properties. Bisimulation invariance states that elements related by an L-bisimulation satisfy the same formulas of the language L. Hennessy-Milner property states that in image-finite (also called finite-branching) models the opposite implication also holds. The restriction to image-finite (or more generally, to ω-saturated) models is essential, since for arbitrary infinite models this opposite implication does not hold. Such adequate notions of bisimulations have been defined for a number of logics (Blackburn, De Rijke, and Venema 2002; Areces, Hoffmann, and Denis 2010; De Rijke 1992; Artale et al. 2021), but despite recent efforts, to the best of our knowledge there is a lack of an adequate bisimulation for a basic modal logic (or basic description logic) extended with definite descriptions. In the case of modal logic, the established bisimulation satisfies the invariance, but not the Hennessy-Milner property (Wałęga and Zawidzki 2023). In the case of description logics, to obtain an adequate notion of the bisimulation the language was extended with the universal modality (Artale et al. 2021). The main challenge in introducing an adequate notion of a bisimulation is due to the specific non-local behaviour of definite descriptions, which is particularly hard to capture with conditions imposed on a bisimulation. Indeed, the known conditions used to define bisimulations do not seem to apply to the case of definite descriptions.

We will aim to close this gap, by introducing a method for constructing adequate bisimulations for logics with definite descriptions. We will introduce such bisimulations (Section 3) for a logic $\mathcal{ML}(\mathsf{DD})$ obtained by extending the basic modal logic with operators $@_\varphi$, and for a logic $\mathcal{BML}(\mathsf{DD})$ which restricts $\mathcal{ML}(\mathsf{DD})$ by allowing only for Boolean formulas in the subscripts φ of operators $@_\varphi$ (both logics are defined in Section 2). Our notions of bisimulations introduce a new type of conditions, whose verification is non-trivial. As we show, however, there is an efficient way of checking these conditions, which we exploit to develop polynomial-time algorithms computing maximal bisimulations between pairs of models (Section 4). Moreover, we apply our notions of bisimulatons to show that definite descriptions do not allow us to define the difference, everywhere, somewhere, and counting operators (Section 5). Finally we briefly conclude the paper (Section 6).

2 Modal Logic of Definite Descriptions

In what follows we present the modal logic of definite descriptions $\mathcal{ML}(\mathsf{DD})$ (Wałęga and Zawidzki 2023), which extends the basic modal language with operators for definite descriptions $@_\varphi$. It allows us to write formulas $@_\varphi\psi$ with intended meaning that 'ψ holds in *the* world in which φ holds'. The logic exploits a Russellian-style semantics of such operators, namely satisfiaction of $@_\varphi\psi$ requires *existence* and *uniqueness* of a world satisfying φ, as we describe in details in what follows.

Syntax. Formulas of $\mathcal{ML}(\mathsf{DD})$ are generated by the grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid @_\varphi\varphi,$$

where p ranges over the set PROP of propositional variables. The grammar above is minimal, which makes the presentation of the logic and proving it's properties more concise, however, it will be sometimes convenient to also use $\bot$, $\top$, $\wedge$, $\rightarrow$, and $\Box$, which stand for standard abbreviations. We let $\mathsf{PROP}(\varphi)$ be the set of all propositional variables occurring in φ. We say that a formula is *flat* if it has no nesting of @-operators. Note that @ can be nested in various ways, for example as in $@_\varphi@_\psi\eta$, or as in $@_{(@_\psi\eta)}\varphi$; flat formulas do not allow for any type of nesting.

Definite descriptions. Formulas of the form $@_\varphi\psi$ aim to express definite descriptions. Observe that φ in the subscript of @ can be complex, namely it can mention Boolean connectives, diamond modal operator $\Diamond$, and @-operator. It turns out that the form of the allowed subscripts of @-operators impacts computational properties of the logic (Wałęga and Zawidzki 2023). Therefore, we will consider in the paper also the restriction of $\mathcal{ML}(\mathsf{DD})$, called $\mathcal{BML}(\mathsf{DD})$, where only Boolean formulas are allowed in the subscripts, that is, formulas φ which do not mention modal operators and @-operators.

Semantics. Semantics of $\mathcal{ML}(\mathsf{DD})$ is given in a Kripke-style, where a *frame* is a pair $\mathcal{F} = (W, R)$ consisting of a non-empty set W of *worlds* and an *accessibility relation* $R \subseteq W \times W$. A *model* based on a frame $\mathcal{F} = (W, R)$ is a tuple $\mathcal{M} = (W, R, V)$, where $V : \mathsf{PROP} \longrightarrow \mathcal{P}(W)$ is a *valuation* assigning a set of worlds to each propositional variable. The *satisfaction relation* $\models$ for $\mathcal{M} = (W, R, V)$ and $w \in W$ is defined inductively as follows:

$\mathcal{M}, w \models p$	iff	$w \in V(p)$, where $p \in \mathsf{PROP}$
$\mathcal{M}, w \models \neg\varphi$	iff	$\mathcal{M}, w \not\models \varphi$
$\mathcal{M}, w \models \varphi \vee \psi$	iff	$\mathcal{M}, w \models \varphi$ or $\mathcal{M}, w \models \psi$
$\mathcal{M}, w \models \Diamond\varphi$	iff	there exists $v \in W$ such that $(w, v) \in R$ and $\mathcal{M}, v \models \varphi$
$\mathcal{M}, w \models @_\varphi\psi$	iff	there exists $v \in W$ such that $\mathcal{M}, v \models \varphi$ and $\mathcal{M}, v \models \psi$, and $\mathcal{M}, v' \not\models \varphi$ for all $v' \neq v$ in W

A formula φ is *true* in a model $\mathcal{M}$, in symbols $\mathcal{M} \models \varphi$, if $\mathcal{M}, w \models \varphi$ for all worlds w in $\mathcal{M}$. We say that φ is *satisfiable* if there exist $\mathcal{M}$ and w such that $\mathcal{M}, w \models \varphi$.

Complexity. It has been shown that satisfiability checking of $\mathcal{BML}(\mathsf{DD})$-formulas is PSpace-complete (Wałęga and Zawidzki 2023, Theorem 5), so not harder than in the basic modal logic $\mathcal{ML}$. In $\mathcal{ML}(\mathsf{DD})$, where we allow for @-operators with arbitrary subscripts, satisfiability checking becomes ExpTime-complete (Wałęga and Zawidzki 2023, Theorem 4). In particular, ExpTime-hardness is obtained by a reduction from $\mathcal{ML}$ enriched with the universal modality, in which satisfiability is known to be ExpTime-complete (Blackburn, De Rijke, and Venema 2002). It is worth to emphasize that this reduction requires using @-operators with non-Boolean subscripts, and that the reduction provides an $\mathcal{ML}(\mathsf{DD})$-formula which is equisatisfiable, but not equivalent to an input $\mathcal{ML}$-formula using universal modality. This observation will be important for our expressive power results from Section 5.

3 Bisimulations

In this section we will introduce bisimulations for $\mathcal{ML}(\mathsf{DD})$ and $\mathcal{BML}(\mathsf{DD})$, as powerful model-theoretic tools for analysing expressive power of definite descriptions. We will aim to construct bisimulations which capture exactly the expressive power of these logics in the sense that our bisimulations will satisfy both the bisimulation invariance and the Hennessy-Milner properties. Recall that bisimulation invariance states that formulas are invariant under the bisimulation and the Hennessy-Milner property states that for image-finite (also called finite-branching) models the opposite implication also holds. A bisimulation which satisfies both of these conditions, allows us to 'capture exactly' the expressive power of a logic.

The task of constructing such bisimulations for logics with definite descriptions is a particularly challenging task that has not been achieved so far despite several attempts. In particular, the previous attempts led to too strong conditions, so that the bisimulation did not satisfy the Hennessy-Milner property (Wałęga and Zawidzki 2023), or required extending the language with other operators, in particular with the universal modality (Artale et al. 2021, Section 4). In what follows we will show what are the main challenges regarding constructing desired bisimulations and how to overcome them.

3.1 Main Challenges

We will identify two main challenges regarding introduction of appropriate bisimulations for logics with definite descriptions. They will allow us to illustrate why the standard bisimulation for $\mathcal{ML}$ is inadequate for $\mathcal{ML}(\mathsf{DD})$ and $\mathcal{BML}(\mathsf{DD})$, as well as what kind of conditions are missing.

A standard bisimulation for $\mathcal{ML}$, which we will call an *$\mathcal{ML}$-bisimulation*, between models $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ is any relation $Z \subseteq W \times W'$ such that whenever $(w, w') \in Z$, the following hold:

Atom: w and w' satisfy the same propositional variables,

Zig: if there is $v \in W$ such that $(w, v) \in R$, then there is $v' \in W'$ such $(v, v') \in Z$ and $(w', v') \in R'$,

Zag: if there is $v' \in W'$ such that $(w', v') \in R'$, then there is $v \in W$ such $(v, v') \in Z$ and $(w, v) \in R$.

We will write $\mathcal{M}, w \;\underline{\leftrightarrow}_{\mathcal{ML}}\; \mathcal{M}', w'$ if there is an $\mathcal{ML}$-bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ such that $(w, w') \in Z$.

Let us now observe two main reasons why the standard bisimulation does not capture the meaning of definite descriptions. First, let us consider models $\mathcal{M}$ and $\mathcal{M}'$ from Figure 1(a) and the relation Z between their worlds. We can observe that Z is an $\mathcal{ML}$-bisimulation, but bisimilar worlds w_1 and w_1' do not satisfy the same $\mathcal{ML}(\mathsf{DD})$-formulas. Indeed, $\mathcal{M}, w_1 \models @_q\top$, but $\mathcal{M}', w_1' \not\models @_q\top$. This example presents a crucial property of definite descriptions, namely although w_1 and w_1' satisfy the same $\mathcal{ML}$-formulas and both of these worlds have 'names'—that is, within their models, they can be unambiguously referred to with some $\mathcal{ML}$-formulas (e.g., q is a name of w_1 in $\mathcal{M}$ and p is a name of w_1' in $\mathcal{M}'$)—these worlds do not satisfy the same $\mathcal{ML}(\mathsf{DD})$-formulas. The reason is that names of w_1 in $\mathcal{M}$ and names of w_1' in $\mathcal{M}'$ are not all the same, for example, q is a name of w_1 in $\mathcal{M}$, but it is not a name of w_1' in $\mathcal{M}'$. The fact that the names of w_1 and w_1' are different, depends on the form of other worlds in the models (w_2, w_3, w_2', and w_3'), which are neither related by the accessibility relation or by Z to worlds w_1 and w_1'. Therefore an adequate definition of bisimulation needs to involve specific 'non-local' conditions, unlike the conditions of the standard $\mathcal{ML}$-bisimulation. As we will show in Section 3.2, requiring that the names in models are the same, is exactly the condition that is needed to obtain an adequate definition of a bisimulation for $\mathcal{ML}(\mathsf{DD})$.

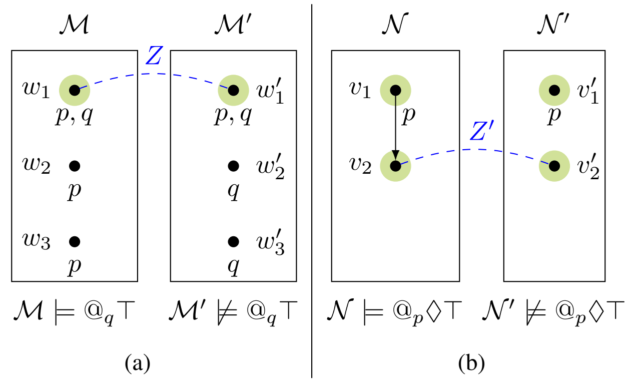

Figure 1: $\mathcal{ML}$-bisimulations Z and Z' which do not preserve satisfiability of $\mathcal{ML}(\mathsf{DD})$- and $\mathcal{BML}(\mathsf{DD})$-formulas, respectively; named worlds (see Definition 1) are marked with circles

This could suggest that in $\mathcal{BML}(\mathsf{DD})$, where definite descriptions are Boolean, to obtain an adequate definition of a bisimulation it is sufficient to assume that Boolean names in two models are the same. Our second observation, however, shows that it is not the case. Indeed, consider models $\mathcal{N}$ and $\mathcal{N}'$ from Figure 1 and a relation Z' between them. We can observe that Z' is an $\mathcal{ML}$-bisimulation and Boolean names of $\mathcal{N}$ and $\mathcal{N}'$ coincide. However, bisimilar worlds v_2 and v_2' do not satisfy the same $\mathcal{BML}(\mathsf{DD})$-formulas; in particular, $\mathcal{N}, v_2 \models @_p\Diamond\top$, but $\mathcal{N}', v_2' \not\models @_p\Diamond\top$. This raises additional challenges regarding an adequate definition of a bisimulation for $\mathcal{BML}(\mathsf{DD})$. In Section 3.3 we will show how to address this difficulty.

3.2 Bisimulation for $\mathcal{ML}(\mathsf{DD})$

Following the observation from Figure 1(a), we will define an $\mathcal{ML}(\mathsf{DD})$-bisimulation by introducing an additional requirement that models have the same 'names'. We start by defining formally the notions of names and named worlds.

Definition 1. *The set Names($\mathcal{M}$), of* names *in a model $\mathcal{M}$, consists of all $\mathcal{ML}$-formulas φ such that φ is satisfied in a unique world of $\mathcal{M}$. The set NamedWorlds($\mathcal{M}$), of* named worlds *in $\mathcal{M}$, consists of all worlds w in $\mathcal{M}$ such that $\mathcal{M}, w \models \varphi$, for some $\varphi \in$ Names($\mathcal{M}$).*

Equivalently, *Names*($\mathcal{M}$) consists of all formulas φ such that $\mathcal{M} \models @_\varphi\top$. For example, q and $p \wedge q$ are names in the model $\mathcal{M}$ from Figure 1(a), because these formulas hold only at w_1. Hence w_1 is a named world. Note that in the model $\mathcal{M}'$ the formula $p \wedge q$ is also a name, but q is not, because it holds in multiple worlds (w_1', w_2', and w_3').

The observation from Figure 1(a) suggests that the existence of a non-empty $\mathcal{ML}(\mathsf{DD})$-bisimulation between $\mathcal{M}$

and $\mathcal{M}'$ should require that $Names(\mathcal{M}) = Names(\mathcal{M}')$. Indeed, if φ is a name in $\mathcal{M}$, but not in $\mathcal{M}'$, then $\mathcal{M} \models @_\varphi\top$, but $\mathcal{M} \not\models @_\varphi\top$. Therefore any bisimilar worlds w and w' from $\mathcal{M}$ and $\mathcal{M}'$, respectively, do not satisfy the same $\mathcal{ML}(\mathsf{DD})$-formulas. What is even more interesting (and which we will prove in this subsection) is that the requirement $Names(\mathcal{M}) = Names(\mathcal{M}')$ is extctly what we need to add to the standard definition of an $\mathcal{ML}$-bisimulation. Thus we define an $\mathcal{ML}(\mathsf{DD})$-bisimulation as follows.

Definition 2. *A relation Z is an $\mathcal{ML}(\mathsf{DD})$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ if $Z = \emptyset$ or both of the following hold:*

- *Z is an $\mathcal{ML}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$,*
- *$Names(\mathcal{M}) = Names(\mathcal{M}')$.*

We write $\mathcal{M}, w \leftrightarrows_{\mathcal{ML}(\mathsf{DD})} \mathcal{M}', w'$ if $(w, w') \in Z$ for some $\mathcal{ML}(\mathsf{DD})$-bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$.

Note that we explicitly let the empty relation be an $\mathcal{ML}(\mathsf{DD})$-bisimulation, since the empty relation falls under the definition of the standard $\mathcal{ML}$-bisimulation and we aim to maintain a close analogy to the standard setting. This, by the fact that $\mathcal{ML}(\mathsf{DD})$-bisimulations are closed under (finite and infinite) unions (Blackburn, De Rijke, and Venema 2002), implies that each pair of models has a unique maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation. An exemplary maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation is depicted in Figure 2. Note that neither the relation from Figure 1(a) nor the one from Figure 1(b) is an $\mathcal{ML}(\mathsf{DD})$-bisimulation, as required. This is because the pairs of models fom these figures have different names.

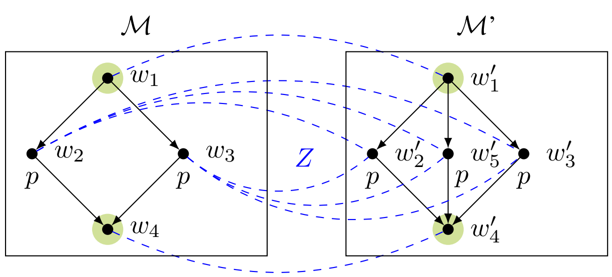

Figure 2: A maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation; named worlds are marked with circles

In the remaining part of this subsection we will prove that our $\mathcal{ML}(\mathsf{DD})$-bisimulation satisfies both the bisimulation invariance and the Hennessy-Milner properties. In the proof of the bisimulation invariance property we will use the fact that $\mathcal{ML}(\mathsf{DD})$-formulas can be transformed to the flat form, where no nesting of @-operators occurs, as shown below.

Lemma 3. *For each $\mathcal{ML}(\mathsf{DD})$-formula there exists an equivalent flat $\mathcal{ML}(\mathsf{DD})$-formula.*

Proof sketch. To flatten a formula φ, we can construct an exponentially long disjunction, where each disjunct is obtained by replacing subformulas of φ with all possible combinations of $\top$ and $\bot$. Our construction is similar to the normalisation procedures for unary modal operators (Areces and Gorín 2010), generalized quantifiers (Van Der Hoek and De Rijke 1993, Theorem 2.11), and counting operators (Areces, Hoffmann, and Denis 2010). ☐

Now we will use Lemma 3 to prove the bisimulation invariance property.

Theorem 4 (Bisimulation invariance property for $\mathcal{ML}(\mathsf{DD})$)**.** *If $\mathcal{M}, w \leftrightarrows_{\mathcal{ML}(\mathsf{DD})} \mathcal{M}', w'$ then w and w' satisfy the same $\mathcal{ML}(\mathsf{DD})$-formulas.*

Proof. Let Z be an $\mathcal{ML}(\mathsf{DD})$-bisimulation between models $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ such that $(w, w') \in Z$, for some $w \in W$ and $w' \in W'$. For an arbitrary $\mathcal{ML}(\mathsf{DD})$-formula φ we will show that $\mathcal{M}, w \models \varphi$ if and only if $\mathcal{M}', w' \models \varphi$. To this end, we will construct an $\mathcal{ML}$-formula ψ such that $\mathcal{M} \models \varphi \leftrightarrow \psi$ and $\mathcal{M}' \models \varphi \leftrightarrow \psi$. Constructing such ψ will finish the proof because Z is an $\mathcal{ML}$-bisimulation, and so, Bisimulation Invariance Lemma for $\mathcal{ML}$ implies the required equivalence.

By Lemma 3 we can assume that φ is flat. Moreover we assume that φ mentions some @-operator, as otherwise it suffices to let $\psi = \varphi$. Hence, φ has a single subformula of the form $@_\eta\vartheta$, for some $\mathcal{ML}$-formulas η and ϑ. We let

$$\psi = \begin{cases} \varphi[\top/@_\eta\vartheta], & \text{if } \{\eta, (\eta \wedge \vartheta)\} \subseteq Names(\mathcal{M}), \\ \varphi[\bot/@_\eta\vartheta], & \text{otherwise,} \end{cases}$$

where $\varphi[\alpha/\beta]$ is obtained from φ by replacing β with α.

Next we will show that $\mathcal{M} \models \varphi \leftrightarrow \psi$. If $\{\eta, (\eta \wedge \vartheta)\} \subseteq Names(\mathcal{M})$, we obtain that $\mathcal{M} \models @_\eta\top$ and $\mathcal{M} \models @_{\eta\wedge\vartheta}\top$. Hence $\mathcal{M} \models @_\eta\vartheta$, and so $\mathcal{M} \models \varphi \leftrightarrow \varphi[\top/@_\eta\vartheta]$. If $\{\eta, (\eta \wedge \vartheta)\} \not\subseteq Names(\mathcal{M})$, we obtain that $\mathcal{M} \not\models @_\eta\top$ or $\mathcal{M} \not\models @_{\eta\wedge\vartheta}\top$. In both of these cases $\mathcal{M} \not\models @_\eta\vartheta$, therefore $\mathcal{M} \models \varphi \leftrightarrow \varphi[\bot/@_\eta\vartheta]$. Consequently $\mathcal{M} \models \varphi \leftrightarrow \psi$.

To show that $\mathcal{M}' \models \varphi \leftrightarrow \psi$ it suffices to observe that, by the definition of an $\mathcal{ML}(\mathsf{DD})$-bisimulation, $Names(\mathcal{M}) = Names(\mathcal{M}')$, and to repeat the argumentation above. ☐

Next we show the Hennessy-Milner property for $\mathcal{ML}(\mathsf{DD})$. Similarly as in the standard Hennessy-Milner Theorem for $\mathcal{ML}$, we will assume that models are image-finite.

Theorem 5 (Hennessy-Milner property for $\mathcal{ML}(\mathsf{DD})$)**.** *Assume that $\mathcal{M}$ and $\mathcal{M}'$ are image-finite models. It holds that $\mathcal{M}, w \leftrightarrows_{\mathcal{ML}(\mathsf{DD})} \mathcal{M}', w'$ if and only if w and w' satisfy the same $\mathcal{ML}(\mathsf{DD})$-formulas.*

Proof. The left-to-right implication follows from Theorem 4. For the opposite implication assume that the same $\mathcal{ML}(\mathsf{DD})$-formulas hold at a world w in $\mathcal{M}$ and at a world w' in $\mathcal{M}'$. Thus the same $\mathcal{ML}$-formulas hold in these worlds, and so, by the Hennessy-Milner Theorem for $\mathcal{ML}$ (Blackburn, De Rijke, and Venema 2002)[Theorem 2.24], there exists an $\mathcal{ML}$-bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ such that $(w, w') \in Z$. We will show that Z is also an $\mathcal{ML}(\mathsf{DD})$-bisimulation, that is, $Names(\mathcal{M}) = Names(\mathcal{M}')$. Indeed, if $\varphi \in Names(\mathcal{M})$, then $\mathcal{M}, w \models @_\varphi\top$. Since the same $\mathcal{ML}(\mathsf{DD})$-formulas hold at w and w', we obtain that $\mathcal{M}', w' \models @_\varphi\top$, and so $\varphi \in Names(\mathcal{M}')$. Analogously, $\varphi \in Names(\mathcal{M}')$ implies that $\varphi \in Names(\mathcal{M})$, so $Names(\mathcal{M}) = Names(\mathcal{M}')$. ☐

As we will show in the next subsection, adapting the definition of $\mathcal{ML}(\mathsf{DD})$-bisimulation to the case of $\mathcal{BML}(\mathsf{DD})$ is not straightforward and requires introducing additional conditions.

3.3 Bisimulation for $\mathcal{BML}(\mathsf{DD})$

We start by observing that the requirement $\mathit{Names}(\mathcal{M}) = \mathit{Names}(\mathcal{M}')$ is too strong do define a bisimulation for $\mathcal{BML}(\mathsf{DD})$. Indeed, worlds related by Z in Figure 3 satisfy the same $\mathcal{BML}(\mathsf{DD})$-formulas, but $\mathit{Names}(\mathcal{M}) \neq \mathit{Names}(\mathcal{M}')$; in particular $\Diamond p$ and $\neg p \wedge \neg \Diamond p$ are names in $\mathcal{M}$ (of the worlds w_1 and w_2, respectively), but they are not names in $\mathcal{M}'$.

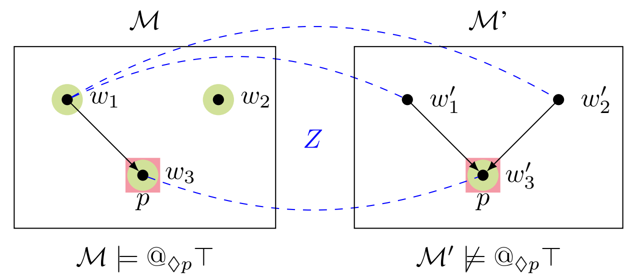

Figure 3: A $\mathcal{BML}(\mathsf{DD})$-bisimulation which is not an $\mathcal{ML}(\mathsf{DD})$-bisimulation; named worlds are marked with circles and worlds with Boolean names are marked with squares

Since $\mathcal{BML}(\mathsf{DD})$-formulas allow in the subscripts of @-operators for Boolean formulas only, it will be useful to consider Boolean names and worlds with Boolean names as defined below.

Definition 6. Boolean names *of a model $\mathcal{M}$ is the set $\mathit{Names}_B(\mathcal{M}) = \{\varphi \in \mathit{Names}(\mathcal{M}) \mid \varphi$ is Boolean$\}$. The set $\mathit{NamedWorlds}_B(\mathcal{M})$, of* worlds in $\mathcal{M}$ with Boolean names, *consists of all worlds w in $\mathcal{M}$ such that $\mathcal{M}, w \models \varphi$, for some $\varphi \in \mathit{Names}_B(\mathcal{M})$.*

Boolean names form a subset of all names; similarly worlds with Boolean names (marked with squares in Figure 3) form a subset of all named worlds. It turns out, however, that replacing *Names* with Names_B in Definition 2 of an $\mathcal{ML}(\mathsf{DD})$-bisimulation does not allow us to obtain an appropriate definition of a bisimulation for $\mathcal{BML}(\mathsf{DD})$, as illustrated in Figure 4. Indeed, Z therein is an $\mathcal{ML}$-bisimulation and $\mathit{Names}_B(\mathcal{M}) = \mathit{Names}_B(\mathcal{M}')$, but Z-related worlds w_3 and w_3' do not satisfy the same $\mathcal{BML}(\mathsf{DD})$-formulas. In particular $\mathcal{M}, w_3 \not\models @_p\Diamond\top$, but $\mathcal{M}', w_3' \models @_p\Diamond\top$.

Thus, the condition $\mathit{Names}(\mathcal{M}) = \mathit{Names}(\mathcal{M}')$ is too strong and the condition $\mathit{Names}_B(\mathcal{M}) = \mathit{Names}_B(\mathcal{M}')$ is too weak for an appropriate definition of an $\mathcal{BML}(\mathsf{DD})$-bisimulation. In what follows we show how to provide conditions appropriate for $\mathcal{BML}(\mathsf{DD})$.

Definition 7. *A relation Z is a* $\mathcal{BML}(\mathsf{DD})$-bisimulation *between models $\mathcal{M}$ and $\mathcal{M}'$ if $Z = \emptyset$ or the following hold:*

- *Z is an $\mathcal{ML}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$,*
- *$\mathit{Names}_B(\mathcal{M}) = \mathit{Names}_B(\mathcal{M}')$,*

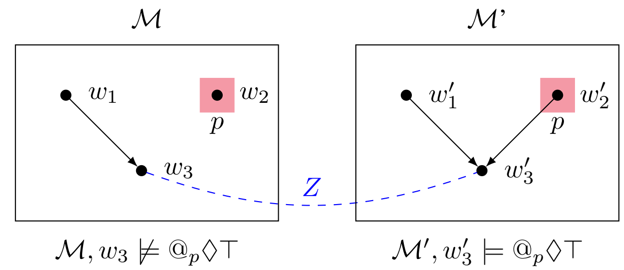

Figure 4: A relation which is not a $\mathcal{BML}(\mathsf{DD})$-bisimulation, although $\mathit{Names}_B(\mathcal{M}) = \mathit{Names}_B(\mathcal{M}')$; worlds with Boolean names are marked with squares

- **Dom:** *the domain of Z contains $\mathit{NamedWorlds}_B(\mathcal{M})$,*
- **Rng:** *the range of Z contains $\mathit{NamedWorlds}_B(\mathcal{M}')$.*

We write $\mathcal{M}, w \leftrightarroweq_{\mathcal{BML}(\mathsf{DD})} \mathcal{M}', w'$ if $(w, w') \in Z$ for some $\mathcal{BML}(\mathsf{DD})$-bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$.

As an example of a $\mathcal{BML}(\mathsf{DD})$-bisimulation consider Z from Figure 3. Next we show that $\mathcal{BML}(\mathsf{DD})$-bisimulations satisfy the bisimulation invariance and Hennessy-Milner properties. It is worth noting that proofs of these properties for $\mathcal{BML}(\mathsf{DD})$ significantly differ from proofs for $\mathcal{ML}(\mathsf{DD})$. In particular our proof of the bisimulation invariance lemma for $\mathcal{ML}(\mathsf{DD})$ heavily relies on the fact that $\mathit{Names}(\mathcal{M})$ can mention arbitrary $\mathcal{ML}$-formulas, whereas the proof of the Hennessy-Milner property for $\mathcal{ML}(\mathsf{DD})$ uses the fact that if $\mathit{Names}(\mathcal{M}) = \mathit{Names}(\mathcal{M}')$, then each $\mathcal{ML}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ is also an $\mathcal{ML}(\mathsf{DD})$-bisimulation. Analogous statements, however, do not hold in $\mathcal{BML}(\mathsf{DD})$.

Theorem 8 (Bisimulation invariance property for $\mathcal{BML}(\mathsf{DD})$)**.** *If $\mathcal{M}, w \leftrightarroweq_{\mathcal{BML}(\mathsf{DD})} \mathcal{M}', w'$ then w and w' satisfy the same $\mathcal{BML}(\mathsf{DD})$-formulas.*

Proof. Assume that Z is a $\mathcal{BML}(\mathsf{DD})$-bisimulation between $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ such that $(w, w') \in Z$, for some $w \in W$ and $w' \in W'$. We will show by induction on the structure of a $\mathcal{BML}(\mathsf{DD})$-formula φ, that $\mathcal{M}, w \models \varphi$ if and only if $\mathcal{M}', w' \models \varphi$. Since each $\mathcal{BML}(\mathsf{DD})$-bisimulation is also an $\mathcal{ML}$-bisimulation, the base case (when φ is a propositional variable) and the inductive step for φ of the forms $\neg\psi$, $\psi \vee \eta$, and $\Diamond\psi$ can be shown in the same way as in the proof of the Bisimulation Invariance Lemma for $\mathcal{ML}$ (Blackburn, De Rijke, and Venema 2002; Blackburn, Van Benthem, and Wolter 2007).

The remaining case in the inductive step, when φ is of the form $@_\psi\eta$, is non-standard. If $\mathcal{M}, w \models @_\psi\eta$, there exists $v \in W$ such that $\mathcal{M}, v \models \psi$, $\mathcal{M}, v \models \eta$, and no world $u \neq v$ in $\mathcal{M}$ satisfies ψ. Since $@_\psi\eta$ is a $\mathcal{BML}(\mathsf{DD})$-formula, ψ is a Boolean formula which belongs to $\mathit{Names}_B(\mathcal{M})$ and $v \in \mathit{NamedWorlds}_B(\mathcal{M})$. Therefore, by Condition Dom, there exists $v' \in W'$ such that $(v, v') \in Z$. By the inductive assumption $\mathcal{M}', v' \models \psi$ and $\mathcal{M}', v' \models \eta$. Hence, to prove that $\mathcal{M}', w' \models @_\psi\eta$, it remains to show that ψ is not satisfied in any world distinct from v' in $\mathcal{M}'$, that is $\psi \in \mathit{Names}_B(\mathcal{M}')$. This, however, follows from the fact that $\psi \in \mathit{Names}_B(\mathcal{M})$ and $\mathit{Names}_B(\mathcal{M}) = \mathit{Names}_B(\mathcal{M}')$. Thus $\mathcal{M}, w \models @_\psi \eta$ implies that $\mathcal{M}', w' \models @_\psi \eta$, and in a symmetric way we can show the opposite implication. □

Theorem 9 (Hennessy-Milner property for $\mathcal{BML}(\mathsf{DD})$)**.** *Assume that $\mathcal{M}$ and $\mathcal{M}'$ are image-finite models. Then $\mathcal{M}, w \leftrightarrows_{\mathcal{BML}(\mathsf{DD})} \mathcal{M}', w'$ if and only if w and w' satisfy the same $\mathcal{BML}(\mathsf{DD})$-formulas.*

Proof. The left-to-right implication follows from Theorem 8. For the opposite implication let $\mathcal{M} = (W, R, V)$, $\mathcal{M}' = (W', R', V')$, and let $\leftrightsquigarrow$ be the $\mathcal{BML}(\mathsf{DD})$-equivalence relation between $\mathcal{M}$ and $\mathcal{M}'$, that is, the relation over $W \times W'$ such that $w \leftrightsquigarrow w'$ if and only if w and w' satisfy the same $\mathcal{BML}(\mathsf{DD})$-formulas. If $\leftrightsquigarrow$ is empty, the implication holds vacuously. Hence, we assume that $v \leftrightsquigarrow v'$, for some $v \in W$ and $v' \in W'$. We will show that $\leftrightsquigarrow$ is a $\mathcal{BML}(\mathsf{DD})$-bisimulation.

We start by observing that $\mathit{Names}_B(\mathcal{M}) = \mathit{Names}_B(\mathcal{M}')$. Indeed, if $\varphi \in \mathit{Names}_B(\mathcal{M})$ and $\varphi \notin \mathit{Names}_B(\mathcal{M}')$, then $\mathcal{M}, v \models @_\varphi \top$, but $\mathcal{M}', v' \not\models @_\varphi \top$. Thus $v \not\leftrightsquigarrow v'$. To show that $\leftrightsquigarrow$ is an $\mathcal{ML}$-bisimulation, we can use the same argumentation as in the proof of Hennessy-Milner property for $\mathcal{ML}$, which shows that the $\mathcal{ML}$-equivalence is an $\mathcal{ML}$-bisimulation (Blackburn, De Rijke, and Venema 2002)[Theorem 2.24]. It remains to show that $\leftrightsquigarrow$ satisfies Conditions Dom and Rng. To show that $\leftrightsquigarrow$ satisfies Condition Dom, we assume that $w \in \mathit{NamedWorlds}_B(\mathcal{M})$, for some $w \in W$. Hence, there exists $\varphi \in \mathit{Names}_B(\mathcal{M})$ which is satisfied only in w. Since $\mathit{Names}_B(\mathcal{M}) = \mathit{Names}_B(\mathcal{M}')$, $\varphi \in \mathit{Names}_B(\mathcal{M}')$, and so, there is a unique world $w' \in W'$ such that $\mathcal{M}', w' \models \varphi$. We claim that $w \leftrightsquigarrow w'$. Indeed, if $\mathcal{M}, w \models \psi$, for some $\mathcal{BML}(\mathsf{DD})$-formula ψ, then $\mathcal{M}, v \models @_\varphi \psi$. As $v \leftrightsquigarrow v'$, we obtain that $\mathcal{M}', v' \models @_\varphi \psi$, and since w' is the only world in $\mathcal{M}'$ which satisfies φ, we obtain that $\mathcal{M}', w' \models \psi$. Similarly we can show that $\mathcal{M}', w' \models \psi$ implies that $\mathcal{M}, w \models \psi$. Hence we conclude that the domain of $\leftrightsquigarrow$ contains $\mathit{NamedWorlds}_B(\mathcal{M})$, that is, $\leftrightsquigarrow$ satisfies Condition Dom. The proof for Condition Rng is symmetric. □

As we have shown, $\mathcal{ML}(\mathsf{DD})$- and $\mathcal{BML}(\mathsf{DD})$-bisimulations satisfy both the bisimulation invariance and the Hennessy-Milner properties. The question arises, however, how to check if a given relation satisfies the new conditions introduced in our definitions of bisimulations. In particular, how can we check if $\mathit{Names}(\mathcal{M}) = \mathit{Names}(\mathcal{M}')$ and $\mathit{Names}_B(\mathcal{M}) = \mathit{Names}_B(\mathcal{M}')$? In the next section we will show not only that these conditions can be checked efficiently (in polynomial time), but also that we can check them by using known algorithms for constructing the standard $\mathcal{ML}$-bisimulation.

4 Algorithms

In this section we will show algorithms which, given a pair of models, compute maximal $\mathcal{ML}(\mathsf{DD})$- and $\mathcal{BML}(\mathsf{DD})$-bisimulations between them. Since models are given as an input, we assume in this section that both of them are finite.

4.1 Algorithm for $\mathcal{ML}(\mathsf{DD})$

In general, our method of computing the maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation between models $\mathcal{M}$ and $\mathcal{M}'$, will consist in constructing the maximal $\mathcal{ML}$-bisimulation Z between them and checking if $\mathit{Names}(\mathcal{M}) = \mathit{Names}(\mathcal{M}')$. If $\mathit{Names}(\mathcal{M}) = \mathit{Names}(\mathcal{M}')$, we will output Z, and otherwise we will output $\emptyset$. Clearly, the main challenge is to check whether $\mathit{Names}(\mathcal{M}) = \mathit{Names}(\mathcal{M}')$. Note that finding a formula which witnesses the fact that $\mathit{Names}(\mathcal{M}) \neq \mathit{Names}(\mathcal{M}')$ cannot be done in a brute force manner, as there are infinitely many $\mathcal{ML}$-formulas which can be a name in one model but not in the other.

The main result of this subsection is that checking if $\mathit{Names}(\mathcal{M}) = \mathit{Names}(\mathcal{M}')$ can be performed efficiently based on the form of Z. In particular, the next theorem shows that checking if $\mathit{Names}(\mathcal{M}) = \mathit{Names}(\mathcal{M}')$ reduces to checking if Z is a total relation and if the restriction of Z to $\mathit{NamedWorlds}(\mathcal{M}) \times \mathit{NamedWorlds}(\mathcal{M}')$ is also total. This provides us with a very efficient and practical approach for checking whether $\mathit{Names}(\mathcal{M}) = \mathit{Names}(\mathcal{M}')$. Note that, for clarity of presentation, we assume in the theorem that $\mathit{Names}(\mathcal{M}) \neq \emptyset$ and $\mathit{Names}(\mathcal{M}') \neq \emptyset$, which can be easily checked, as we will discuss afterwards.

Theorem 10. *Let $\mathcal{M}$ and $\mathcal{M}'$ be finite models such that $\mathit{Names}(\mathcal{M}) \neq \emptyset$ and $\mathit{Names}(\mathcal{M}') \neq \emptyset$. Then the following statements are equivalent:*

1. *$\mathit{Names}(\mathcal{M}) = \mathit{Names}(\mathcal{M}')$,*
2. *the maximal $\mathcal{ML}$-bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ is a total relation and the restriction of Z to $\mathit{NamedWorlds}(\mathcal{M}) \times \mathit{NamedWorlds}(\mathcal{M}')$ is also total*[1].

Proof. Assume first that Statement 2 holds. To show that Statement 1 holds, fix $\varphi \in \mathit{Names}(\mathcal{M})$. Hence $\mathcal{M}, w \models \varphi$, for some $w \in \mathit{NamedWorlds}(\mathcal{M})$. By Statement 2 the restriction of Z to $\mathit{NamedWorlds}(\mathcal{M}) \times \mathit{NamedWorlds}(\mathcal{M}')$ is a total relation, so there is $w' \in \mathit{NamedWorlds}(\mathcal{M}')$ such that $(w, w') \in Z$. Hence, by Bisimulation Invariance Lemma for $\mathcal{ML}$, we obtain that $\mathcal{M}', w' \models \varphi$. To prove that $\varphi \in \mathit{Names}(\mathcal{M}')$ it remains to show that there is no v' in $\mathcal{M}'$ which is distinct from w' and such that $\mathcal{M}', v' \models \varphi$. Suppose towards a contradiction that there exists such v'. By Statement 2 relation Z is total, so there exists v in $\mathcal{M}$ such that $(v, v') \in Z$, and since Z is an $\mathcal{ML}$-bisimulation, $\mathcal{M}, v \models \varphi$. Moreover, since $\varphi \in \mathit{Names}(\mathcal{M})$, we obtain that $v = w$. Hence, we have shown that $(w, w') \in Z$ and $(w, v') \in Z$ which, by the Bisimulation Invariance Lemma for $\mathcal{ML}$, implies w' and v' satisfy the same $\mathcal{ML}$-formulas. Therefore, $w' \notin \mathit{NamedWorlds}(\mathcal{M})$, which raises a contradiction. We can show analogously that $\varphi \in \mathit{Names}(\mathcal{M}')$ implies that $\varphi \in \mathit{Names}(\mathcal{M})$. Hence Statement 1 holds.

For the opposite implication assume that Statement 1 holds and let Z be the maximal $\mathcal{ML}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$. To show Statement 2, we need to show that Z and its restriction to $\mathit{NamedWorlds}(\mathcal{M}) \times \mathit{NamedWorlds}(\mathcal{M}')$ are total. For this, it suffices to show:

[1]In the paper by a total relation we mean a relation which is both left-total and right-total.

(i) for any world w in $\mathcal{M}$, if $w \in$ *NamedWorlds*$(\mathcal{M})$, then there is $w' \in$ *NamedWorlds*$(\mathcal{M}')$ such that $(w, w') \in Z$, and if $w \notin$ *NamedWorlds*$(\mathcal{M})$, there is some w' in $\mathcal{M}'$ such that $(w, w') \in Z$, and

(ii) for any world w' in $\mathcal{M}'$, if $w' \in$ *NamedWorlds*$(\mathcal{M}')$, there is $w \in$ *NamedWorlds*$(\mathcal{M})$ such that $(w, w') \in Z$, and if $w' \notin$ *NamedWorlds*$(\mathcal{M}')$, there is some world w in $\mathcal{M}$ such that $(w, w') \in Z$.

In what follows we will focus on showing Statement (i) since the proof of Statement (ii) is analogous.

Assume that $w \in$ *NamedWorlds*$(\mathcal{M})$. Hence there is $\varphi \in$ *Names*$(\mathcal{M})$ such that $\mathcal{M}, w \models \varphi$. By Statement 1 we obtan that $\varphi \in$ *Names*$(\mathcal{M}')$, so $\mathcal{M}', w' \models \varphi$ for some $w' \in$ *NamedWorlds*$(\mathcal{M}')$. We will show that $(w, w') \in Z$. By the Hennessy-Milner property for $\mathcal{ML}$ and the fact that $\mathcal{M}$ and $\mathcal{M}'$ are finite (so in particular image-finite), it suffices to show that w and w' satisfy the same $\mathcal{ML}$-formulas ψ. If $\mathcal{M}, w \models \psi$, then $\varphi \wedge \psi$ holds in $\mathcal{M}$ only at w, and so, $(\varphi \wedge \psi) \in$ *Names*$(\mathcal{M})$. Thus, by Statement 1, $(\varphi \wedge \psi) \in$ *Names*$(\mathcal{M}')$, and so, $\mathcal{M}', w' \models \varphi \wedge \psi$. Analogously $\mathcal{M}', w' \models \psi$ implies that $\mathcal{M}, w \models \psi$, so w and w' indeed satisfy the same $\mathcal{ML}$-formulas.

Next we assume that $w \notin$ *NamedWorlds*$(\mathcal{M})$. Since *Names*$(\mathcal{M}') \neq \emptyset$, there is some $v' \in$ *NamedWorlds*$(\mathcal{M}')$. We let $U = W \cup W'$ be the union of worlds W in $\mathcal{M}$ and worlds W' in $\mathcal{M}'$ (we assume that W and W' are disjoint) and we let $\leftrightsquigarrow$ be the $\mathcal{ML}$-equivalence relation over U. For each $u \in U$ we let φ_u be an arbitrarily chosen $\mathcal{ML}$-formula which holds at u (in the model containing this world), but which does not hold at any $u' \in U$ such that $u' \not\leftrightsquigarrow u$ (the existence of φ_u follows from the fact that both W and W' are finite). We define $\varphi = \bigwedge_{u \in W' \setminus \{v'\}} \neg \varphi_u$. Since $v' \in$ *NamedWorlds*$(\mathcal{M}')$, we obtain that $v' \not\leftrightsquigarrow u$ for each $u \in W' \setminus \{v'\}$, and so, $\mathcal{M}', v' \models \varphi$. Moreover, v' is the only world in $\mathcal{M}'$ which satisfies φ, so $\varphi \in$ *Names*$(\mathcal{M}')$. Hence, by Statement 1, $\varphi \in$ *Names*$(\mathcal{M})$. Now suppose towards a contradiction that there is no $w' \in W'$ such that $(w, w') \in Z$. Thus, by the Hennessy-Milner property for $\mathcal{ML}$, we obtain that $w \not\leftrightsquigarrow w'$, for all $w' \in W'$. Therefore, by the definition of φ, we obtain that $\mathcal{M}, w \models \varphi$. Hence $w \in$ *NamedWorlds*$(\mathcal{M})$, which raises a contradiction. ☐

Equipped with Theorem 10 we are ready to provide Algorithm 1, which computes the maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation between a pair of models. The algorithm computes three maximal $\mathcal{ML}$-bisimulations: Z between $\mathcal{M}$ and $\mathcal{M}'$ (Line 1), Z_1 between $\mathcal{M}$ and itself (Line 2), and Z_2 between $\mathcal{M}'$ and itself (Line 3). Bisimulations Z_1 and Z_2 are used to compute, respectively, $N =$ *NamedWorlds*$(\mathcal{M})$ in Line 4 and $N' =$ *NamedWorlds*$(\mathcal{M}')$ in Line 5. If $N = N' = \emptyset$, then neither $\mathcal{M}$ or $\mathcal{M}'$ have named worlds. Hence, the maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation is Z, and so, the algorithm returns it in Line 6. If one of N and N' is empty but the other is not, then *NamedWorlds*$(\mathcal{M}) \neq$ *NamedWorlds*$(\mathcal{M}')$, and so the maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation is the empty relation. Hence, the algorithm returns $\emptyset$ in Line 7. Otherwise, the algorithm checks conditions from Statement 2 in Theorem 10. If they are satisfied, Z is returned in Line 8. If not, the maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation is the empty relation, which is returned in Line 9.

Algorithm 1: Maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation

Input: models $\mathcal{M} = (W, R, V)$, $\mathcal{M}' = (W', R', V')$
Output: the maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$

1 $Z := \textsc{MaxBsimML}(\mathcal{M}, \mathcal{M}')$;
2 $Z_1 := \textsc{MaxBsimML}(\mathcal{M}, \mathcal{M})$;
3 $Z_2 := \textsc{MaxBsimML}(\mathcal{M}', \mathcal{M}')$;
4 $N := \{w \in W \mid (w, v) \notin Z_1$ for all $v \neq w$ in $W\}$;
5 $N' := \{w \in W' \mid (w, v) \notin Z_2$ for all $v \neq w$ in $W'\}$;
6 **if** $N = N' = \emptyset$ **then return** Z;
7 **if** $N = \emptyset \neq N'$ *or* $N' \neq \emptyset = N'$ **then return** $\emptyset$;
8 **if** Z *is total over* $W \times W'$ *and* $Z \cap (N \times N')$ *is total over* $N \times N'$ **then return** Z;
9 **else return** $\emptyset$;

Theorem 11. *Algorithm 1 outputs the maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$.*

Proof. Since Z computed in the algorithm is the maximal $\mathcal{ML}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$, the output of the algorithm needs to be Z if *Names*$(\mathcal{M}) =$ *Names*$(\mathcal{M}')$, and $\emptyset$ otherwise. Since $\mathcal{M}$ is finite, by the Hennessy-Milner property for $\mathcal{ML}$, the set *Names*$(\mathcal{M})$ consists of all worlds in $\mathcal{M}$ which are not $\mathcal{ML}$-bisimilar with any other world in $\mathcal{M}$. Thus N computed in Line 4 coincides with *Names*$(\mathcal{M})$, whereas the set N' from Line 5 is *Names*$(\mathcal{M}')$. If $N = N' = \emptyset$ then *Names*$(\mathcal{M}) =$ *Names*$(\mathcal{M}')$, so Z is correctly returned in Line 6. If one of N and N' is empty but the other is not, then *NamedWorlds*$(\mathcal{M}) \neq$ *NamedWorlds*$(\mathcal{M}')$, so $\emptyset$ is correctly returned in Line 7. Otherwise, *Names*$(\mathcal{M}) \neq \emptyset$ and *Names*$(\mathcal{M}') \neq \emptyset$. Hence, *Names*$(\mathcal{M}) =$ *Names*$(\mathcal{M}')$ if and only if Statement 2 from Theorem 10 holds true. The conditions checked in Line 8 coincides with Statement 2 so if they hold, the algorithm correctly returns Z in Line 8, and otherwise it correctly returns $\emptyset$ in Line 9 ☐

It is worth to observe that our computation of the maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation is feasible in polynomial time. Indeed, computing the maximal $\mathcal{ML}$-bisimulation between two models is feasible in polynomial time and so are all additional computations of Algorithm 1.

Theorem 12. *Algorithm 1 terminates in polynomial time.*

Importantly, Algorithm 1 not only terminates in polynomial time, but also most of its computations consist in constructing standard $\mathcal{ML}$-bisimulations. This allows us to delegate most of the work to an off-the-shelf approach for computing the standard $\mathcal{ML}$-bisimulation.

4.2 Algorithm for $\mathcal{BML}(\mathsf{DD})$

To construct the maximal $\mathcal{BML}(\mathsf{DD})$-bisimulation between models $\mathcal{M}$ and $\mathcal{M}'$, it is crucial to check whether *Names*$_B(\mathcal{M}) =$ *Names*$_B(\mathcal{M}')$. Each element of *Names*$_B(\mathcal{M}) \cup$ *Names*$_B(\mathcal{M})$ is a Boolean formula over the signature of $\mathcal{M}$ and $\mathcal{M}'$ (i.e., over the set of propositional variables satisfied in $\mathcal{M}$ or $\mathcal{M}'$). There is a bounded number of non-equivalent formulas among them, so one could determine which of them belong to $Names_B(\mathcal{M})$ and which to $Names_B(\mathcal{M}')$. The number of the formulas to check, however, is exponentially large. Instead, we will introduce a polynomial procedure which exploits the following modification of Theorem 10, where $\mathcal{PC}$ is propositional calculus and we say that two worlds are $\mathcal{PC}$-equivalent if they satisfy the same Boolean formulas.

Theorem 13. *Let $\mathcal{M}$ and $\mathcal{M}'$ be finite models such that $Names_B(\mathcal{M}) \neq \emptyset$ and $Names_B(\mathcal{M}') \neq \emptyset$. Then the following statements are equivalent:*

1. *$Names_B(\mathcal{M}) = Names_B(\mathcal{M})$,*
2. *the $\mathcal{PC}$-equivalence relation between worlds in $\mathcal{M}$ and $\mathcal{M}'$ is a total relation, and its restriction to $NamedWorlds_B(\mathcal{M}) \times NamedWorlds_B(\mathcal{M}')$ is also total.*

Proof sketch. The proof is similar to the one from Theorem 10. The main difference is that we replace *Names* with $Names_B$, *NamedWorlds* with $NamedWorlds_B$, and $\mathcal{ML}$-bisimulation with $\mathcal{PC}$-equivalence. □

We exploit Theorem 13 to compute, in polynomial time, the maximal $\mathcal{BML}(\mathsf{DD})$-bisimulation between a pair of input models $\mathcal{M}$ and $\mathcal{M}'$. Our approach is presented in Algorithm 2. The algorithm computes the set $N = NamedWorlds_B(\mathcal{M})$ in Line 2 and the set $N' = NamedWorlds_B(\mathcal{M}')$ in Line 3. However, unlike exploiting $\mathcal{ML}$-bisimulations Z_1 and Z_2 in Algorithm 1, the sets N and N' are computed in Algorithm 2 directly, namely by comparing the sets $\mathsf{PROP}(w)$ of propositional variables which hold in a given world w. Then, in Line 6, the $\mathcal{PC}$-equivalence E between worlds of $\mathcal{M}$ and $\mathcal{M}'$ is computed. Relation E and Theorem 13 are used in Line 6, similarly as Z and Theorem 10 are used in Algorithm 1.

Algorithm 2: Maximal $\mathcal{BML}(\mathsf{DD})$-bisimulation

Input: models $\mathcal{M} = (W, R, V)$, $\mathcal{M}' = (W', R', V')$
Output: the maximal $\mathcal{BML}(\mathsf{DD})$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$

1 $Z \coloneqq$ MAXBSIMML$(\mathcal{M}, \mathcal{M}')$;
2 $N \coloneqq \{w \in W \mid \mathsf{PROP}(w) \neq \mathsf{PROP}(v)$ for all $v \neq w$ in $W\}$;
3 $N' \coloneqq \{w' \in W' \mid \mathsf{PROP}(w') \neq \mathsf{PROP}(v')$ for all $v' \neq w'$ in $W'\}$;
4 **if** $N = N' = \emptyset$ **then return** Z;
5 **if** $N = \emptyset \neq N'$ *or* $N' \neq \emptyset = N'$ **then return** $\emptyset$;
6 $E \coloneqq \{(w, w') \mid \mathsf{PROP}(w) = \mathsf{PROP}(w')$ where $w \in W$ and $w' \in W'\}$;
7 **if** *E is total over $W \times W'$ and $E \cap (N \times N')$ is total over $N \times N'$* **then return** Z;
8 **else return** $\emptyset$;

Theorem 14. *Algorithm 2 outputs the maximal $\mathcal{BML}(\mathsf{DD})$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$.*

Proof. Algorithm 2 has a similar structure to Algorithm 1, so after showing that N is $NamedWorlds_B(\mathcal{M})$, N' is $NamedWorlds_B(\mathcal{M}')$, and E is the $\mathcal{PC}$-equivalence between worlds in $\mathcal{M}$ and $\mathcal{M}'$, we can apply an analogous argumentation as in Theorem 11. To show that N is $NamedWorlds_B(\mathcal{M})$, assume that $w \in N$. Therefore, by Line 2, $\mathsf{PROP}(w) \neq \mathsf{PROP}(v)$, for all $v \in W$ distinct from w. Hence, $\bigwedge_{p \in \mathsf{PROP}(w)} p \wedge \bigwedge_{p \in \mathsf{PROP} \setminus \mathsf{PROP}(w)} \neg p$ is a name of w, where PROP is the signature of $\mathcal{M}$ (i.e., the set of all propositional variables which are satisfied in some worlds of $\mathcal{M}$). Thus, $w \in NamedWorlds_B(\mathcal{M})$. If $w \in NamedWorlds_B(\mathcal{M})$, then there is a Boolean formula φ which holds only in w, and so, there is no $v \neq w$ in $\mathcal{M}$ such that $\mathsf{PROP}(w) = \mathsf{PROP}(v)$. Therefore $w \in N$. Similarly we can show that $N' = NamedWorlds_B(\mathcal{M}')$. Finally, $\mathsf{PROP}(w) = \mathsf{PROP}(w')$ means that w and w' are $\mathcal{PC}$-equivalent, so E computed in Line 8 is the $\mathcal{PC}$-equivalence relation, as required. □

We observe that all the computations in Algorithm 2 are feasible in polynomial time. This, in particular, includes computing the maximal $\mathcal{ML}$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$, sets N and N', as well as the relation E. Thus the following result holds.

Theorem 15. *Algorithm 2 terminates in polynomial time.*

4.3 Comparison of Bisimulations

We can observe that, as one would expect from appropriate notions of $\mathcal{ML}(\mathsf{DD})$-, $\mathcal{BML}(\mathsf{DD})$-, and $\mathcal{ML}$-bisimulations, $\mathcal{ML}(\mathsf{DD})$-bisimilar worlds are also $\mathcal{BML}(\mathsf{DD})$-bisimilar, and $\mathcal{BML}(\mathsf{DD})$-bisimilar worlds are also $\mathcal{ML}$-bisimilar. Moreover, which is also desired, we can show that none of the opposite implications hold.

Proposition 16. *The following hold:*

if $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}}_{\mathcal{ML}(\mathsf{DD})} \mathcal{M}', w'$, then $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}}_{\mathcal{BML}(\mathsf{DD})} \mathcal{M}', w'$,

if $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}}_{\mathcal{BML}(\mathsf{DD})} \mathcal{M}', w'$, then $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}}_{\mathcal{ML}} \mathcal{M}', w'$,

for any model $\mathcal{M}$ with a world w and any model $\mathcal{M}'$ with a world w'. However, none of the opposite implications holds.

Proof. By the definition, every $\mathcal{BML}(\mathsf{DD})$-bisimulation is an $\mathcal{ML}$-bisimulation, so $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}}_{\mathcal{BML}(\mathsf{DD})} \mathcal{M}', w'$ implies that $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}}_{\mathcal{ML}} \mathcal{M}', w'$. The example from Figure 1 (a) shows that the opposite implication does not hold and the example from Figure 3 shows that $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}}_{\mathcal{BML}(\mathsf{DD})} \mathcal{M}', w'$ does not imply $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}}_{\mathcal{ML}(\mathsf{DD})} \mathcal{M}', w'$.

It remains to show that $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}}_{\mathcal{ML}(\mathsf{DD})} \mathcal{M}', w'$ implies that $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}}_{\mathcal{BML}(\mathsf{DD})} \mathcal{M}', w'$. If $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}}_{\mathcal{ML}(\mathsf{DD})} \mathcal{M}', w'$, then $Names(\mathcal{M}) = Names(\mathcal{M}')$ and there exists an $\mathcal{ML}(\mathsf{DD})$-bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ such that $(w, w') \in Z$. If $Names(\mathcal{M}) = Names(\mathcal{M}') = \emptyset$, then $Names_B(\mathcal{M}) = Names_B(\mathcal{M}') = \emptyset$, and so, Conditions Dom and Rng hold vacuously. Hence, Z is a $\mathcal{BML}(\mathsf{DD})$-bisimulation and $\mathcal{M}, w \mathrel{\underline{\leftrightarrow}}_{\mathcal{BML}(\mathsf{DD})} \mathcal{M}', w'$. If $Names(\mathcal{M}) = Names(\mathcal{M}') \neq \emptyset$, then we let Z' be the maximal $\mathcal{ML}(\mathsf{DD})$-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$. Hence, by Theorem 10, the restriction of Z' to *NamedWorlds*$(\mathcal{M}) \times$ *NamedWorlds*$(\mathcal{M}')$ is total. Since *NamedWorlds*$_B(\mathcal{M}) \subseteq$ *NamedWorlds*$(\mathcal{M})$ and *NamedWorlds*$_B(\mathcal{M}') \subseteq$ *NamedWorlds*$(\mathcal{M}')$, Conditions Dom and Rng hold, so Z' is a $\mathcal{BML}(\mathsf{DD})$-bisimulation. Moreover $(w, w') \in Z'$, so $\mathcal{M}, w \leftrightarrows_{\mathcal{ML}(\mathsf{DD})} \mathcal{M}', w'$. □

Note that in this section we assumed that models are finite, which we used in the proof above. In particular we used this assumption to apply Theorem 10. Our hypothesis is that in the case of infinite models $\mathcal{M}, w \leftrightarrows_{\mathcal{ML}(\mathsf{DD})} \mathcal{M}', w'$ also implies that $\mathcal{M}, w \leftrightarrows_{\mathcal{BML}(\mathsf{DD})} \mathcal{M}', w'$, but we leave verifying this hypothesis for future work.

5 Application of the New Bisimulations

In this section we will apply the newly introduced bisimulations to show what operators are not definable in $\mathcal{ML}(\mathsf{DD})$ and $\mathcal{BML}(\mathsf{DD})$. To show that some operator O is not expressible in a language L, it suffices to provide an L-bisimulation relating a world w to a world w' such that a formula with operator O holds in w but not in w'. In this way we will show that $\mathcal{ML}(\mathsf{DD})$ (and so, also $\mathcal{BML}(\mathsf{DD})$) cannot define the following broadly studied modal operators: the *difference* operator D, the *everywhere* (also known as the universal) operator A, its dual *somewhere* operator E, and the counting operators $\exists_{\geq n}$ for all positive integers n. Semantics of these operators is as follows:

$\mathcal{M}, w \models \mathsf{D}\varphi$	iff	$\mathcal{M}, v \models \varphi$, for some $v \neq w$,
$\mathcal{M}, w \models \mathsf{A}\varphi$	iff	$\mathcal{M}, v \models \varphi$, for all worlds v,
$\mathcal{M}, w \models \mathsf{E}\varphi$	iff	$\mathcal{M}, v \models \varphi$, for some world v,
$\mathcal{M}, w \models \exists_{\geq n}\varphi$	iff	$\mathcal{M}, v \models \varphi$, for at least n distinct v.

Theorem 17. *The following are not definable in $\mathcal{ML}(\mathsf{DD})$: the difference operator D, the everywhere operator A, the somewhere operator E, and counting operators $\exists_{\geq n}$ for all positive integers n.*

Proof. To show the D, A, and E are not definable in $\mathcal{ML}(\mathsf{DD})$ we construct models $\mathcal{M}$ and $\mathcal{M}'$ together with an $\mathcal{ML}(\mathsf{DD})$ bisimulation Z between them, as depicted in Figure 5(a). It suffices to observe that $\mathcal{M}, w_1 \models \mathsf{A}p$, but $\mathcal{M}', w_1' \not\models \mathsf{A}p$. Also $\mathcal{M}, w_1 \not\models \mathsf{D}\neg p$, but $\mathcal{M}', w_1' \models \mathsf{D}\neg p$. Moreover, $\mathcal{M}, w_1 \not\models \mathsf{E}\neg p$, but $\mathcal{M}', w_1' \models \mathsf{E}\neg p$.

Next we show that $\exists_{\geq n}$, for any $n \geq 1$, is not definable in $\mathcal{ML}(\mathsf{DD})$. If $n = 1$ then $\exists_{\geq n}$ coincides with E which, as we have shown, is not definable in $\mathcal{ML}(\mathsf{DD})$. To show that $\exists_{\geq n}$ with $n \geq 2$ is also not definable, we construct models $\mathcal{N}$ and $\mathcal{N}'$, as well as an $\mathcal{ML}(\mathsf{DD})$-bisimulation Z' between them, as presented in Figure 5(b). Since $(v_1, v_1') \in Z'$ and $\mathcal{N}, v_1 \not\models \exists_{\geq n}p$, but $\mathcal{N}', v_1' \models \exists_{\geq n}p$, the result follows. □

Theorem 17 reveals an interesting relation between definite descriptions and the everywhere operator A. On the one hand, satisfiability checking in modal logic with A reduces in logarithmic space to satisfiability checking in $\mathcal{ML}(\mathsf{DD})$ and both problems are ExpTime-complete (Wałęga and Zawidzki 2023, Theorem 4). On the other hand, as shown in Theorem 17, there is no equivalence preserving translation from modal logic with A to $\mathcal{ML}(\mathsf{DD})$.

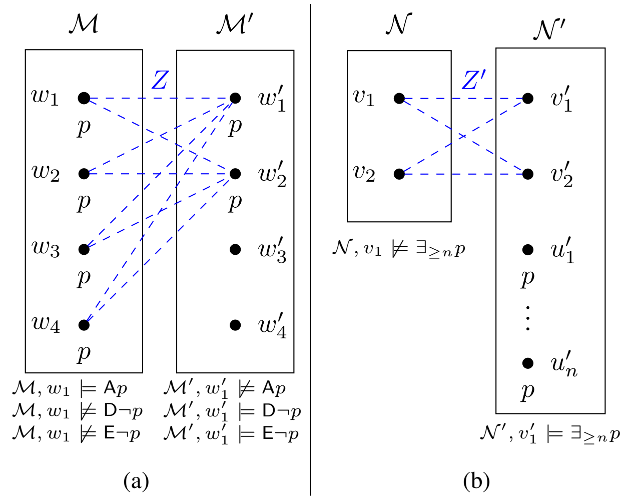

Figure 5: $\mathcal{ML}(\mathsf{DD})$-bisimulations Z and Z' witnessing non-definability of D, A, E, and $\exists_{\geq n}$ in $\mathcal{ML}(\mathsf{DD})$

6 Conclusions

In this paper we have introduced bisimulations adequate for the propositional modal logic with definite descriptions; in particular, for the logic $\mathcal{ML}(\mathsf{DD})$ where definite description operators $@_\varphi$ can mention arbitrarily complex formulas φ, and for the logic $\mathcal{BML}(\mathsf{DD})$ where subscripts φ of the definite description operators can be Boolean formulas only. Our notions of $\mathcal{ML}(\mathsf{DD})$- and $\mathcal{BML}(\mathsf{DD})$-bisimulations satisfy both the bisimulation invariance and Hennessy-Milner properties, which makes them important tools for studying expressive power of the corresponding logics.

To establish desired notions of bisimulations, we have introduced a new type of conditions, which requires that models have the same names. Checking if models $\mathcal{M}$ and $\mathcal{M}'$ have the same names is a non-trivial task, but we managed to show that it can be verified by inspecting the form of the maximal $\mathcal{ML}$-bisimualtion between $\mathcal{M}$ and $\mathcal{M}'$. This allowed us to provide polynomial-time algorithms for computing maximal $\mathcal{ML}(\mathsf{DD})$- and $\mathcal{BML}(\mathsf{DD})$-bisimulations between a pair of models. Most of the computations performed by these algorithms consists in constructing standard $\mathcal{ML}$-bisimulations, which can be delegated to off-the-shelf methods. Finally, we have applied our notions of bisimulations, to show that neither $\mathcal{ML}(\mathsf{DD})$ nor $\mathcal{BML}(\mathsf{DD})$ is expressive enough to define the difference, everywhere, somewhere, and counting operators.

To the best of our knowledge, requiring that models have the same names is a novel condition, which has not been considered in the research on bisimulations (Blackburn, De Rijke, and Venema 2002; Areces, Hoffmann, and Denis 2010; De Rijke 1992; Artale et al. 2021). We believe that this condition can be useful for introducing bisimulations for a wider range of modal logics with definite descriptions, including description, temporal, and spatial logics. In future we plan to verify this hypothesis.

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