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Total Preorders vs. Ranking Functions under Belief Revision – the Dynamics of Empty Layers

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Abstract

Total preorders and Spohn's ranking functions are most popular semantic structures in nonmonotonic reasoning and belief revision. Each ranking function uniquely induces a total preorder, while each total preorder corresponds to infinitely many ranking functions because of the empty layers that ranking functions may have. In this paper, we adopt a dynamic perspective and investigate the role of empty layers in belief revision scenarios. We strengthen the notion of (inferential) equivalence of ranking functions by introducing revision equivalence which postulates the equivalence of ranking functions after (most general) revision operations. Moreover, we single out so-called linearly equivalent ranking functions as prototypes of ranking functions with regularly inserted empty layers. Such ranking functions are most suitable to provide an invariance property for revision equivalence which claims that linear equivalence should be preserved. We show that strategic c-revisions ensure (conditional) revision equivalence of linearly equivalent ranking functions if the strategies are adequately chosen, whereas the Darwiche-Pearl postulates for iterated revision alone are not enough to guarantee revision equivalence of ranking functions. We evaluate various other iterated revision approaches from the literature with respect to revision equivalence and preserving linear equivalence under revision. Furthermore, we present an approach to defining equivalence preserving revision operators for ranking functions from revision operators for total preorders.

1 Introduction

Total preorders on possible worlds and Spohn's ranking functions (Spohn 1988) are broadly used in nonmonotonic reasoning and belief revision approaches. Indeed, for AGM belief revision theory (Alchourrón, Gärdenfors, and Makinson 1985), total preorders are a basic requirement (Katsuno and Mendelzon 1991), and for nonmonotonic reasoning, they provide inference relations of high quality (Makinson 1989; Kraus, Lehmann, and Magidor 1990). Ranking functions assign a natural number to each possible world and thus implement a convenient representation of total preoders that makes it easy to specify changes and validate inferences. Ranking functions and total preorders clearly correspond to one another, but ranking functions can have empty layers, i.e., not each natural number is assigned to possible worlds, while total preorders cannot. The benefit of such empty layers that naturally arise when working with ranking functions is not clear prima facie, and they appear to be a bit arbitrary and even obsolete. Moreover, representation issues are encountered — which ranking function would be most adequate to represent total preorders in a revision scenario? And in which cases is the chosen ranking representation irrelevant, and any ranking function representing the total preorder would yield the same revision result in the end?

In this paper, we argue that empty layers show their power and relevance (only) when changing ranking functions. We introduce the formal notion of *revision equivalence*[1] to establish a strong equivalence relation among ranking functions that guarantees the equivalence of ranking functions also under (arbitrary) revisions. We show that strategic c-revisions (Kern-Isberner 2004; Kern-Isberner, Sezgin, and Beierle 2023) can ensure revision equivalence of so-called linearly equivalent ranking functions while general iterated revisions according to the Darwiche-Pearl framework (Darwiche and Pearl 1997) cannot. This reveals that empty layers which are inserted into a ranking function in a most regular way do not influence the qualitative results (i.e., with respect to the induced total preorder) of iterated revision operations when the arithmetics of ranking functions is thoroughly used, as c-revisions do. This is a crucial insight regarding the representation of total preorders by ranking functions in belief revision: Any two revision equivalent ranking functions may be taken as representations for a given total preorder to be (suitably) c-revised, the induced total preorders after change would be the same.

However, empty layers which are inserted into ranking functions in arbitrary ways may heavily influence the qualitative outcomes of change. This is illustrated in various examples, using c-revisions and also other approaches from the literature. On the other hand, we present a general approach to defining revision operators for ranking functions which are able to preserve equivalence from (well-known) approaches to revising total preorders. In this way, our methodology allows for incorporating other approaches to

[1]We would like to thank Hans van Ditmarsch for mentioning this term informally in his presentation at ACLAI'23 in Malaga, Spain, initiating this research work in this way.

iterated revision from the literature easily.

This paper is organized as follows: Section 2 recalls formal basics and notations which are used in the paper. We briefly point out links to related work in Secion 3, and give a more comprehensive description of previous work on iterated belief revision in Section 4. Section 5 introduces our concept of revision equivalence for ranking functions, and Section 6 sharpens this concept to linear revision equivalence. In Section 7, we present our approach to define revision operators for ranking functions from revision operators for total preorders, and we conclude in Section 8.

2 Formal Preliminaries

Let $\mathcal{L}$ be a finitely generated propositional language over an alphabet Σ with atoms $a, b, c, \dots$ and with formulas $A, B, C, \dots$, equipped with the standard connectives $\wedge, \vee, \neg$. For conciseness of notation, we will omit the logical and-connector, writing AB instead of $A \wedge B$, and overlining formulas will indicate negation, i.e., $\overline{A}$ means $\neg A$. Logical equivalence is denoted by $\equiv$, and the set of classical logical consequences of $\mathcal{A} \subseteq \mathcal{L}$ by $Cn(\mathcal{A})$, with $\top$ denoting an arbitrary propositional tautology. The set of all propositional interpretations resp. possible worlds over Σ is denoted by Ω. $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world $\omega \in \Omega$; then ω is called a *model* of A, and the set of all models of A is denoted by $\mathit{Mod}\,(A)$. Similarly, for sets of propositions $\mathcal{S} \subseteq \mathcal{L}$, $\mathit{Mod}\,(\mathcal{S})$ denotes the set of possible worlds that satisfy all elements of $\mathcal{S}$. For propositions $A, B \in \mathcal{L}$, $A \models B$ holds iff $Mod(A) \subseteq Mod(B)$, as usual. Analogously, for sets of propositions $\mathcal{A}, \mathcal{B} \subseteq \mathcal{L}$, $\mathcal{A} \models \mathcal{B}$ holds iff $\mathit{Mod}\,(\mathcal{A}) \subseteq \mathit{Mod}\,(\mathcal{B})$. By slight abuse of notation, we will use ω both for the model and the corresponding conjunction of all positive or negated atoms. This will allow us to ease notation a lot. Since $\omega \models A$ means the same for both readings of ω, no confusion will arise.

We also consider conditionals $(B|A) \in (\mathcal{L}|\mathcal{L})$ which express statements like *"If A then plausibly B"*. A *conditional belief base* Δ is a finite set of conditionals. Though considering contradictions within conditionals is interesting and relevant in principle, we focus on non-contradictory formulas and conditionals here to make the general techniques clearer, leaving the general case for future work. This means that we presuppose for each conditional $(B|A)$ dealt with in this paper that $A, AB \not\equiv \perp$ hold.

In this paper, we focus on epistemic states Ψ which are represented by total preorders (TPO) on Ω: $\Psi = (\Omega, \preceq_\Psi)$. Total preorders $\preceq_\Psi$ stand for plausibility orderings on the set of possible worlds, and are transitive and reflexive total relations. As usual, $\omega_1 \prec_\Psi \omega_2$ if $\omega_1 \preceq_\Psi \omega_2$, but not $\omega_2 \preceq_\Psi \omega_1$, and $\omega_1 \approx_\Psi \omega_2$ if both $\omega_1 \preceq_\Psi \omega_2$ and $\omega_2 \preceq_\Psi \omega_1$. The most plausible worlds are located in the lowermost layer of $\preceq_\Psi$ which we denote by $\min(\Omega, \preceq_\Psi)$. More generally, if $\tilde{\Omega} \subseteq \Omega$ is a subset of possible worlds, $\min(\tilde{\Omega}, \preceq_\Psi)$ denotes the set of minimal worlds in $\tilde{\Omega}$ according to $\preceq_\Psi$. A proposition A is believed in Ψ, $\Psi \models A$, if for all $\omega \in \min(\Omega, \preceq_\Psi)$ it holds that $\omega \models A$, i.e. if $\min(\Omega, \preceq_\Psi) \models A$; the set of all believed propositions in Ψ is denoted by $\mathit{Bel}\,(\Psi)$. The preorder $\preceq_\Psi$ is lifted to a relation between propositions in the usual way: $A \preceq_\Psi B$ if there is $\omega \models A$ such that $\omega \preceq_\Psi \omega'$ for all $\omega' \models B$; equivalently, whenever $\mathit{Mod}\,(A), \mathit{Mod}\,(B)$ are both not empty, if $\min(\mathit{Mod}\,(A), \preceq_\Psi) \preceq_\Psi \min(\mathit{Mod}\,(B), \preceq_\Psi)$. For conditionals $(B|A)$, $(B|A)$ is accepted in Ψ, denoted by $\Psi \models (B|A)$, if $AB \prec_\Psi A\overline{B}$. Note that A is plausibly believed in Ψ iff the conditional $(A|\top)$ is accepted by Ψ. This allows us to subsume plausible beliefs in terms of conditional beliefs, which supports a more coherent view on reasoning and revision.

Ordinal Conditional Functions (OCF, also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$ were firstly introduced by Spohn (Spohn 1988) and implement total preorders by ranks in the ordinals, here natural numbers. These ranks express degrees of implausibility, or surprise. The degree of (im)plausibility of a formula A is defined by $\kappa(A) := \min\{\kappa(\omega) | \omega \models A\}$. Hence, due to $\kappa^{-1}(0) \neq \emptyset$, at least one of $\kappa(A), \kappa(\overline{A})$ must be 0. A proposition A is believed in κ, denoted by $\kappa \models A$, if $\omega \models A$ for all ω such that $\kappa(\omega) = 0$; this is equivalent to saying that $\kappa(\overline{A}) > 0$; the set of all believed propositions in κ is denoted by $\mathit{Bel}\,(\kappa)$. This notion can be extended in a natural way to assign ranks to sets of formulas $\mathcal{S} \subseteq \mathcal{L}$ via $\kappa(\mathcal{S}) = \min\{\kappa(\omega) \mid \omega \models \mathcal{S}\}$. Conditionals are accepted in the epistemic state represented by κ, written as $\kappa \models (B|A)$, if $\kappa(AB) < \kappa(A\overline{B})$. For a subset of possible worlds $\tilde{\Omega} \subseteq \Omega$, $\min(\mathit{Mod}\,(\tilde{\Omega}), \kappa)$ denotes the set of minimal worlds in $\tilde{\Omega}$ according to their ranks in κ. Note that these definitions are in full compliance with corresponding definitions for total preorders.

Both ranking functions and total preorders are organized in *layers*, i.e., subsets $\Omega_0, \Omega_1, \dots$ of worlds which are equivalent with respect to the total preorder resp. have the same rank. Ω_0 is the lowermost layer containing the most plausible worlds. By slight abuse of notation, we sometimes write $\Omega_0 \prec \Omega_1 \prec \dots$ to indicate the ordering of the layers. In contrast to total preorders, ranking functions usually have empty layers, i.e., there are ranks $r \in \mathbb{N}$ with $\kappa^{-1}(r) = \emptyset$.

Total preorders and ranking functions both provide convenient representations of epistemic states for nonmonotonic reasoning and belief revision.

They are fully compatible in that each ranking function induces (uniquely) a total preorder, and each total preorder can be associated with some ranking function. In the paper (Kern-Isberner, Sezgin, and Beierle 2023), this relationship between total preorders and ranking functions are made precise via transformations. We recall the corresponding definitions and technical results in an adapted form here.

Definition 1 ((Kern-Isberner, Sezgin, and Beierle 2023))**.** *Let κ be a ranking function on Ω. The* transformation operator *τ maps κ to an epistemic state $\Psi_\kappa = (\Omega, \preceq_\kappa)$, $\tau : \kappa \mapsto \Psi_\kappa$, such that for all $\omega_1, \omega_2 \in \Omega$*

$$\omega_1 \preceq_\kappa \omega_2 \textit{ iff } \kappa(\omega_1) \leqslant \kappa(\omega_2) \tag{1}$$

holds. The transformation operator *ρ maps each epistemic state $\Psi = (\Omega, \preceq_\Psi)$ to a ranking function κ_Ψ, $\rho : \Psi \mapsto \kappa_\Psi$, by setting*

$$\kappa_\Psi(\omega) = \min_{\kappa \in \tau^{-1}(\Psi)} \{\kappa(\omega)\}. \tag{2}$$

ρ is a well-defined operator such that

$$\kappa_\Psi(\omega_1) \leqslant \kappa_\Psi(\omega_2) \text{ iff } \omega_1 \preceq_\Psi \omega_2 \tag{3}$$

holds, i.e. $\kappa_\Psi \in \tau^{-1}(\Psi)$. We have that $\tau \circ \rho = id$, but $\rho \circ \tau \neq id$ in general.

3 Related Work

Revision equivalence is similar in spirit to uniform equivalence resp. strong equivalence in answer set programming (ASP) (Eiter et al. 2005) which postulates that logic programs yield the same (respective parts of) answer sets even if further facts resp. rules are added. However, while in ASP, the role of a stronger notion of equivalence is to ensure modularity of (parts of) a logic program, we are more interested in guaranteeing representation invariance regarding the TPO structure in the context of this paper.

Our axiomatization of revision equivalence for ranking functions is very similar to the equivalence axiom (E) in forgetting (Gonçalves, Knorr, and Leite 2016; Kern-Isberner et al. 2019). In this paper, we explore equivalence in the framework of iterated revision operators.

4 AGM-based Iterated Revision

We recall basics of iterated revision according to (Darwiche and Pearl 1997) and of c-revisions (Kern-Isberner 2004) in this section.

4.1 Darwiche-Pearl (DP) Revision and Elementary Operators

The DP framework (Darwiche and Pearl 1997) extends the original AGM framework (Alchourrón, Gärdenfors, and Makinson 1985; Katsuno and Mendelzon 1991) by taking iterated belief revision into account. To this end, they utilize total preorders to identify AGM revision operators for epistemic states.

Proposition 1 ((Darwiche and Pearl 1997)). *A revision operator $*$ that assigns a posterior epistemic state $\Psi * A$ to a prior state Ψ and a proposition A is an AGM revision operator for epistemic states iff there exists a total preorder $\preceq_\Psi$ on Ω with $Mod(Bel(\Psi)) = \min(\Omega, \preceq_\Psi)$ such that*

$$Mod(Bel(\Psi * C)) = \min(Mod(C), \preceq_\Psi)$$

holds for every proposition C.

The following postulates for the revision of an epistemic state Ψ equipped with a total preorder $\preceq_\Psi$ with a proposition C have been proposed in (Darwiche and Pearl 1997):

(DP1) If $\omega_1, \omega_2 \models C$, then $\omega_1 \preceq_\Psi \omega_2$ iff $\omega_1 \preceq_{\Psi * C} \omega_2$.

(DP2) If $\omega_1, \omega_2 \not\models C$, then $\omega_1 \preceq_\Psi \omega_2$ iff $\omega_1 \preceq_{\Psi * C} \omega_2$.

(DP3) If $\omega_1 \models C$ and $\omega_2 \not\models C$, then $\omega_1 \prec_\Psi \omega_2$ implies $\omega_1 \prec_{\Psi * C} \omega_2$.

(DP4) If $\omega_1 \models C$ and $\omega_2 \not\models C$, then $\omega_1 \preceq_\Psi \omega_2$ implies $\omega_1 \preceq_{\Psi * C} \omega_2$.

An AGM revision operator for epistemic states (in the sense of Proposition 1) that satisfies postulates (DP1–4) will be called a DP revision operator in this paper. A crucial insight from (Katsuno and Mendelzon 1991; Darwiche and Pearl 1997) is that total preorders (and in particular ranking functions) are adequate representations of epistemic states in the context of (iterated) AGM revision, and we work on these representations. In the following, we often speak of total preorders and ranking functions as epistemic states, tacitly assuming their representative characteristic.

The term *elementary revision operators* (Chandler and Booth 2020) stands for a group of three basic operators for iterated belief revision of epistemic states Ψ (represented by total preorders) with a single proposition A: natural revision $\bullet_n$ (Boutilier 1993), lexicographic revision $\bullet_\ell$ (Nayak, Pagnucco, and Peppas 2003), and restrained revision $\bullet_r$ (Booth and Meyer 2006). Those operators have been characterized by axioms in (Chandler and Booth 2023). We focus on $\bullet_n$ and $\bullet_\ell$ here which can be characterized by the following properties.

(NR) $\omega \preceq_{\Psi \bullet_n A} \omega'$ iff (1) $\omega \in \min(Mod(A), \preceq_\Psi)$, or (2) $\omega, \omega' \notin \min(Mod(A), \preceq_\Psi)$ and $\omega \preceq_\Psi \omega'$.

(LR) $\omega \preceq_{\Psi \bullet_\ell A} \omega'$ iff (1) $\omega \models A$ and $\omega' \not\models A$, or (2) ($\omega \models A$ iff $\omega' \models A$) and $\omega \preceq_\Psi \omega'$.

4.2 Strategic C-Revisions

C-revisions have been introduced in (Kern-Isberner 2001; Kern-Isberner 2004) and are defined for ranking functions as follows (we recall definitions from (Kern-Isberner 2004; Kern-Isberner, Sezgin, and Beierle 2023) here):

Definition 2 (C-revisions for OCFs; $cr^{\Delta}_{\kappa,i}$, $CR(\kappa,\Delta)$)**.** *Let κ be an OCF and $\Delta = \{(B_1|A_1), \dots, (B_n|A_n)\}$ a set of conditionals. Then a* c-revision of κ by Δ *is an OCF $\kappa^* = \kappa * \Delta$ of the form*

$$\kappa^*(\omega) = \kappa_0 + \kappa(\omega) + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B}_i}} \eta_i \tag{4}$$

with nonnegative integers η_i for each $(B_i|A_i)$, satisfying

$$(cr^{\Delta}_{\kappa,i}) \qquad \eta_i > \min_{\omega \models A_i B_i} \Big\{ \kappa(\omega) + \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \eta_j \Big\} - \min_{\omega \models A_i \overline{B}_i} \Big\{ \kappa(\omega) + \sum_{\substack{j \neq i \\ \omega \models A_j \overline{B}_j}} \eta_j \Big\} \tag{5}$$

The constraint satisfaction problem for c-revisions of κ by Δ, *denoted by $CR(\kappa,\Delta)$, is given by the set of constraints $cr^{\Delta}_{\kappa,i}$, for $i \in \{1, \dots, n\}$, where the η_i are constraint variables taking values in $\mathbb{N}$.*

(5) ensures that $\kappa^* \models \Delta$, and the integer κ_0 is a normalizing term, i.e., ensuring that $\kappa^*(\top) = 0$. Each c-revision is an iterated revision in the sense of Darwiche and Pearl (Darwiche and Pearl 1997) because the DP-postulates are implied by the principle of conditional preservation (Kern-Isberner 2004; Kern-Isberner 2018). A solution of $CR(\kappa,\Delta)$ is an n-tuple $\vec{\eta} = (\eta_1, \dots, \eta_n)$ of natural numbers, and each solution defines a proper c-revision $\kappa^*_{\vec{\eta}}$ according to (4) (Kern-Isberner, Sezgin, and Beierle 2023).

Since (4) and (5) provide a general schema for revision operators, many c-revisions are possible. With a selection strategy (Beierle and Kern-Isberner 2021), we can select single, well-defined solutions for any revision problem:

Definition 3 (Selection strategy σ, strategic c-revision $*_\sigma$)**.** *A* selection strategy (for c-revisions) *is a function*

$$\sigma : (\kappa, \Delta) \mapsto \vec{\eta}$$

assigning to each pair of an OCF κ and a (consistent) set of conditionals Δ an impact vector $\vec{\eta}$ that solves $CR(\kappa, \Delta)$. If $\sigma(\kappa, \Delta) = \vec{\eta}$, the c-revision of κ by Δ determined by σ *is $\kappa^*_{\vec{\eta}}$, denoted by $\kappa *_\sigma \Delta$, and $*_\sigma$ is a* strategic c-revision.

5 Revision Equivalence of Ranking Functions

We start with considering (static) inferential equivalence and then broaden our view to take revision operations into account. We also present some first results on when to expect revision equivalence.

5.1 Inferential Equivalence

As can be found, e.g., in (Beierle and Kutsch 2019), two ranking functions κ, κ' are (inferentially) equivalent, $\kappa \cong \kappa'$, iff for all $\omega_1, \omega_2 \in \Omega$ it holds that $\kappa(\omega_1) \leq \kappa(\omega_2)$ iff $\kappa'(\omega_1) \leq \kappa'(\omega_2)$. In particular, $\kappa(\omega_1) = \kappa(\omega_2)$ iff $\kappa'(\omega_1) = \kappa'(\omega_2)$, i.e., equivalent ranking functions have exactly the same layers, but may assign different ranks to possible worlds resp. formulas. Nevertheless, they preserve the property of a world being a minimal model for a formula.

Proposition 2. *Let $\kappa_1 \cong \kappa_2$ be two equivalent OCFs, let $A \in \mathcal{L}_\Sigma$. Then a model ω of A is a minimal model of A with respect to κ_1 iff it is a minimal model of A with respect to κ_2. In particular, $Mod(Bel(\kappa_1)) = Mod(Bel(\kappa_2))$, and for any model ω of A, $\kappa_1(\omega) = \kappa_1(A)$ iff $\kappa_2(\omega) = \kappa_2(A)$.*

Proof. Let $\omega \models A$ be a minimal model of A with respect to κ_1, i.e., $\kappa_1(\omega) \leq \kappa_1(\omega')$ for all models ω' of A. Then also $\kappa_2(\omega) \leq \kappa_2(\omega')$ for all models ω' of A holds, and therefore ω is also a minimal model of A with respect to κ_2.

Since $Mod(Bel(\kappa_1)), Mod(Bel(\kappa_2))$ are the respective minimal models of a tautology $\top$, $Mod(Bel(\kappa_1)) = Mod(Bel(\kappa_2))$ follows as a special case.

Finally, observing that $\kappa_i(\omega) = \kappa_i(A)$ holds if and only if ω is a minimal model of A with respect to κ_i, $i \in \{1, 2\}$, this yields $\kappa_1(\omega) = \kappa_1(A)$ iff $\kappa_2(\omega) = \kappa_2(A)$. ☐

As an immediate consequence of Proposition 2, we obtain that $Bel(\kappa)$ is identical for all inferentially equivalent κ.

Proposition 3. *If $\kappa_1 \cong \kappa_2$ for OCFs κ_1, κ_2, then $Bel(\kappa_1) = Bel(\kappa_2)$.*

Note that two ranking functions κ_1, κ_2 are equivalent iff $\tau(\kappa_1) = \tau(\kappa_2)$, which can be seen immediately from (1). We have that $\rho \circ \tau \neq id$ in general. However, κ and $\rho \circ \tau(\kappa)$ are equivalent.

ω	$\kappa_1(\omega)$	$\kappa_2(\omega)$	$\kappa_1^*(\omega)$	$\kappa_2^*(\omega)$
ab	0	0	1	2
$a\overline{b}$	3	6	3	8
$\overline{a}b$	2	4	2	2
$\overline{a}\overline{b}$	1	2	0	0

Table 1: Ranking functions κ_1, κ_2 and their DP-revisions for Example 1

5.2 Revision Equivalence – Definitions

Equivalent rankings induce the same total preorder but can assign different ranks to possible worlds due to having empty layers at different positions. For the (static) conditional/nonmonotonic inferences they yield, these empty layers have no effect. However, when revising the ranking functions by the same proposition, or the same conditional, empty layers may crucially influence the revised ranking function.

Example 1. *Let κ_1, κ_2 be ranking functions over $\Sigma = \{a, b\}$ as given in Table 1. It is straightforward to see that κ_1, κ_2 are equivalent with $Bel(\kappa_1) = Bel(\kappa_2) = Cn(ab)$. Now, we revise both ranking functions by $\overline{a}$ according to the so-called DP-principles for iterated revision from (Darwiche and Pearl 1997), yielding κ_1^*, κ_2^* (see Table 1). Actually, κ_2^* is a strategic c-revision $\kappa_2^* = \kappa_2 *_\sigma \overline{a}$ with $\sigma(\kappa_2, \{\overline{a}\}) = (4)$, and c-revisions are known to satisfy the DP-principles (Kern-Isberner 2018); for κ_1^*, those principles are easily checked.*

While for both prior ranking functions, we have $\kappa_i(ab) < \kappa_i(\overline{a}b)$, i.e., $\kappa_i \models (a|b)$, $i \in \{1, 2\}$, this still holds for κ_1^*, but not for κ_2^*. In more detail, we have $\kappa_1^*(ab) = 1 < 2 = \kappa_1^*(\overline{a}b)$, but $\kappa_2^*(ab) = 2 = \kappa_2^*(\overline{a}b)$. Therefore, κ_1^* and κ_2^* are no longer equivalent because the position of empty layers in κ_2 affects the revision outcome. Note that $\sigma(\kappa_2, \{\overline{a}\}) = (4)$ has not been chosen minimally. The corresponding constraint system (5) yields $\eta_i > 2$, so also $\sigma(\kappa_2, \{\overline{a}\}) = (3)$ would have been possible, providing a minimal c-revision which would still accept $(a|b)$. So, this is not a failure of c-revisions per se, but amounts to freedom of choice which should be used deliberately. In this case, there is no explicit justification for choosing a number higher than $\sigma(\kappa_2, \{\overline{a}\}) = (3)$. Nevertheless, κ_2^* still fully complies with the DP-framework.

In this example, we observe how empty layers may influence revision operations of equivalent ranking functions. This has important consequences for the representation of total preorders by ranking functions in belief revision scenarios. Both ranking functions represent the total preorder $ab \prec \overline{a}\overline{b} \prec \overline{a}b \prec a\overline{b}$ but revision of κ_1 by $\overline{a}$ yields $\overline{a}\overline{b} \prec ab \prec \overline{a}b \prec a\overline{b}$, while the revision of κ_2 by $\overline{a}$ produces $\overline{a}\overline{b} \prec ab \approx \overline{a}b \prec a\overline{b}$. Note that both revisions are DP-revisions, i.e., even the DP-framework cannot guarantee equivalent revision outcomes of equivalent ranking functions. Since this framework lifts AGM revision (Alchourrón, Gärdenfors, and Makinson 1985) to the epistemic level, it is clear that also AGM revision cannot address this point.

To establish a representation invariance of total preorders (which are fundamental to AGM belief change theory (Katsuno and Mendelzon 1991)) with respect to ranking functions under belief revision, we introduce a stronger notion of equivalence. As a general prerequisite and as indicated in Section 2, we presuppose that all occurring propositions and conditionals resp. sets thereof are consistent. This helps us to focus on the core ideas of the methodology while the more general case of allowing inconsistencies is left for future work.

Definition 4 (Revision equivalence)**.** *Let κ_1, κ_2 be two ranking functions (over the same signature), let $*$ be an iterated revision operator taking ranking functions and (sets of) propositions resp. conditionals as input, and returning a revised ranking function as output.*

- κ_1, κ_2 *are* (propositionally) revision equivalent with respect to $*$*, in symbols* $\kappa_1 \cong^* \kappa_2$*, if* $\kappa_1 * A \cong \kappa_2 * A$ *for all* $A \in \mathcal{L}$*.*
- κ_1, κ_2 *are* conditionally revision equivalent with respect to $*$*, in symbols* $\kappa_1 \cong^{c*} \kappa_2$*, if* $\kappa_1 * (B|A) \cong \kappa_2 * (B|A)$ *for all* $(B|A) \in (\mathcal{L}|\mathcal{L})$*.*
- κ_1, κ_2 *are* universally propositionally/conditionally revision equivalent with respect to $*$*, in symbols* $\kappa_1 \cong^*_u \kappa_2$ *resp.* $\kappa_1 \cong^{c*}_u \kappa_2$*, if* $\kappa_1 * \mathcal{S} \cong \kappa_2 * \mathcal{S}$ *for all* $\mathcal{S} \subset \mathcal{L}$*, resp.* $\kappa_1 * \Delta \cong \kappa_2 * \Delta$ *for all* $\Delta \subset (\mathcal{L}|\mathcal{L})$*.*

In spite of the many postulates that, e.g., the AGM theory, or the DP framework provide, it is clear that (particularly) iterated revision operators dealing with ranking functions and total preorders can be quite arbitrary. To restrict arbitrariness a bit, we often postulate that any revision operator $*$ to be considered for revision equivalence in the following satisfies a stability postulate which is inspired by AGM revision resp. expansion:

(Stability) Let φ be a proper input for a revision operator $*$ for ranking functions, i.e., φ may be a proposition, or a conditional, or even sets thereof. If $\kappa \models \varphi$, then $\kappa * \varphi = \kappa$.

(Stability) claims that the prior ranking function is not changed unnecessarily, i.e., if it already satisfies the success condition of revision. For strategic c-revisions, (Stability) can be ensured by postulating the following for strategies:

(Stab) $\sigma(\kappa, \Delta) = (0, \ldots, 0)$ if $\kappa \models \Delta$.

Since our Definition 4 of revision equivalence is most general and can be applied to any (advanced) revision operator, we also presuppose a compatibility of revising by conditionals with (multiple) propositional revision via the following postulate.

Propositional Compatibility (PC) $\kappa * (A|\top) = \kappa * A$ for all $A \in \mathcal{L}$ and all κ.

In this way, we ensure that conditional revision operators can be used to define (iterated) propositional revisions. E.g., the (strategic) c-revisions recalled in Sec. 4.2 can also be taken as (multiple) propositional revision operators.

Before we explore the technical properties of revision equivalence in more detail, we want to show that (propositional) revision equivalence generalizes (inferential) equivalence, and that conditional revision equivalence generalizes propositional revision equivalence, when the two postulates above are presupposed.

Proposition 4. *Let $*$ be an iterated revision operator taking ranking functions and propositions resp. conditionals as input, and returning a revised ranking function as output.*

- *Revision equivalent ranking functions with respect to $*$ are (inferentially) equivalent if $*$ satisfies (Stability).*
- *Conditionally revision equivalent ranking functions with respect to $*$ are (propositionally) revision equivalent with respect to $*$ equivalent if $*$ satisfies (Propositional Compatibility).*

The proof of this proposition is straightforward, and it is clear that (Propositional Compatibility) can also be applied to sets of propositions, i.e., to multiple revisions (Delgrande and Jin 2012; Kern-Isberner and Huvermann 2017).

This proposition shows that the notions of conditional revision equivalence, propositional revision equivalence, and inferential equivalence are downwards compatible. It helps us with our investigations which ranking functions are revision equivalent with respect to a given revision operator because we can focus on considering (inferentially) equivalent ranking functions.

5.3 Some Negative Results and a First Theorem

It is clear that (conditional) revision equivalence is highly desirable because it allows to take any ranking function that implements a given total preorder via the transformation operator τ for purposes of revision, and any such revision would yield the same total preorder. Hence, revision operators for ranking functions could be unambiguously used for defining (maybe complex) revisions of total preorders. For instance, c-revisions are able to revise ranking functions by sets of conditionals, which is far beyond what is currently possible for total preorders (see, e.g., (Kern-Isberner, Sezgin, and Beierle 2023) for first approaches). However, the next proposition shows that this is not realistic in general, not even in simple cases when revising by just one proposition.

Proposition 5. *Let κ_1, κ_2 be two different, but (inferentially) equivalent ranking functions which both have at least two layers such that their lowermost layer Ω_0 has more than one element. Then there is a strategic c-revision operator $*_\sigma$ and $A \in \mathcal{L}$ such that $\kappa_1 *_\sigma A \not\cong \kappa_2 *_\sigma A$.*

Proof. Let $\kappa_1 \cong \kappa_2$ be as specified in the proposition. Then we can choose $\omega_1 \in \Omega_0$, and ω_2 in another layer, which means that $\kappa_i(\omega_1) = 0$ and $\kappa_i(\omega_2) > 0$ for both ranking functions ($i \in \{1, 2\}$). We c-revise by $A = \omega_1 \vee \omega_2$, i.e., we investigate $\kappa_i^* = \kappa_i *_\sigma A$ for $i \in \{1, 2\}$ with a strategic c-revision $*_\sigma$ such that $\eta^1 = \sigma(\kappa_1, A) = \kappa_1(\omega_2)$ and $\eta^2 = \sigma(\kappa_2, A) = \kappa_2(\omega_2) + 1$. Both κ_1^*, κ_2^* have the form (4), more precisely, for $i \in \{1, 2\}$,

$$\kappa_i^*(\omega) = \kappa_i(\omega) + \begin{cases} 0 & \textit{if } \omega \in \{\omega_1, \omega_2\} \\ \eta^i & \textit{if } \omega \notin \{\omega_1, \omega_2\} \end{cases}$$

and both constraint variables η^1, η^2 comply with the constraint (5) which yields $\eta^i > \kappa_i(\omega_1) = 0$ in this case. Now we consider models of $B = \vee_{\omega\in\Omega_0,\omega\neq\omega_1}\, \omega$. Due to the prerequisite that Ω_0 has more than one element, $B \not\equiv \bot$. For all $\omega \models B$, we have $\kappa_i^*(\omega) = \eta^i$, $i \in \{1,2\}$. According to the selection of η^1, η^2, we now have $\kappa_1^*(\omega) = \kappa_1(\omega_2)$, but $\kappa_2^*(\omega) = \kappa_2(\omega_2) + 1 > \kappa_2(\omega_2)$. Therefore, κ_1^* and κ_2^* cannot be equivalent. □

Proposition 5 states that any two ranking functions as specified in the proposition cannot be (propositionally) revision equivalent under c-revisions. Since any c-revision fully complies with the DP-postulates, such ranking functions cannot be (propositionally) revision equivalent under DP-revisions.

Nevertheless, exploring the prerequisites of Proposition 5 and its proof, we find a specific type of ranking functions that are (even conditionally) revision equivalent with respect to c-revisions.

Proposition 6. *If κ_1, κ_2 are two equivalent ranking functions having exactly two layers Ω_0, Ω_1 such that $\Omega_0 = \{\omega_0\}$ contains exactly one element, then*

$$\kappa_1 *_\sigma (B|A) \cong \kappa_2 *_\sigma (B|A)$$

*for any $A, B \in \mathcal{L}$, and any strategic c-revision $*_\sigma$ whose strategy σ satisfies (**Stab**).*

The quite technical proof of this proposition has been moved to the appendix due to limited space.

Since the uniform ranking function κ_u defined by $\kappa_u(\omega) = 0$ for all $\omega \in \Omega$ is the only ranking function with just one layer, we summarize the insights of this section in the following theorem.

Theorem 7. *Let κ_1, κ_2 be two different ranking functions. Then κ_1, κ_2 are conditionally revision equivalent with respect to strategic c-revisions complying with (**Stab**) iff both κ_1, κ_2 have exactly two layers Ω_0, Ω_1 such that $\Omega_0 = \{\omega_0\}$ contains exactly one element.*

Theorem 7 shows that revision equivalence of ranking functions even under c-revisions can be expected only in very special cases. Therefore, in the next section, we refine both the notion of revision equivalence and the strategies for c-revisions further.

6 Linear Revision Equivalence

As the previous section shows, revision equivalence is hard to achieve in general. In this section, we equip the notion of equivalence of ranking functions with more numerical structure by introducing *linear revision equivalence*. We define this novel concept and show how strategic c-revisions can be made compatible with it. Afterwards, we also evaluate other approaches from the literature regarding linear revision equivalence.

6.1 Definition and Basic Properties

Revision equivalence is a property of ranking functions, but is crucially parameterized by the used revision operator. If the (type of) revision operator is fixed, then the research question is in which cases equivalence of ranking functions can be preserved under revision. Example 1 showed that DP-revision operators are too general to guarantee revision equivalence, and Proposition 5 revealed that also c-revisions cannot preserve equivalence of ranking functions in general. Nevertheless, Proposition 6 gives rise to some hope that we can have revision equivalence under c-revisions for specific classes of ranking functions. Therefore, we focus on c-revisions first, and we also narrow down the notion of equivalence of ranking functions to the specific case that we observed in Example 1: there, we have $\kappa_2 = 2\kappa_1$, yielding a specific form of equivalence where empty layers are inserted in a very regular way. We call this *linear equivalence*.

Definition 5. *Two OCFs κ_1, κ_2 over Ω_Σ are* linearly equivalent*, in symbols $\kappa_1 \cong_\ell \kappa_2$, if there is a positive rational number q such that $\kappa_2 = q \cdot \kappa_1$.*

Note that in Definition 5, we allow q to be rational, but $q \cdot \kappa$ must be an OCF in the end, yielding only natural numbers for ranks. It is obvious, that κ and any of its multiples $q \cdot \kappa$ are equivalent. Clearly, $\cong_\ell$ is also an equivalence relation on OCFs. Suitable representatives of the equivalence classes are, e.g., those κ where the greatest common divisor of all $\kappa(\omega)$ is 1. Note that $\cong_\ell$ is a subrelation of $\cong$, so all statements of Proposition 2 also hold for linearly equivalent OCFs.

Proposition 2 states that for any equivalent OCFs κ_1, κ_2 and for any model ω of A, $\kappa_1(\omega) = \kappa_1(A)$ iff $\kappa_2(\omega) = \kappa_2(A)$. From this, however, we cannot derive a relationship between $\kappa_1(A)$ and $\kappa_2(A)$ in general, mainly because of empty layers being possibly present in the OCFs. For the special case of linearly equivalent OCFs, a useful result can be shown here.

Lemma 8. *If $\kappa_2 = q \cdot \kappa_1$, then $\kappa_2(A) = q \cdot \kappa_1(A)$ for any formula A.*

Proof. Let ω be a minimal model of A with respect to κ_2, i.e., $\kappa_2(A) = \kappa_2(\omega) = q \cdot \kappa_1(\omega)$. Due to Proposition 2, $\kappa_1(\omega) = \kappa_1(A)$, and the statement of the lemma follows immediately. □

A crucial property of linearly equivalent ranking functions is that they insert empty layers in a very regular way – the distance between any two successive layers, i.e., the number of empty layers between them, is constant. The following lemma makes this more precise, its proof is straightforward.

Lemma 9. *Let κ_1, κ_2 be two equivalent ranking functions with layers Ω_j, $j \in \{0, \ldots, m\}$, i.e., $\kappa_i(\omega) = r_i^j$ for all $\omega \in \Omega_j$, $0 = r_i^0 < r_i^1 < \ldots < r_i^m$, $i \in \{1,2\}, j \in \{0, \ldots, m\}$. κ_1, κ_2 are equivalent iff there is $q \in \mathbb{Q}$ such that*

$$r_2^j - r_2^{j-1} = q \cdot (r_1^j - r_1^{j-1})\ (j \in \{1, \ldots, m\}).$$

In particular, if Ψ is an epistemic state and $\kappa_1 = \kappa_\Psi$ its minimal ranking representation according to Definition 1, and $\kappa_2 = q \cdot \kappa_\Psi$, then we have $r_2^j - r_2^{j-1} = q$.

6.2 Strategies for Linear Revision Equivalence

Regarding the negative result with respect to DP-revision and even linearly equivalent ranking functions, a first hypothesis might be that c-revisions in general do a better job

ω	$\kappa_1(\omega)$	$\kappa_2(\omega)$	$\kappa_1 *_\sigma \overline{a}(\omega)$	$\kappa_2 *_\sigma \overline{a}(\omega)$	$\kappa_2 *_{\sigma'} \overline{a}(\omega)$
ab	0	0	1	3	2
$a\overline{b}$	3	6	4	9	8
$\overline{a}b$	2	4	1	2	2
$\overline{a}\overline{b}$	1	2	0	0	0

Table 2: Ranking functions κ_1, κ_2 and their strategic c-revisions for Example 2

preserving equivalence under revision. However, the next example shows that also this is not the case.

Example 2. *Let κ_1, κ_2 be the same ranking functions over $\Sigma = \{a, b\}$ as in Example 1, i.e., we have that $\kappa_2 = 2\kappa_1$, hence $\kappa_1 \cong_\ell \kappa_2$. Now, we strategically c-revise both ranking functions by $\overline{a}$, choosing $\sigma(\kappa_1, \{\overline{a}\}) = (2)$, and $\sigma(\kappa_2, \{\overline{a}\}) = (5)$. However, we are again loosing equivalence under revision, as Table 2 clearly shows. Now we have $(\kappa_1 *_\sigma \overline{a})(ab) = 1 = (\kappa_1 *_\sigma \overline{a})(\overline{a}b)$, but $(\kappa_2 *_\sigma \overline{a})(ab) = 3 > 2 = (\kappa_1 *_\sigma \overline{a})(\overline{a}b)$.*

Nevertheless, we might find that our selection strategy σ is too arbitrary by setting $\sigma(\kappa_2, \{\overline{a}\}) = (5)$. We know that $\kappa_2 = 2\kappa_1$, so choosing $\sigma'(\kappa_2, \{\overline{a}\}) = 2\sigma(\kappa_1, \{\overline{a}\}) = 2(2) = (4)$ seems to be more suitable. And indeed, as Table 2 shows, we now have equivalence of the revised ranking functions if we keep $\sigma'(\kappa_1, \{\overline{a}\}) = \sigma(\kappa_1, \{\overline{a}\}) = (2)$. Even more, we find that the revised ranking functions are still linearly equivalent with the same factor 2, i.e., $\kappa_2 *_{\sigma'} \overline{a} = 2 \cdot (\kappa_1 *_{\sigma'} \overline{a})$.

The crucial insight from this example is that indeed, strategic c-revisions yield revision equivalence of linearly equivalent ranking functions if the strategies respect multiples of ranking functions. This motivates the following postulate for strategic c-revisions:

(Mult^c) $\sigma(q \cdot \kappa, \Delta) = q \cdot \sigma(\kappa, \Delta)$.

Using this postulate, we are now able to formulate a first general positive result which even holds for universal conditional revision:

Theorem 10. *Linearly equivalent ranking functions are universally conditionally revision equivalent under any strategic c-revision $*_\sigma$ where the strategy σ satisfies (Mult^c). More precisely, if $\kappa_2 = q \cdot \kappa_1$, and σ satisfies (Mult^c), then it holds that $\kappa_2 *_\sigma \Delta = q \cdot (\kappa_1 *_\sigma \Delta)$ for any (consistent) set of conditionals $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$.*

Proof. Let $\kappa_2 = q \cdot \kappa_1$ and let σ satisfy (Mult^c). Now let $\kappa_1^* = \kappa_1 *_\sigma \Delta$ and $\kappa_2^* = \kappa_2 *_\sigma \Delta$ for a set of conditionals $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$. Furthermore, let $\sigma(\kappa_1, \Delta)_i$ and $\sigma(\kappa_2, \Delta)_i$ denote the i-th element of $\sigma(\kappa_1, \Delta)$ and $\sigma(\kappa_2, \Delta)$, respectively, belonging to the i-th conditional $(B_i|A_i)$. Then for all possible worlds ω it holds that

$$\kappa_2^*(\omega) = \nu_2 + \kappa_2(\omega) + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \sigma(\kappa_2, \Delta)_i$$

ω	$\kappa_1(\omega)$	$\kappa_2(\omega)$	$(\kappa_1 *_{\mathrm{DJ}} \mathcal{S}(\omega)$	$(\kappa_1 *_{\mathrm{DJ}} \mathcal{S})(\omega)$
abc	2	4	1	2
$ab\overline{c}$	3	6	0	0
$a\overline{b}c$	5	10	4	7
$a\overline{b}\overline{c}$	4	8	3	5
$\overline{a}bc$	5	10	4	7
$\overline{a}b\overline{c}$	4	8	3	5
$\overline{a}\overline{b}c$	1	2	5	8
$\overline{a}\overline{b}\overline{c}$	0	0	4	6

Table 3: Ranking functions κ_1, κ_2 and their DJ-revisions with $\mathcal{S} = \{a, b\}$ for Example 3.

where ν_2 is the normalizing integer defined as

$$\nu_2 = -\min_{\omega \in \Omega} \Big\{ \kappa_2(\omega) + \sum_{\substack{1 \leqslant i \leqslant n, \\ \omega \models A_i \overline{B_i}}} \sigma(\kappa_2, \Delta)_i \Big\}.$$

By substituting κ_2 with $q \cdot \kappa_1$ and applying (Mult^c) as well as distributivity (with respect to q) in both equations above, we obtain

$$\kappa_2^*(\omega) = q \cdot \Big(\nu_1 + \kappa_1(\omega) + \sum_{\substack{1 \leqslant i \leqslant n \\ \omega \models A_i \overline{B_i}}} \sigma(\kappa_1, \Delta)_i \Big)$$

where ν_1 is the corresponding normalizing integer for κ_1 defined as

$$\nu_1 = -\min_{\omega \in \Omega} \Big\{ \kappa_1(\omega) + \sum_{\substack{1 \leqslant i \leqslant n, \\ \omega \models A_i \overline{B_i}}} \sigma(\kappa_1, \Delta)_i \Big\}.$$

Therefore, $\kappa_2^*(\omega) = q \cdot \kappa_1^*(\omega)$, which was to be shown. □

This theorem shows that strategic c-revision $*_\sigma$ where the strategy σ satisfies (Mult^c) respects linear equivalence in a perfect way, i.e., the factor q which is crucially related to number and position of empty layers (see Lemma 9) is the same for the revised ranking functions. This yields a clear invariance property that can be checked to ensure revision equivalence of ranking functions in a precise numerical manner.

Definition 6. *A revision operator $*$* preserves linear equivalence *if for any linearly equivalent κ_1, κ_2 such that $\kappa_2 = q \cdot \kappa_1$ and for any proper input φ, it holds that*

$$\kappa_2 * \varphi = q \cdot (\kappa_1 * \varphi).$$

Theorem 10 then states that strategic c-revisions $*_\sigma$ where the strategy σ satisfies (Mult^c) preserve linear equivalence under revision by sets of conditionals.

6.3 Linear Revision Equivalence for Other Approaches from the Literature

Besides c-revisions, there are other OCF revision operators in the literature. One such operator is the parallel OCF revision operator proposed in (Delgrande and Jin 2012), which we denote by $*_{\mathrm{DJ}}$ here. In order to define $*_{\mathrm{DJ}}$, we first need to introduce two operations on sets of formulas.

ω	$\kappa_1(\omega)$	$\kappa_2(\omega)$	$(\kappa_1 *_{\text{DJ}} \{a\})(\omega)$	$(\kappa_1 *_{\text{DJ}} \{a\})(\omega)$
ab	3	6	1	2
$a\overline{b}$	2	4	0	0
$\overline{a}b$	1	2	2	3
$\overline{a}\overline{b}$	0	0	1	1

Table 4: Ranking functions κ_1, κ_2 and their DJ-revisions with $S = \{a\}$ for Example 4.

Definition 7. *Let $\mathcal{S}$ be a set of formulas and $\mathcal{S}' \subseteq \mathcal{S}$. Then the* completion *of $\mathcal{S}'$ with respect to $\mathcal{S}$ is defined as*

$$\lceil \mathcal{S}' \rceil^{\mathcal{S}} = \mathcal{S}' \cup \{\neg A \mid A \in (\mathcal{S} \setminus \mathcal{S}')\}.$$

Furthermore, the reduction *of $\mathcal{S}$ to a possible world ω is defined as*

$$\lfloor \mathcal{S} \rfloor_\omega = \{A \in \mathcal{S} \mid \omega \models A\}.$$

The revision operator $*_{\text{DJ}}$ can now be defined via three conditions, which can be utilized to successively set the ranks of different sets of possible worlds in the resulting OCF.

Definition 8 ($*_{\text{DJ}}$ (Delgrande and Jin 2012))**.** *Let κ be an OCF and let $\mathcal{S}$ be a (finite and consistent) set of propositions. Then $\kappa^* = (\kappa *_{DJ} \mathcal{S})$ is constructed as follows.*

- *For all $\omega_1 \in \min(Mod\,(\mathcal{S}), \kappa)$, set $\kappa^*(\omega_1) = 0$.*
- *For $1 \leq i \leq |\mathcal{S}|$ (successively): Let $\mathcal{S}'' \subset \mathcal{S}$ with $|\mathcal{S}| - |\mathcal{S}''| = i$. For all $\omega_2 \in \min(Mod\,(\lceil \mathcal{S}'' \rceil^{\mathcal{S}}), \kappa)$, set*

$$\kappa^*(\omega_2) = 1 + \max\{\kappa^*(\lceil \mathcal{S}' \rceil^{\mathcal{S}}), \kappa^*(\lceil \mathcal{S}' \rceil^{\mathcal{S}}) + \kappa(\lceil \mathcal{S}'' \rceil^{\mathcal{S}}) - \kappa(\lceil \mathcal{S}' \rceil^{\mathcal{S}}) \mid \mathcal{S}'' \subset \mathcal{S}' \subseteq \mathcal{S} \text{ with } |\mathcal{S}'| - |\mathcal{S}''| = 1\}.$$

- *For $\omega_3 \notin \min(Mod\,(\lceil \lfloor \mathcal{S} \rfloor_{\omega_3} \rceil^{\mathcal{S}}), \kappa)$, set*

$$\kappa^*(\omega_3) = \kappa^*(\lceil \lfloor \mathcal{S} \rfloor_{\omega_3} \rceil^{\mathcal{S}}) + \kappa(\omega_3) - \kappa(\lceil \lfloor \mathcal{S} \rfloor_{\omega_3} \rceil^{\mathcal{S}}).$$

As this operator $*_{\text{DJ}}$ takes an OCF and a set of propositional formulas as input, one might be interested in checking whether $*_{\text{DJ}}$ preserves (linear) equivalence, i.e. if $\kappa_1 \cong \kappa_2$ implies $\kappa_1 *_{\text{DJ}} \mathcal{S} \cong \kappa_2 *_{\text{DJ}} \mathcal{S}$. The following example demonstrates that this is not the case.

Example 3. *Let κ_1, κ_2 be ranking functions over $\Sigma = \{a, b, c\}$ as defined in Table 3. Note that $\kappa_2 = 2 \cdot \kappa_1$ and, therefore, $\kappa_1 \cong_\ell \kappa_2$. Now let $\kappa_1^* = \kappa_1 *_{DJ} \{a, b\}$ and $\kappa_2^* = \kappa_2 *_{DJ} \{a, b\}$. Table 3 shows that $\kappa_1^*(\overline{a}\overline{b}\overline{c}) = 4 = \kappa_1^*(\overline{a}bc)$ while $\kappa_2^*(\overline{a}\overline{b}\overline{c}) = 6 < 7 = \kappa_2^*(\overline{a}bc)$. Hence, $\kappa_1^* \not\cong \kappa_2^*$.*

Even when revising by only a single proposition A via $*_{\text{DJ}}$, equivalence among OCFs is not necessarily preserved, as the following example shows.

Example 4. *Let κ_1, κ_2 be ranking functions over $\Sigma = \{a, b\}$ as defined in Table 4. Note that $\kappa_2 = 2 \cdot \kappa_1$ and, therefore, $\kappa_1 \cong_\ell \kappa_2$. Now let $\kappa_1^* = \kappa_1 *_{DJ} \{a\}$ and $\kappa_2^* = \kappa_2 *_{DJ} \{a\}$. Table 4 shows that $\kappa_1^*(\overline{a}\overline{b}) = 1 = \kappa_1^*(ab)$ while $\kappa_2^*(\overline{a}\overline{b}) = 1 < 2 = \kappa_2^*(ab)$. Hence, $\kappa_1^* \not\cong \kappa_2^*$.*

7 Equivalence via TPO Revisions

In principle, every revision operator $\bullet$ for total preorders can be used to define a revision operator for ranking functions κ by utilizing the transformation functions τ and ρ. There are two approaches to this:

- We transform κ into a TPO, perform the TPO revision directly, and then transform the result back into an OCF. This corresponds to defining an OCF-revision operator $\circledast$ by
$$\kappa \circledast \varphi = \rho(\tau(\kappa) \bullet \varphi), \tag{6}$$
where φ is an appropriate input for $\bullet$.
- We design an OCF-revision operator $*$ that mimics the behavior of $\bullet$ and check the correspondence to $\bullet$ afterwards. More precisely, we define $*$ accordingly from scratch such that
$$\Psi \bullet \varphi = \tau(\kappa * \varphi) \tag{7}$$
holds for every epistemic state Ψ properly represented by a total preorder, every $\kappa \in \tau^{-1}(\Psi)$, and every new information φ.

Note that in Equation (7), the operator $*$ is required to work for all $\kappa \in \tau^{-1}(\Psi)$ instead of just the minimal κ_Ψ from Definition 1. Equations (6) and (7) are compatible in the sense that $\circledast$ from (6) satisfies (7).

Lemma 11. *Let $\bullet$ be a TPO-revision operator and let $\circledast$ be the OCF-revision operator induced by $\bullet$ via* (6)*. Then for every epistemic state Ψ equipped with a TPO, every $\kappa \in \tau^{-1}(\Psi)$, and every new information φ, it holds that*

$$\Psi \bullet \varphi = \tau(\kappa \circledast \varphi).$$

Proof. According to (6), it holds that $\tau(\kappa \circledast \varphi) = \tau(\rho(\tau(\kappa) \bullet \varphi)) = \tau(\kappa) \bullet \varphi = \Psi \bullet \varphi$. □

Now we will take a closer look at the first approach.

Proposition 12. *Let $\bullet$ be a revision operator for total preorders, and let φ be an appropriate input for $\bullet$. Then the OCF-revision operator $\circledast$ induced by $\bullet$ via Equation* (6) *preserves equivalence, i.e., any two equivalent ranking functions are also (universally/propositionally/conditionally) revision equivalent.*

Proof. Let κ_1, κ_2 be equivalent OCFs. Then it follows immediately from Definition 1 that $\tau(\kappa_1) = \tau(\kappa_2)$. Hence, the revision using $\circledast$ delivers not just equivalent, but identical results for both OCFs, i.e. $\kappa_1 \circledast \varphi = \kappa_2 \circledast \varphi$. □

Proposition 12 allows us to obtain OCF-revision operators from TPO-revision operators via Equation (6) where the revision result on the side of the total preorder does not depend on the chosen ranking representation of the total preorder.

It should be noted that $\circledast$ does not satisfy (Stability) in general, even if $\bullet$ does not alter the preorder when revising with already known information. This is an artifact due to the loss of all information about empty layers of the previous ranking function when applying $\rho \circ \tau$.

From the proof of the proposition above, it is obvious that transformation functions can also be used to preserve linear equivalence among OCFs. For two equivalent OCFs κ_1, κ_2

with $\kappa_2 = q \cdot \kappa_1$ (and $q > 1$), one simply needs to scale the result of the revision $\kappa_2 \circledast \varphi$ with q. The following proposition demonstrates how such a scale-aware revision operator can be defined.

Proposition 13. *Let κ be an OCF, let $\circledast$ be constructed from a revision operator $\bullet$ for total preorders as described in Equation* (6)*, and let φ be an appropriate input for $\bullet$. Then the OCF-revision operator $\otimes$ defined by*

$$\kappa \otimes \varphi = (\kappa \circledast \varphi) \cdot \min\{\kappa(\omega) \mid \omega \in \Omega, \kappa(\omega) > 0\}$$

preserves linear equivalence.

Proof. Let κ_1, κ_2 be equivalent OCFs with $\kappa_2 = q \cdot \kappa_1$ and let $q > 1$. For $i \in \{1,2\}$, let $r_i = \min\{\kappa_i(\omega) \mid \omega \in \Omega, \kappa_i(\omega) > 0\}$. Then we have $r_2 = q \cdot r_1$ due to Lemma 9. As a result, $\kappa_2 \otimes \varphi = (\kappa_2 \circledast \varphi) \cdot r_2 = \rho(\tau(\kappa_2) \bullet \varphi) \cdot r_2 = \rho(\tau(\kappa_1) \bullet \varphi) \cdot r_1 \cdot q = (\kappa_1 \circledast \varphi) \cdot r_1 \cdot q = (\kappa_1 \otimes \varphi) \cdot q$. □

Now we examine the second approach of defining OCF-revision operators that behave like TPO-revision operators. We provide two such operators that originate from well-known elementary revision operators here.

Definition 9 ($*_n$, $*_\ell$)**.** *Let κ be an OCF and A be a proposition. The* natural OCF-revision *operator $*_n$ is defined by*

$$(\kappa *_n A)(\omega) = \begin{cases} 0 & \textit{iff } \omega \models A \textit{ and } \kappa(\omega) = \kappa(A), \\ 1 + \kappa(\omega) & \textit{otherwise.} \end{cases}$$

The lexicographic OCF-revision *operator $*_n$ is defined by*

$$(\kappa *_\ell A)(\omega) = \kappa(\omega) - \kappa(A) + \begin{cases} 0 & \textit{iff } \omega \models A, \\ 1 + \max\limits_{\omega \models A}\{\kappa(\omega)\} & \textit{otherwise.} \end{cases}$$

Note that $*_n$ does not satisfy (Stability), although $\bullet_n$ does not alter preorders when the lowermost layer already contains only models of the new information. This is an artifact that could be avoided by introducing an additional case $(\kappa *_n A)(\omega) = \kappa(\omega)$ iff $\kappa \models A$ in the definition, but we opt for the more concise definition here.

For $*_\ell$, however, the violation of (Stability) is not an artifact, since $\bullet_\ell$ deliberately makes all models of the new information A more plausible, even if A is already believed.

We now show that the operators $*_n$ and $*_\ell$ defined above are indeed suitable OCF-realizations of the TPO-revision operators $\bullet_n$ and $\bullet_\ell$, respectively.

Proposition 14. *The operator $*_n$ complies with (NR) in the sense that it fulfills Equation* (7) *with respect to $\bullet_n$. Analogously, the operator $*_\ell$ complies with (LR) in the sense that it fulfills Equation* (7) *with respect to $\bullet_\ell$.*

Proof. Let Ψ be an epistemic state equipped with a total preorder $\preceq_\Psi$ and let $\kappa \in \tau^{-1}(\Psi)$. Furthermore, let A be a proposition and let ω, ω' be possible worlds.

We first prove that $\Psi \bullet_n A = \tau(\kappa *_n A)$ by showing that $\Psi^\bullet = \tau(\kappa *_n A)$ complies with the equivalence stated in (NR). "⇐": (1) If $\omega \in \min(Mod(A), \preceq_\Psi)$, we have $\kappa(\omega) = \kappa(A)$. This results in $(\kappa *_n A)(\omega) = 0 \leq (\kappa *_n A)(\omega')$ for all $\omega' \in \Omega$. Therefore, $\omega \preceq_{\Psi^\bullet} \omega'$. (2) If $\omega, \omega' \notin \min(Mod(A), \preceq_\Psi)$ and $\omega \preceq_\Psi \omega'$, we have $\kappa(\omega) \leq \kappa(\omega')$. Therefore, $(\kappa *_n A)(\omega) = \kappa(\omega) + 1$ and $(\kappa *_n A)(\omega') = \kappa(\omega') + 1$. Hence, $\omega \preceq_{\Psi^\bullet} \omega'$.

"⇒": We prove the other direction by contraposition. Let $\omega \notin \min(Mod(A), \preceq_\Psi)$. This results in $(\kappa *_n A)(\omega) = \kappa(\omega) + 1 > 0$. (i) If $\omega' \in \min(Mod(A), \preceq_\Psi)$, we have $(\kappa *_n A)(\omega') = 0$ and consequently $\omega' \prec_{\Psi^\bullet} \omega$. (ii) If $\omega' \notin \min(Mod(A), \preceq_\Psi)$, the negation of the second condition in (NR) means we have $\omega' \prec_\Psi \omega$. As a result, $(\kappa *_n A)(\omega') = \kappa(\omega') + 1 < (\kappa *_n A)(\omega)$ and $\omega' \prec_{\Psi^\bullet} \omega$.

Now we prove that $\Psi \bullet_\ell A = \tau(\kappa *_\ell A)$. Let $\Psi^\bullet = \tau(\kappa *_\ell A)$ and let $c = \max_{\omega \models A}\{\kappa(\omega)\}$ be the maximal rank of models of A in κ. We need to show that $\Psi^\bullet$ complies with the equivalence stated in (LR). "⇐": (1) If $\omega \models A$ and $\omega' \not\models A$, we have $\kappa(\omega) \leq c$. Therefore, $\kappa(\omega) < 1 + c + \kappa(\omega')$. Hence, $(\kappa *_\ell A)(\omega) < (\kappa *_\ell A)(\omega')$ and $\omega \prec_{\Psi \bullet_\ell A} \omega'$. (2) If ($\omega \models A$ iff $\omega' \models A$ and $\omega \preceq_\Psi \omega'$, we have $\kappa(\omega) \leq \kappa(\omega')$. During the revision, the same constant value is added to both sides of this inequation. Therefore, $(\kappa *_\ell A)(\omega) \leq (\kappa *_\ell A)(\omega')$ and $\omega \preceq_{\Psi \bullet_\ell A} \omega'$.

"⇒": We prove this direction by contraposition. Let $\omega \not\models A$ and $\omega' \models A$. Analogously to the first case in the other direction, the revision results in $\omega' \prec_{\Psi^\bullet} \omega$. □

Moreover, both operators $*_n$ and $*_\ell$ preserve equivalence among OCFs. In order to prove this, we show more generally that compliance with (7) is enough for an OCF-revision operator to guarantee preservation of equivalence.

Proposition 15. *Let $*$ be a revision operator for ranking functions that satisfies* (7) *for some revision operator $\bullet$ for total preorders, and let φ be an appropriate input for $\bullet$. Let κ_1, κ_2 be equivalent OCFs. Then*

$$\kappa_1 * \varphi \cong \kappa_2 * \varphi.$$

Proof. Let $\bullet$ be a revision operator for total preorders such that (7) holds with respect to $*$ and $\bullet$, and let φ be a suitable input for $\bullet$. It holds that $\kappa_1, \kappa_2 \in \tau^{-1}(\Psi)$ for $\Psi = \tau(\kappa_1) = \tau(\kappa_2)$. Therefore, $\Psi \bullet \varphi = \tau(\kappa_1 * \varphi) = \tau(\kappa_2 * \varphi)$ according to Lemma 11, which means that $(\kappa_1 * \varphi) \cong (\kappa_2 * \varphi)$. □

Proposition 15 implies that when using ranking functions to implement a TPO-revision operator applied to a total preorder Ψ, any ranking function κ with $\tau(\kappa) = \Psi$ that satisfies (7) can be taken as a suitable representation, yielding a unique revision result.

We are now able to formulate the following corollary, the proof of which is immediate from Proposition 14 together with Proposition 15.

Corollary 16. *Let κ_1, κ_2 be equivalent OCFs. Then κ_1 and κ_2 are (propositionally) revision equivalent with respect to both $*_n$ and $*_\ell$.*

However, although they preserve inferential equivalence, the two OCF operators $*_n$ and $*_\ell$ do not preserve linear equivalence among OCFs.

Example 5. *Table 5 shows two linearly equivalent ranking functions κ_1, κ_2 with $\kappa_2 = 2 \cdot \kappa_1$. We can see that both $(\kappa_1 *_n a) \cong (\kappa_2 *_n a)$ and $(\kappa_1 *_\ell a) \cong (\kappa_2 *_\ell a)$. However, there is no $q \in \mathbb{Q}$ such that $(\kappa_2 *_n a) = q \cdot (\kappa_1 *_n a)$ or $(\kappa_2 *_\ell a) = q \cdot (\kappa_1 *_\ell a)$.*

ω	κ_1	κ_2	$(\kappa_1 *_n a)$	$(\kappa_2 *_n a)$	$(\kappa_1 *_\ell a)$	$(\kappa_2 *_\ell a)$
ab	1	2	0	0	0	0
$a\overline{b}$	3	6	4	7	2	4
$\overline{a}b$	0	0	1	1	3	5
$\overline{a}\overline{b}$	2	4	3	5	5	9

Table 5: Ranking functions κ_1, κ_2 and their natural and lexicographic OCF revisions with a for Example 5.

8 Conclusion

In this paper, we introduced the novel concept of revision equivalence for ranking functions to ensure that empty layers of ranking functions do not affect the induced qualitative total preorder after revision. This allows for using the convenient framework of ranking functions for revising total preorders, as then any ranking representation of a total preorder can be used for revision. However, as our investigations showed, revision equivalence is not easy to achieve in general, e.g., under iterated revisions according to the DP-framework (Darwiche and Pearl 1997). Therefore, we introduced more structural information into the notion of equivalence by considering linearly equivalent ranking functions and found that c-revisions (Kern-Isberner 2004) are perfectly adequate to preserve linear equivalence even for revising by sets of conditionals. We also presented a general approach to make use of revision operators for total preorders such as natural (Boutilier 1993) or lexicographic (Nayak, Pagnucco, and Peppas 2003) revision to define revision operators for ranking functions which preserve equivalence.

As part of our future work, we plan to evaluate and make use of more approaches to iterated revision from the literature, both revision operators for ranking functions and for total preorders, and also to consider iterated revision frameworks beyond the basic DP-framework. In particular, we will investigate implications of the axioms (Ind) (Jin and Thielscher 2004) and (P) (Delgrande and Jin 2012), and of those axioms that have been used for characterizing the elementary revision operators in (Chandler and Booth 2023) for our framework. Moreover, the problem of ensuring the preservation of equivalence is also relevant for improvement operators (Konieczny and Perez 2008), whose behavior may heavily depend on empty layers as well.

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