Axiomatization of Approximate Exclusion

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Abstract

We define and axiomatize approximate exclusion atoms in the team semantic setting. A team is a set of assignments, which can be seen as a mathematical model of a uni-relational database. We say that an approximate exclusion atom is satisfied in a team if the corresponding usual exclusion atom is satisfied in a large enough subteam. We consider the implication problem for a set of approximate exclusion atoms and show that it is axiomatizable for consequences with a degree of approximation that is not too large. We prove the completeness theorem for usual exclusion atoms, currently missing from the literature, and generalize it to the approximate case. We also provide a polynomial time algorithm for the implication problems. The results can also be applied to exclusion dependencies in database theory.

1 Introduction

Team semantics was introduced in (Hodges 1997a; Hodges 1997b) and further developed in (Väänänen 2007) with the introduction of dependence logic. Team semantics is suited for examining expressions about relationships between variables, such as different dependencies from database theory, since formulas are evaluated in a finite set of assignments, called a *team*, instead of a single assignment. In the team semantics setting, functional dependencies are captured by dependence atoms (Väänänen 2007), and exclusion dependencies introduced in (Casanova and Vidal 1983) were adapted as exclusion atoms in (Galliani 2012), together with inclusion atoms.

Approximate functional dependencies were defined in (Kivinen and Mannila 1995), and later defined as approximate dependence atoms and axiomatized in (**?**). We define, analogously, that an *approximate exclusion atom* is satisfied in a team if there exists a large enough subteam that satisfies the corresponding usual exclusion atom. The definition is motivated by dependence and exclusion atoms both being downward closed, i.e., when a team satisfies an atom, so do its subteams. The approximate atoms are suitable when it is permitted that the team has some, typically small, bounded degree of error.

We consider the following implication problem: Does a (possibly infinite) set of approximate exclusion atoms imply a given approximate exclusion atom? We show that the implication problem is axiomatizable for consequences whose approximation allows less than half of the team to be faulty, and for assumption sets for which there exists a gap between the approximations larger than the one in the consequence and the approximation in the consequence. The former restriction still allows all cases where the natural interpretation of the consequence is "almost exclusion", and the latter is trivially satisfied for finite assumption sets.

We define a complete set of rules for exclusion atoms, whose explicit axiomatization is currently missing from the literature. Some of the rules are from the system for exclusion and inclusion combined introduced in (Casanova and Vidal 1983), with a necessary additional rule. We then generalize the system and the counterexample team in the completeness proof for usual exclusion atoms to prove completeness for approximate exclusion atoms. We also provide a polynomial time algorithm showing that the finite implication problems for (approximate) exclusion atoms are decidable.

The results in this paper can immediately be transferred to the database setting by reading "uni-relational database" instead of "team" and "exclusion dependency" instead of "exclusion atom".

2 Exclusion Atoms

We recall basic definitions of team semantics and exclusion atoms as defined in (Galliani 2012).

A team T is a finite set of assignments $s : \mathcal{V} \longrightarrow M$, where $\mathcal{V}$ is a set of variables and M is a set of values. We write $x_i, y_i, \ldots$ for individual variables and $x, y, \ldots$ for finite (possibly empty) tuples of variables. Given a tuple x, we denote its i:th variable by x_i. Let $x = \langle x_1, \ldots, x_n \rangle$ and $y = \langle y_1, \ldots, y_m \rangle$. We write $s(x)$ as shorthand for $\langle s(x_1), \ldots, s(x_n) \rangle$. The concatenation xy is the tuple $\langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$. The tuples x and y are equal if and only if they are of the same length, denoted $|x| = |y|$, and $x_1 = y_1, \ldots, x_n = y_n$. We define Var to be a function such that $Var(x) = \{x_i : 1 \leq i \leq |x|\}$.

Exclusion atoms are written as $x|y$, where $|x| = |y|$. We recall the semantics of the usual exclusion atom:

$$T \models x|y \text{ iff for all } s_1, s_2 \in T : s_1(x) \neq s_2(y).$$

It follows that exclusion atoms are downward closed: $T \models x|y$ implies that $T' \models x|y$ for all subteams $T' \subseteq T$. Exclusion atoms also have the empty team property since all exclusion atoms are satisfied by the empty team. We call atoms of the form $x|x$ *contradictory* since they are only satisfied in the empty team.

We write $x|y \models u|v$ if for all teams T, $T \models x|y$ implies $T \models u|v$. If both $x|y \models u|v$ and $u|v \models x|y$, we say that $u|v$ and $x|y$ are semantically equivalent and write $x|y \equiv u|v$.

We define the rules for exclusion atoms and prove that the system they form is sound and complete.

Definition 1. *The rules for exclusion atoms are:*

(E1) If $x|x$, then $y|z$
(E2) If $x|y$, then $y|x$
(E3) If $x|y$, then $xu|yv$
(E4) If $xuu|yvv$, then $xu|yv$
(E5) If $xyz|uvw$, then $xzy|uwv$ ($|x| = |u|$ and $|y| = |v|$)
(E6) If $xw|yw$, then $zz|xy$.

Lemma 1 (Soundness)**.** *Let Σ be a set of exclusion atoms. If $\Sigma \vdash x|y$, then $\Sigma \models x|y$.*

Proof. The proofs are straightforward. For $E6$, if $T \models xw|yw$, then for each $s \in T$ there is no tuple a such that $s(xy) = aa$. On the other hand, for all $s \in T$, $s(zz) = bb$ for some tuple b, so we conclude $T \models zz|xy$. □

Let us comment on the rules in relation to the system for exclusion and inclusion combined in (Casanova and Vidal 1983). The rules $E1, E2, E3, E5$ are all included in their system and, assuming they do not distinguish between dependencies like $xuu|yvv$ and $xu|yv$, the rule $E4$ does not apply. Rule $E6$ is new and we show that it is not derivable from the rules in (Casanova and Vidal 1983). Recall the definition of inclusion atoms $Inc(x, y)$, with $|x| = |y|$,

$$T \models Inc(x, y) \text{ iff for all } s \in T, \text{ there exists } s' \in T \text{ such that } s(x) = s'(y).$$

Remark 1. *The system for inclusion and exclusion in (Casanova and Vidal 1983) includes the rules $E1, E2, E3, E5$ together with the rules below*[1]*, and is not complete for exclusion consequences: Consider $x_1 \neq y_1$, then $x_1w_1w_2|y_1w_1w_2$ is not contradictory. We have that $x_1w_1w_2|y_1w_1w_2 \models z_1z_1|x_1y_1$ with $|x_1y_1| \neq |x_1w_1w_2|$, but no rule in this system allows such a change in arity.*

(IE1) $Inc(x, x)$
(IE2) If $Inc(xyz, uvw)$, then $Inc(xzy, uwv)$ ($|x| = |u|$ and $|y| = |v|$)
(IE3) If $Inc(xu, yv)$, then $Inc(x, y)$
(IE4) If $Inc(x, z)$ and $Inc(z, y)$, then $Inc(x, y)$
(IE5) If $x|x$, then $Inc(y, z)$
(IE6) If $Inc(x, u)$, $Inc(y, v)$ and $u|v$, then $x|y$.

We define a function to identify exclusion atoms with sets of ordered pairs, such that two exclusion atoms are identified with the same set only if they are semantically equivalent. We also define sets such that for a given variable in an exclusion atom, the set contains exactly those variables that appear on the other side of the exclusion symbol in the same position as the given variable.

[1]In (Casanova and Vidal 1983), $IE2$ and $IE3$ correspond to one rule.

Definition 2. *Let $|x| = |y| = n$ and $x|y$ be any exclusion atom. Define the atom's set representation $S(x|y)$ by:*

$$S(x|y) = \{\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \dots, \langle x_n, y_n\rangle\}.$$

For each x_i and y_i we define the correspondence sets

$$\mathcal{C}_{x_i} = \{y_j : \langle x_i, y_j\rangle \in S(x|y)\}$$
$$\mathcal{C}_{y_i} = \{x_j : \langle x_j, y_i\rangle \in S(x|y)\}.$$

We give examples of these definitions for the exclusion atom $x|y = v_1v_1v_2v_3|u_1u_2v_1u_3$:

$$S(x|y) = \{\langle v_1, u_1\rangle, \langle v_1, u_2\rangle, \langle v_2, v_1\rangle, \langle v_3, u_3\rangle\},$$

$\mathcal{C}_{x_1} = \mathcal{C}_{x_2} = \{u_1, u_2\}$, $\mathcal{C}_{x_3} = \{v_1\}$, and $\mathcal{C}_{y_1} = \{v_1\}$.

Let us also define the *end-constant form* for atoms $u|v$ as the semantically equivalent atom $u'c|v'c$ where $S(u|v) = S(u'c|v'c)$ and for all $\langle u_i, v_i\rangle \in S(u|v)$, if $u_i = v_i$, then $\langle u_i, v_i\rangle \in S(c|c)$ and $\langle u_i, v_i\rangle \notin S(u'|v')$.

The next lemma shows how the rules correspond to exclusion atoms' set representations. By item (i) in Lemma 2, together with soundness, it follows that if two exclusion atoms are identified with the same set, then they are semantically equivalent. Item (ii) gives a set representation condition in a derivation where the rule $E6$ is used.

Lemma 2. *Suppose that $|x| = |y| = n$ and $u \neq v$. Let $uc|vc$ be in end-constant form.*

(i) $u|v \vdash_{\{E3,E4,E5\}} x|y$ iff $S(u|v) \subseteq S(x|y)$.
(ii) Suppose $S(uc|vc) \not\subseteq S(x|y)$ and $S(vc|uc) \not\subseteq S(x|y)$, then $uc|vc \vdash x|y$ iff there is $d \in \{x, y\}$ such that

$$S(uc|vc) \subseteq \bigcup_{1\le i\le n} \mathcal{C}_{d_i} \times \mathcal{C}_{d_i} \cup \{\langle w_l, w_l\rangle : w_l \in \mathcal{V}\}.$$

Proof. (i) Immediate by the rules $E3$, $E4$ and $E5$.

(ii) Let $|u| = |v| = m$. By the assumptions and item (i), we need to use rule $E6$ in the derivation. We note that one application of rule $E6$ is sufficient, since after that all variables in uv have already been moved to the same side. We check the case when $d = x$ and the derivation is of the form $uc|vc \vdash_{\{E6\}} zz|uv \vdash_{\{E3,E4,E5\}} x|y$, the case when $d = y$ can be reduced to this one via rule $E2$.

$$\begin{aligned}
&uc|vc \vdash_{\{E6\}} zz|uv \vdash_{\{E3,E4,E5\}} x|y\\
&\iff S(zz|uv) \subseteq S(x|y) \text{ by item (i)}\\
&\iff \text{For all } 1 \le j \le m, \langle z_j, u_j\rangle, \langle z_j, v_j\rangle \in S(x|y)\\
&\iff \text{For all } 1 \le j \le m, \langle u_j, v_j\rangle \in \mathcal{C}_{x_i} \times \mathcal{C}_{x_i} \text{ for some } 1 \le i \le n\\
&\iff S(uc|vc) \subseteq \bigcup_{1\le i\le n} \mathcal{C}_{x_i} \times \mathcal{C}_{x_i} \cup \{\langle w_l, w_l\rangle : w_l \in \mathcal{V}\}.
\end{aligned}$$

□

	x_1	...	x_n	y_1	...	y_n	z_1	z_2	...
s_1	a_1	$-$	a_n		...				...
s_2		...		a_1	$-$	a_n			...

Table 1: Counterexample team for exclusion atoms.[2]

We are now ready to prove the completeness theorem for exclusion atoms, by constructing a counterexample team.

Theorem 1 (Completeness). *Let $\Sigma \cup \{x|y\}$ be a set of exclusion atoms with $|x| = |y| = n$. If $\Sigma \models x|y$ then $\Sigma \vdash x|y$.*

Proof. We assume $\Sigma \not\vdash x|y$ and show $\Sigma \not\models x|y$. Let $X = Var(x)$, $Y = Var(y)$ and $Z = \{z_1, z_2, \dots\}$ be the variables in Σ that are not in $X \cup Y$. Construct a team $T = \{s_1, s_2\}$ with values from $\mathbb{N}$ as follows. Define $s_1(x) = \langle a_1, \dots, a_n\rangle = s_2(y)$, such that for all $i, j \in \{1, \dots, n\}$, $a_i = a_j$ if and only if $x_i = x_j$ or $y_i = y_j$. All other assigned values are values from $\mathbb{N}$ occurring only once.

Clearly, $T \not\models x|y$. Let $u|v \in \Sigma$. First note that $u|v$ with $u = v$ or $S(u|v)$ satisfying any of the cases in Lemma 2 would allow us to prove a contradiction.

For the remaining $u|v \in \Sigma$, we show $T \models u|v$ by going through cases one by one (noting that symmetrical variants are similar) while excluding the previous cases. First, consider when $Var(uv) \cap Z \neq \emptyset$: If there is $\langle u_i, z_j\rangle \in S(u|v)$, and $u_i \neq z_j$, then u_i and z_j have no common values in the team. Otherwise, all variables from Z appear in the form $\langle z_j, z_j\rangle \in S(u|v)$. Then one of the following two cases must hold.

(a) There is some $\langle x_i, y_k\rangle \in S(u|v)$ with $x_i \neq y_k$. If $i = k, x_i = x_k$ or $y_i = y_k$, there is a possible shared value for x_i and y_k at $s_1(x_i) = s_2(y_k)$, but $s_1(z_j) \neq s_2(z_j)$.

(b) There is some $\langle x_i, x_k\rangle \in S(u|v)$ with $x_i \neq x_k$ (or similarly $\langle y_i, y_k\rangle \in S(u|v)$), then also $y_i \neq y_k$ by Lemma 2 (ii), so x_i and x_k share no values.

Now suppose that $Var(uv) \subseteq X \cup Y$. The case when $Var(uv) \subseteq X$ (or similarly for Y) is similar to item (b).

(c) If $Var(u) \subseteq X$ and $Var(v) \subseteq Y$, then there is $\langle x_i, y_k\rangle \in S(u|v)$ with $x_i \neq x_k$ and $y_i \neq y_k$ (by Lemma 2 (i)), so x_i, y_k have no common values.

(d) If $Var(u) \subseteq X \setminus Y$ (or similarly for $Y \setminus X$) and $Var(v) \cap X$ and $Var(v) \cap Y$ are both nonempty, then there are $\langle x_i, x_k\rangle, \langle x_l, y_m\rangle \in S(u|v)$ such that $x_k \notin Y$ and $y_m \notin X$. The only possible shared value for x_l and y_m is at $s_1(x_l) = s_2(y_m)$, but $s_1(x_i) \neq s_2(x_k)$.

(e) If the intersections $Var(v) \cap X, Var(v) \cap Y, Var(u) \cap X$, and $Var(u) \cap Y$ are all nonempty, either there are $\langle x_i, y_k\rangle, \langle y_l, x_m\rangle \in S(u|v)$ with $x_i \notin Y$ and $y_k \notin X$. Then the only possible shared value for x_i and y_k is at $s_1(x_i) = s_2(y_k)$, but $s_1(y_l) \neq s_2(x_m)$. Or, there are $\langle x_i, x_k\rangle, \langle y_l, y_m\rangle \in S(u|v)$ and the only possible shared value for x_i and x_k is at $s_1(x_i) = s_1(x_k)$, but $s_1(y_l) \neq s_1(y_m)$.

□

[2]If $x_i = y_j$ for some indices i and j, then they correspond to one column in the team with $s_1(x_i) = a_i$ and $s_2(x_i) = a_j$.

We show in Theorem 4 that the implication problem for finite sets of approximate exclusion atoms is decidable by constructing a polynomial time algorithm, from which the corresponding result follows for usual exclusion atoms.

Theorem 2 (Decidability). *Let $\Sigma \cup \{x|y\}$ be a finite set of exclusion atoms. The implication problem for whether $\Sigma \vdash x|y$ is decidable.*

3 Approximate Exclusion Atoms

We define approximate exclusion atoms in an analogous way to the approximate dependence atoms in (**?**).

Definition 3. *Let p be a real number such that $0 \leq p \leq 1$. $T \models x|_p y$ iff there is a subteam $T' \subseteq T$, $|T'| \leq p \cdot |T|$, such that $T \setminus T' \models x|y$.*

Thus $p = 0$ coincides with the usual exclusion atoms, while an approximate exclusion atom with $p = 1$ is always satisfied by the empty team property. For small approximations p, a team satisfying the approximate exclusion atom $x|_p y$ corresponds to the usual exclusion atom $x|y$ *almost* being satisfied in the team. Thus the statement "almost all individuals in last year's top 50 ranking are not in this year's top 50 ranking" can be formalized as, e.g., $x_1|_{\frac{3}{50}} y_1$, meaning that for a team $T = \{s_1, s_2, \dots, s_{50}\}$ such that for all $i \in \{1, 2, \dots, 50\}$, $s_i(x_1)$ is the name of the individual in last year's i:th place and $s_i(y_1)$ is the name of the individual in this year's i:th place, $T \models x_1|_{\frac{3}{50}} y_1$ holds if and only if at most three individuals remained in this year's top 50 ranking.

We extend the definition of *contradictory* atoms to include approximate exclusion atoms of the form $x|_p x$, $p < 1$, since they too are only satisfied in the empty team.[3]

We define the rules for approximate exclusion atoms and show that they are sound.

Definition 4. *The rules for approximate exclusion atoms are:*

(A1) *For $p < 1$, if $x|_p x$, then $y|_0 z$*
(A2) *If $x|_p y$, then $y|_p x$*
(A3) *If $x|_p y$, then $xu|_p yv$*
(A4) *If $xuu|_p yvv$, then $xu|_p yv$*
(A5) *If $xyz|_p uvw$, then $xzy|_p uwv$ ($|x| = |u|$ and $|y| = |v|$)*
(A6) *If $xw|_p yw$, then $zz|_p xy$*
(A7) *For $q \leq p \leq 1$, if $x|_q y$, then $x|_p y$*
(A8) *$x|_1 y$.*

Lemma 3 (Soundness). *The rules A1-A8 are sound.*

[3]Like for approximate dependence atoms in (**?**), it is easy to show that locality fails for approximate exclusion atoms.

Proof. The rules $A1$-$A6$ are the approximate versions of the rules $E1$-$E6$ in Definition 1, and soundness follows similarly. For $A7$, if already $T \models x|_q y$, then for $p \geq q$, $T \models x|_p y$. For $A8$, we have that $T \models x|_1 y$ always holds by the empty team property. □

We extend the definition of the *set representation* to approximate exclusion atoms by defining $S(x|_p y) = S(x|y)$. Now two approximate exclusion atoms with the same degree of approximation have the same set representation only if they are semantically equivalent. We also extend the definition of the *end-constant form* to approximate exclusion atoms $u|_q v$ as $u'c|_q v'c$ analogously.

Lemma 4. *Suppose that $|x| = |y| = n$, $u \neq v$ and $q \leq p < 1$. Let $uc|_q vc$ be in end-constant form.*

(i) $u|_q v \vdash_{\{A3,A4,A5,A7\}} x|_p y$ iff $S(u|_q v) \subseteq S(x|_p y)$.

(ii) Suppose that $S(uc|_q vc) \not\subseteq S(x|_p y)$ and $S(vc|_q uc) \not\subseteq S(x|_p y)$, then $uc|_q vc \vdash x|_p y$ iff there is $d \in \{x, y\}$ such that

$$S(uc|_q vc) \subseteq \bigcup_{1 \leq i \leq n} \mathcal{C}_{d_i} \times \mathcal{C}_{d_i} \cup \{\langle z_l, z_l \rangle : z_l \in \mathcal{V}\}.$$

Proof. (i) Immediate by the rules $A3$, $A4$, $A5$ and $A7$.

(ii) We note that in a derivation $uc|_q vc \vdash x|_p y$ with $q \leq p$, w.l.o.g., we can let $q = p$, thus the proof can be reduced to the one for Lemma 2 (ii). □

Remark 2. *Note that any set Σ of non-contradictory approximate exclusion atoms is satisfied by a unary team where all variables obtain values that occur only once in the team.*

Next, we generalize the counterexample team in Theorem 1 and show completeness for consequences with approximations $p < \frac{1}{2}$, thus the rule $A8$ can be omitted. We note that the counterexample team constructed in Theorem 1 is the most general type of team that does not satisfy the consequence and does not depend on the assumption set. When generalized to the approximate setting, we must consider the approximations in both the consequence and the assumption set.

Theorem 3 (Completeness). *Let $\Sigma \cup \{x|_p y\}$ be a set of approximate exclusion atoms with $|x| = |y| = n$, $0 \leq p < \frac{1}{2}$ such that if there are $u|_q v \in \Sigma$ with $q > p$, then $r = min\{q > p : u|_q v \in \Sigma\}$ exists.*[4] *If $\Sigma \models x|_p y$ then $\Sigma \vdash x|_p y$.*

Proof. Let $0 \leq p < \frac{1}{2}$ and assume that $\Sigma \nvdash x|_p y$. We show that $\Sigma \not\models x|_p y$. Construct a team T of size k such that $p < \frac{l}{k} \leq r$, where $r = min\{q > p : u|_q v \in \Sigma\}$, for some positive integer l.[5] Construct the team T with values from $\mathbb{N}$ occurring only once, except for $s_1(x) = s_{l+1}(y), \ldots, s_l(x) = s_{2l}(y)$, i.e., to satisfy $x|y$ we would have to remove at least l lines, and for all $e \in \{1, \ldots, l\}$, $e' \in \{l+1, \ldots, 2l\}$ and $i, j \in \{1, \ldots, n\}$, $s_e(x_i) = s_e(x_j)$ and $s_{e'}(y_i) = s_{e'}(y_j)$ if and only if $x_i = x_j$ or $y_i = y_j$.

Clearly, $T \not\models x|_p y$. We show that $T \models u|_q v$ for all $u|_q v \in \Sigma$. Let $u|_q v \in \Sigma$. If $u = v$ and $0 \leq q < 1$, then we derive a contradiction. If $S(u|_q v)$ fulfils any of the cases in Lemma 4 and $q \leq p$, then we can derive a contradiction, and if $q > p$, then we are allowed to remove at least l lines, so $T \models u|_q v$.

For all other cases, we can show similarly to the proof of Theorem 1, that $T \models u|_0 v \models u|_q v$. □

[4] As for approximate dependence atoms in (?), we need to avoid infinite consequences of the form $\{x|_{\frac{1}{n}} y : n \in \mathbb{N}\} \models x|y$, since for a recursive assumption set the decidability of the logical consequence allows us to encode the halting problem.

[5] If no atom in Σ has approximation $q > p$, then $l = 1$ and $k = 2$ suffices.

	x_1	...	x_n	y_1	...	y_n	z_1	z_2	...
s_1	a_1	–	a_n		...				...
s_2	b_1	–	b_n		...				...
s_3		...		a_1	–	a_n			...
s_4		...		b_1	–	b_n			...
s_5		...			...				...

Table 2: Counterexample team for approximate exclusion atoms where the set $\Sigma \cup \{x|_p y\}$ and $r = min\{q > p : u|_q v \in \Sigma\}$ are such that $p < \frac{2}{5} \leq r$.

	x_1	x_2	x_3	v_1	y_3	z_1	z_2	z_3	...
s_1	a_1	a_1	a_2	1	2	3	4	5	...
s_2	b_1	b_1	b_2						...
s_3	c_1	c_1	c_2						...
s_4				a_1	a_2				...
s_5				b_1	b_2				...
s_6				c_1	c_2				...
s_7									...
s_8									...

Table 3: Counterexample team for the consequence $x_1x_2x_3|_{\frac{1}{4}} v_1v_1y_3$ and assumption set Σ with $r = min\{q > p : u|_q v \in \Sigma\} \geq \frac{3}{8}$.

As a direct consequence of the form of the complete proof system for approximate exclusion atoms, we obtain consequence compactness in a very strong sense: Let $\Sigma \cup \{x|_p y\}$ be as in Theorem 3, if $\Sigma \models x|_p v$, then there is some $u|_q v \in \Sigma$ such that $u|_q v \models x|_p y$.

We define Algorithm 1 and show that the finite implication problem for approximate exclusion atoms is decidable in polynomial time.

Theorem 4 (Decidability). *Let $\Sigma \cup \{x|_p y\}$, $p < \frac{1}{2}$, be a finite set of approximate exclusion atoms with $|x| = |y| = n$. The implication problem for whether $\Sigma \vdash x|_p y$ is decidable.*

Proof. We show that Algorithm 1 is sound and complete. First note that the algorithm halts for any input $\Sigma \cup \{x|_p y\}$, since Σ is finite. If the algorithm returns TRUE, then $\Sigma \vdash x|_p y$ follows by the rules $A1$-$A7$ and Lemma 4. If the algorithm returns FALSE at step 3, then by Remark 2 and Theorem 3, $\Sigma \nvdash x|_p y$. If the algorithm returns FALSE at step 13, then by Lemma 4, $\Sigma \nvdash x|_p y$. □

Algorithm 1 $\Sigma \vdash x|_p y$?

Input: Finite set $\Sigma \cup \{x|_p y\}$ of approximate exclusion atoms with $|x| = |y| = n$ and $0 \leq p < \frac{1}{2}$
Output: TRUE if $\Sigma \vdash x|_p y$, FALSE otherwise

```
if x|_q y ∈ Σ or y|_q x ∈ Σ with q ≤ p then return TRUE
if there exists u|_q v ∈ Σ with u = v and q < 1 then return TRUE
if x = y then return FALSE
V := {w_l : w_l occurs in Σ ∪ {x|_p y}}
S(x|_p y) := {⟨x_1, y_1⟩, ..., ⟨x_n, y_n⟩}
C_{x_i} := {y_j : ⟨x_i, y_j⟩ ∈ S(x|_p y)} for all 1 ≤ i ≤ n
C_{y_i} := {x_j : ⟨x_j, y_i⟩ ∈ S(x|_p y)} for all 1 ≤ i ≤ n
for u|_q v ∈ Σ with q ≤ p do
    S(u|_q v) := {⟨u_1, v_1⟩, ..., ⟨u_n, v_n⟩}
    if S(u|_q v) ⊆ S(x|_p y) then return TRUE
    if S(v|_q u) ⊆ S(x|_p y) then return TRUE
    if S(u|_q v) ⊆ ⋃_{1≤i≤n} C_{d_i} × C_{d_i} ∪ {⟨w_l, w_l⟩ : w_l ∈ V} for some d ∈ {x, y} then return TRUE
return FALSE
```

4 Conclusion and Further Research

We axiomatized exclusion atoms and showed that the finite implication problem is decidable. First-order logic with exclusion atoms ($FO(|)$) is expressively equivalent to dependence logic, which has partial axiomatizations (Kontinen and Väänänen 2013; Yang 2019), and one could consider doing the same for $FO(|)$.

On the propositional side, the expressive power of propositional logic with exclusion atoms coincides with propositional dependence logic, but its axiomatization is missing. Furthermore, the system for exclusion atoms presented in this paper is not complete in the propositional setting. For instance, $x_1|y_1, y_1|z_1, z_1|x_1 \models u|v$ holds when we restrict the domain of the model to only two values.

We defined and axiomatized approximate exclusion atoms for consequences with an approximation $p < \frac{1}{2}$. Axiomatizing approximate versions of other downward closed dependencies such as the degenerated dependency in (Thalheim 1991) can be considered. The approximate versions of non-downward closed dependencies, like inclusion, likely have to be captured with a different definition.

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