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Possibility of Conditionals and Conditional Possibilities: From a Triviality Result to Possibilistic Imaging

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Abstract

The present paper extends to possibility theory the classical Lewis triviality result that arises when one tries to identify the probability of conditionals and conditional probability, and its way-outs provided by imaging and generalized imaging updating rules. Precisely, after showing that a triviality result, similar to Lewis's, holds for conditional possibility measures, we start exploring an imaging updating rule within the setting of this uncertainty theory and we prove that imaged possibility indeed represents the possibility of those conditionals that can be formalized in Lewis's logics called C2 (where Stalnaker's conditional are representable) and C1 (that Lewis himself described as an appropriate logic for counterfactual conditionals). Leveraging on recently introduced algebraic models for the aforementioned logics C2 and C1, we finally show that imaged possibilities and possibilities of (counterfactual) conditionals can be uniformly treated in that setting. Indeed, the canonical way of seeing conditional possibility measures as simple measures on those algebras offers a privileged perspective through which one can look at conditional possibility, imaged possibility and the possibility of conditionals.

1 Introduction

Historically, a fundamental question that has been addressed at the intersection of knowledge representation and probability theory is: can we find a logical connective $\rhd$ whose probability aligns exactly with the corresponding conditional probability? More precisely, given a probability P on a Boolean algebra $\mathbf{A}$, the question is if it possible to define a binary operator $\rhd$: $\mathbf{A} \times \mathbf{A} \to \mathbf{A}$ such that for all $a, b \in A$ with $b \neq \bot$

$$P(b \rhd a) = P(a \mid b) \tag{1}$$

This question delves beyond technical considerations, prompting a deeper examination of conditional probability's nature within knowledge representation. Specifically, it investigates whether conditional probability can be interpreted and reduced to the probability of a true conditional statement.

Initially, Stalnaker's conditional (Stalnaker 1968) appeared promising in fulfilling this role, see (Stalnaker 1970). However, subsequent work by Lewis (1976) and Hájek (1989) demonstrated that no (truth-functional) conditional connective within the same Boolean algebra can fulfill this function without trivializing the probability function. These "triviality results" hold significant weight, revealing that conditional probability cannot be directly interpreted as the probability of a conditional relation definable within the original Boolean framework.

Building on these insights, Lewis (1976) and Gärdenfors (1982) showed that the probability of a Stalnaker conditional $b \mathrel{\Box\!\!\rightarrow} a$ can be characterized using a more general update rule for probabilities, called "imaged probability":

$$P(b \mathrel{\Box\!\!\rightarrow} a) = P_b(a). \tag{2}$$

Here, $P_b(\cdot)$ represents a new probability measure, reflecting the scenario where b is true and resulting from transferring all the probability mass of where b is false to the relevant worlds where b is true. More precisely, a Stalnaker conditional, or equivalently a conditional in the logic C2 in (Lewis 1971), is interpreted with respect to a possible worlds model $\Sigma = (\Omega, \mathscr{S}, v)$ where $\mathscr{S}$ is a sphere system that provides a similarity ordering among worlds, and v is a valuation function; here $b \mathrel{\Box\!\!\rightarrow} a$ is deemed true at a world w whenever a (the consequent) is true in *the most similar* world where b (the antecedent) is true. Hence, P_b is the result of transferring the mass of each non-b-world to its most similar b-world; in order to perform the imaging update procedure, a similarity structure over our sample space is needed.

This imaging procedure has become a powerful alternative to conditionalization for updating probabilities with new information, e.g. (Pearl 2017). Also, imaging has been recently employed within the setting of probabilistic belief change (Rens and Meyer 2017). Notably, it allows for an interpretation as the probability of a suitable conditional statement being true. For instance, Rosella and Flaminio (2023), following Dubois and Prade (1994), show how to extend the imaging procedure to Dempster-Shafer belief functions in order to characterize the probability of Lewis counterfactuals, i.e. conditionals in Lewis's logic C1 (Lewis 1971). Finally, in the frame of Theory Change, while (Bayesian) conditioning can be seen as a sort of a numerical AGM revision operator (Gärdenfors 1988), imaging seems to be closer to the notion of update operator in the sense of Katsuno and Meldelzon (1991), see (Dubois and Prade 1994) for a discussion on this.

This paper explores a similar question within the framework of possibility theory (Dubois and Prade 1988), where "possibility measures" are a formal tool to represent uncertainty and knowledge which, unlike probability, rank events based on their plausibility. Interestingly, possibility theory also accounts for conditionalization and conditional possibility measures. Hence, analogous questions arise for possibility measures: is it possible to represent conditional possibilities as possibilities of conditionals? If not, how can we characterize the possibility of conditional logical operators?

We address these questions by proving first a triviality result for possibility theory demonstrating that no truth-conditional operator in a Boolean setting can directly capture (a reasonable notion of) conditional possibilities. Building on this triviality result, we show that the possibility of well-known conditionals (like Stalnaker conditional and Lewis counterfactual) can be characterized using a generalization of "imaged possibility measures" introduced in (Dubois and Prade 1994). This establishes a deep connection between logical conditional operators and the imaging update procedure within possibility theory. We then build upon those in (Flaminio, Godo, and Ugolini 2021), where conditional possibilities are represented as "canonically extended possibility measures" within the so-called *Boolean algebras of conditionals* (BACs). Our results allow us to represent the conditional operators we analysed, along with their induced imaged possibilities, in a highly expressive algebraic framework.

The main contributions of the present paper can be summarized as follows:

1. **Possibilistic Triviality Result**. We establish a fundamental limitation by proving that no conditional operator $\rhd$: $\mathbf{A} \times \mathbf{A} \to \mathbf{A}$ within a Boolean algebra can satisfy the equation $\Pi(b \rhd a) = \Pi(a \mid b)$, where Π denotes a possibility measure.
2. **Possibilities of Conditionals vs Conditional Possibilities**. We define possibility and necessity measures over Lewis-Stalnaker sphere models. This allows us to compute the possibility of conditionals within these frameworks and study their logical properties from a numerical perspective.
3. **Representation of Possibilistic Imaging**. We refine the definition of possibilistic imaging induced by Lewis-Stalnaker models, originally proposed by Dubois and Prade (1994). This refinement allows us to express the possibility of Lewis-Stalnaker conditionals in terms of the corresponding imaged possibility ($\Pi(b \rhd a) = \Pi_b(a)$).
4. **Unified Algebraic Framework**. By combining the results of (Flaminio, Godo, and Hosni 2020) and (Rosella, Flaminio, and Bonzio 2023), we establish a unified algebraic representation for conditional possibility measures and Lewis-Stalnaker conditionals within the novel framework of Lewis algebras. We further develop an algebraic interpretation of the possibility of Lewis-Stalnaker conditionals in terms of a corresponding canonically extended possibility measure defined on a suitable Lewis algebra. This connection reveals that possibilities of conditionals, characterized as imaged possibilities, ultimately correspond to the possibility of modal conditionals within the Lewis algebras framework.

The paper is structured as follows. Section 2 presents the triviality result for possibility measures. Section 3 explores Lewis-Stalnaker semantics, imaged possibility measures, and characterizes the possibility of a relevant class of conditionals. Sections 4 and 5 move to the algebraic setting, applying recent results on conditional possibility and BACs. Finally, Section 6 concludes the paper.

2 Possibilities, Conditional Possibilities and a Lewis-style Triviality Result

We assume the reader to be acquainted with the theory of finite Boolean algebras. For otherwise we recall that for every finite Boolean algebra $\mathbf{A}$ there exists a finite set Ω such that $\mathbf{A} = (A, \wedge, \vee, \neg, \top, \bot)$ is isomorphic to the powerset algebra $(\mathcal{P}(\Omega), \cap, \cup, ^c, \Omega, \emptyset)$ where $\mathcal{P}(\Omega)$ is the powerset of Ω and $\cap, \cup, ^c$ denotes the set-theoretic operations of intersection, union, and complementation, respectively. As we will further elaborate in Remark 3.2 below, the set Ω can actually be understood as a set of possible worlds. Recall from (Givant and Halmos 2009) that every finite Booolean algebra is *atomic* and hence, in the powerset representation of $\mathbf{A}$, the set Ω can be taken to be the set $\mathbb{at}(\mathbf{A})$ of the atoms of $\mathbf{A}$. As a further notation, for every $a \in A$, $\mathbb{at}_{\leq}(a)$ denotes the set of atoms of $\mathbf{A}$ that stand below a.

Notation 1. To ease the reading and without danger of confusion, for $x, y \in [0, 1]$ we will sometimes write $x \vee y$ and $x \wedge y$ instead of $\max\{x, y\}$ and $\min\{x, y\}$ respectively. Similarly, if $X \subseteq [0, 1]$ we will write $\bigvee X = \bigvee_{x \in X} x$ and $\bigwedge X = \bigwedge_{x \in X} x$ to respectively denote $\sup\{x \mid x \in X\}$ and $\inf\{x \mid x \in X\}$.

Possibility theory deals with a type of uncertainty that is alternative to the one handled by probability theory. Possibility measures, the mathematical models of such uncertainty theory, were introduced first in (Zadeh 1978) in the context of fuzzy sets. Later on, the theory has been further extended and developed by Dubois, Prade and colleagues in a series of publications (see e.g., (Dubois and Prade 1988; Dubois and Prade 1991; Dubois and Prade 2014)).

A *possibility measure* on a Boolean algebra $\mathbf{A} = (A, \wedge, \vee, \neg, \top, \bot)$ is a mapping $\Pi : \mathbf{A} \to [0, 1]$ such that

(Π1) $\Pi(\top) = 1$ and $\Pi(\bot) = 0$;

(Π2) $\Pi(\bigvee_{i \in I} a_i) = \sup\{\Pi(a_i) : i \in I\}$, for $\{a_i\}_{i \in I} \subseteq A$.

If the Boolean algebra $\mathbf{A}$ is finite the above (Π2) can be equivalently written as

(Π3) $\Pi(a \vee b) = \max\{\Pi(a), \Pi(b)\}$.

Given an algebra $\mathbf{A}$ we will denote by $\mathcal{P}(\mathbf{A})$ (or simply $\mathcal{P}$ whenever $\mathbf{A}$ is clear by the context) the set of possibility measures on $\mathbf{A}$.

The dual of a possibility measure $\Pi : \mathbf{A} \to [0, 1]$ is the mapping $N : \mathbf{A} \to [0, 1]$, called *necessity measure*, such that, for all $a \in A$,

$$N(a) = 1 - \Pi(\neg a).$$

In the finite setting, possibility and necessity measures are completely determined by their corresponding (normalized) possibility distributions on the set of atoms $\mathfrak{at}(\mathbf{A})$ of the algebra $\mathbf{A}$. Namely, $\Pi : \mathbf{A} \to [0,1]$ is a possibility measure iff there is a mapping $\pi : \mathfrak{at}(\mathbf{A}) \to [0,1]$ such that $\bigvee_{\alpha\in\mathfrak{at}(\mathbf{A})} \pi(\alpha) = 1$ and for all $a \in A$,

$$\Pi(a) = \bigvee_{\alpha\leq a} \pi(\alpha) \quad \text{and} \quad N(a) = \bigwedge_{\alpha\leq\neg a} 1 - \pi(\alpha).$$

As for *conditional* possibility measures, there have been several proposals in the literature, see e.g. (Dubois and Prade 1990; Baets, Tsiporkova, and Mesiar 1999; Coletti and Vantaggi 2009) and (Walley and de Cooman 1999) for a survey.

To introduce conditional possibility measures in a general setting, we first need to recall the following notion.

Definition 2.1. A function $* : [0,1]\times[0,1] \to [0,1]$ is called a *t-norm* if it is such that:

(Commutativity): $x * y = y * x$;

(Associativity): $x * (y * z) = (x * y) * z$;

(Monotonicity): $x * z \leq y * z$, whenever $x \leq y$;

(Neutral element): $1 * x = x$.

A t-norm $*$ is said to be *(left-)continuous* if so is with respect to the usual topology of $[0,1]$. Also $*$ is said to be *without zero-divisors* if for all $x, y \in [0,1]$, $x * y = 0$ implies $x = 0$ or $y = 0$.

Main examples of continuous t-norms are the Łukasiewicz t-norm $x * y = \max(x + y - 1, 0)$, the product t-norm $x * y = x \cdot y$, and the minimum t-norm $x * y = \min(x, y)$. The last two are without zero-divisors.

T-norms are $[0,1]$-valued *conjunctive* operators in the sense that they are extensions of the $\{0,1\}$-valued Boolean conjunction truth-function. For every t-norm $*$ one can define an implication-like operation $\Rightarrow: [0,1] \times [0,1] \to [0,1]$ as follows: for all $x, y \in [0,1]$

$$x \Rightarrow y = \bigvee\{z \in [0,1] \mid x * z \leq y\}.$$

When $*$ is left-continuous, the supremum above is in fact a maximum, and $\Rightarrow$ is called the *residuum* of $*$ since then the following residuation condition holds: $x * y \leq z$ iff $x \leq y \Rightarrow z$. In such a case, $(*, \Rightarrow)$ is called a *residuated pair* (see e.g. the monograph (Klement, Mesiar, and Pap 2000) for further details). In the rest of this paper, and without loss of generality, we will assume $*$ to be continuous so that its residuum $\Rightarrow$ always exists. Notice that if $*$ is without zero-divisors, then $x \Rightarrow 0 = 0$ iff $x > 0$.

For a given continuous t-norm $*$, let us now recall our working definition of $*$-conditional possibility function on $\mathbf{A}$ as a primitive notion, which was originally introduced in (Bouchon-Meunier, Coletti, and Marsala 2002), see also (Coletti and Vantaggi 2007; Coletti and Vantaggi 2009).

From now on, for every Boolean algebra $\mathbf{A}$ we will denote by A' the set $A \setminus \{\bot\}$ of its positive elements.

Definition 2.2. Given a t-norm $*$, a $*$-conditional possibility measure on a Boolean algebra $\mathbf{A}$ is a binary map $\overline{\Pi}(\cdot \mid \cdot) : A \times A' \to [0,1]$ satisfying the following conditions:

($C\Pi1$) $\overline{\Pi}(a \mid b) = \overline{\Pi}(a \wedge b \mid b)$, for all $a \in A, b \in A'$

($C\Pi2$) $\overline{\Pi}(\cdot \mid b)$ is a possibility measure on $\mathbf{A}$, for all $b \in A'$

($C\Pi3$) $\overline{\Pi}(a \wedge b \mid c) = \overline{\Pi}(b \mid a \wedge c) * \overline{\Pi}(a \mid c)$, for all $a, b, c \in A$ such that $a \wedge c \in A'$.

We will denote by $\mathcal{CP}_*(\mathbf{A})$ (or simply $\mathcal{CP}_*$ when the algebra $\mathbf{A}$ is clear by the context) the set of all $*$-conditional possibilities on $\mathbf{A}$. Also, for any $b \in A'$ we will write $\overline{\Pi}^b(\cdot)$ to denote the possibility measure on $\mathbf{A}$ as ensured by ($C\Pi2$).

In addition to the axiomatic definition given in Definition 2.2 above, conditional possibilities can be defined from an unconditional one and a residuated pair. Indeed, given a possibility measure $\Pi : \mathbf{A} \to [0,1]$ and a residuated pair $(*, \Rightarrow)$, one can define the following mapping $\Pi_* : A \times A' \to [0,1]$ as follows:

$$\Pi_*(a \mid b) = \begin{cases} \Pi(b) \Rightarrow \Pi(a \wedge b), & \text{if } a \wedge b \neq \bot \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

As shown in (Coletti, Petturiti, and Vantaggi 2013), Π_* is a $*$-conditional possibility in the sense of Definition 2.2, that is, $\Pi_* \in \mathcal{CP}_*(\mathbf{A})$.

Lemma 2.3. *For every possibility measure Π on $\mathbf{A}$ and for every t-norm $*$, Π_* defined as above is a $*$-conditional possibility measure, whence it belongs to $\mathcal{CP}_*(\mathbf{A})$.*

Now, for the same possibility measure Π on $\mathbf{A}$, let

$$\mathcal{CP}_*(\Pi) = \{\overline{\Pi} \in \mathcal{CP}_* \mid \overline{\Pi}(\cdot \mid \top) = \Pi(\cdot)\}$$

be the subset of $\mathcal{CP}_*(\mathbf{A})$ of those $*$-conditional measures agreeing with Π on $\mathbf{A}$ when fixing the conditional event to $\top$. It is easy to check that, for any $a \in A$,

$$\Pi_*(a \mid \top) = \Pi(\top) \Rightarrow \Pi(\top \wedge a) = 1 \Rightarrow \Pi(a) = \Pi(a),$$

and hence $\Pi_* \in \mathcal{CP}_*(\Pi)$. Moreover, in (Flaminio, Godo, and Ugolini 2021) it is shown that Π_* is the greatest measure in $\mathcal{CP}(\Pi)$.

Lemma 2.4. *For every possibility measure Π on $\mathbf{A}$ and a continuous t-norm $*$, $\Pi_* \in \mathcal{CP}_*(\Pi)$. Moreover, if $\overline{\Pi} \in \mathcal{CP}_*(\Pi)$ then $\overline{\Pi} \leq \Pi_*$.*

Therefore, Π_* can be intuitively regarded as the least informative $*$-conditional measure extending Π.

Now we can prove the main result of this section.

Theorem 2.5. *Let $*$ be a t-norm without zero-divisors and let $\mathbf{A}$ be a finite Boolean algebra. Assume there exists a binary connective $\rhd$ in $\mathbf{A}$ such that, for all $a, c \in A$, $\Pi(a \rhd c) = \Pi_*(c \mid a)$ for all $\Pi \in \mathcal{P}(\mathbf{A})$. Then,*

$$\Pi(a \rhd c) = \Pi(c)$$

for all $a, c \in A$ such that $a \wedge c \neq \bot$ and $\Pi(a \wedge \neg c) > 0$.

Proof. Let us start observing that $(a \rhd c) = (a \rhd c) \wedge \top = (a \rhd c) \wedge (c \vee \neg c) = ((a \rhd c) \wedge c) \vee ((a \rhd c) \wedge \neg c)$. Therefore, by ($\Pi3$)

$$\begin{aligned} \Pi(a \rhd c) &= \Pi(((a \rhd c) \wedge c) \vee ((a \rhd c) \wedge \neg c)) \\ &= \max\{\Pi((a \rhd c) \wedge c), \Pi((a \rhd c) \wedge \neg c)\}. \end{aligned}$$

By ($C\Pi3$), since $\Pi_* \in \mathcal{CP}_*(\boldsymbol{\Pi})$, by Lemma 2.3 the two factors in the above expressions are

$$\begin{aligned} \Pi((a \rhd c) \wedge c) &= \Pi_*((a \rhd c) \wedge c \mid \top) \\ &= \Pi_*(a \rhd c \mid c) * \Pi_*(c \mid \top) \\ &= \Pi_*(a \rhd c \mid c) * \Pi(c), \end{aligned}$$

and

$$\begin{aligned}\Pi((a \rhd c) \wedge \neg c) &= \Pi_*((a \rhd c) \wedge \neg c \mid \top) \\ &= \Pi_*(a \rhd c \mid \neg c) * \Pi_*(c \mid \top) \\ &= \Pi_*(a \rhd c \mid \neg c) * \Pi(\neg c).\end{aligned}$$

Therefore,

$$\Pi(a \rhd c) = \max(\Pi_*(a \rhd c \mid c)*\Pi(c), \Pi_*(a \rhd c \mid \neg c)*\Pi(\neg c)) \tag{4}$$

As anticipated above, for every $d \in A \setminus \{\bot\}$, let us write $\Pi^d(\cdot)$ for $\Pi_*(\cdot \mid d)$. Then, Π^c and $\Pi^{\neg c}$ are possibility measures and hence by hypothesis we have:

$$\Pi_*(a \rhd c \mid c) = \Pi^c(a \rhd c) = (\Pi^c)_*(c \mid a),$$
$$\Pi_*(a \rhd c \mid \neg c) = \Pi^{\neg c}(a \rhd c) = (\Pi^{\neg c})_*(c \mid a).$$

Let us compute e.g. $(\Pi^c)_*$. By definition, for all $x, y \in A$ with $y \neq \bot$,

$$(\Pi^c)_*(x \mid y) = \begin{cases} \Pi^c(y) \Rightarrow \Pi^c(x \wedge y), & \text{if } x \wedge y \neq \bot \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, assuming $a \wedge c \neq \bot$ and using $(C\Pi 1)$, we have:

$$\begin{aligned}(\Pi^c)_*(c \mid a) &= \Pi^c(a) \Rightarrow \Pi^c(a \wedge c) \\ &= \Pi_*(a \mid c) \Rightarrow \Pi_*(a \wedge c \mid c) \\ &= \Pi_*(a \mid c) \Rightarrow \Pi_*(a \mid c) = 1.\end{aligned}$$

Analogously, by definition of Π_*, we have:

$$\begin{aligned}(\Pi^{\neg c})_*(c \mid a) &= \Pi^{\neg c}(a) \Rightarrow \Pi^{\neg c}(a \wedge c) \\ &= \Pi_*(a \mid \neg c) \Rightarrow \Pi_*(a \wedge c \mid \neg c) \\ &= \Pi_*(a \mid \neg c) \Rightarrow 0.\end{aligned}$$

Since $*$ has no zero-divisors and $\Pi_*(a \mid \neg c) > 0$, one gets $\Pi_*(a \mid \neg c) \Rightarrow 0 = 0$, whence $(\Pi^{\neg c})_*(c \mid a) = 0$. Summing up, we have:

$$\begin{aligned}&\Pi(a \rhd c) = \\ &\max(\Pi_*(a \rhd c \mid c) * \Pi(c), \Pi_*(a \rhd c \mid \neg c) * \Pi(\neg c)) = \\ &\max((\Pi^c)_*(c \mid a) * \Pi(c), (\Pi^{\neg c}_*(c \mid a) * \Pi(\neg c)) = \\ &\max(1 * \Pi(c), 0 * \Pi(\neg c)) = \Pi(c).\end{aligned}$$

That is to say, $\Pi(a \rhd c) = \Pi(c)$. □

The above leads to a possibilistic version of Lewis's triviality result. Indeed, let us call a possibility measure Π on $\mathbf{A}$ *trivial* if there is no pair of elements $a, c \in A$ such that $a \wedge c \neq \bot$ and $\Pi(a \wedge \neg c) > 0$ (and thus $a \wedge \neg c \neq \bot$). Actually, as next lemma shows, trivial possibility measures are those that assign value 1 to only one atom and 0 to the rest of the atoms.

Lemma 2.6. *A (normalised) possibility measure on* $\mathbf{A}$ *is trivial iff there exists an atom* $\alpha \in \mathbb{at}(\mathbf{A})$ *such that* $\Pi(\alpha) = 1$ *and* $\Pi(\neg\alpha) = 0$.

Proof. Let $\mathrm{Supp}(\Pi) = \{\alpha \in \mathbb{at}(\mathbf{A}) \mid \Pi(\alpha) > 0\}$. To prove the left-to-right direction, notice that by hypothesis $|\mathrm{Supp}(\Pi)| = 1$ and hence it is clear that Π is trivial.

As for the right-to-left direction, by absurdum, assume that $|\mathrm{Supp}(\Pi)| > 1$, and let $\alpha_1, \alpha_2 \in \mathrm{Supp}(\Pi)$. Further let $a = \alpha_1 \vee \alpha_2$, and $b = \neg\alpha_2$. It follows that $a \wedge b = \alpha_1 \geq \bot$, while $\Pi(a \wedge \neg b) = \Pi(\alpha_2) > 0$, and hence Π is not trivial.

Therefore, the only trivial possibility measures Π are exactly those such that $|\mathrm{Supp}(\Pi)| = 1$. □

Finally, we have below a corollary which is the possibilistic analogue to (Hájek 2011, Theorem in §5). We adapt from it the following notions: 1) a class $\mathcal{C}$ of possibility measures on $\mathbf{A}$ is *closed under $*$-conditioning* whenever if $\Pi \in \mathcal{C}$ then $\Pi_*(\cdot \mid a) \in \mathcal{C}$ as well for every $a \in A'$; and 2) a conditional $\rhd$ is a *possibility $*$-conditional* for $\mathcal{C}$ if, for any $\Pi \in \mathcal{C}$, $\Pi(a \rhd c) = \Pi_*(c \mid a)$ holds for all $a, c \in A$.

Corollary 2.7. *Let $*$ be a t-norm without zero-divisors. If a class $\mathcal{C}$ of possibility measures on $\mathbf{A}$ is closed under $*$-conditioning, then there is no possibility $*$-conditional $\rhd$ for $\mathcal{C}$ unless the class $\mathcal{C}$ consists entirely of trivial possibility measures.*

By Theorem 6 in (Flaminio, Godo, and Marchioni 2012), in fact, trivial possibility measures are exactly the Boolean homomorphisms of $\mathbf{A}$ to the two-elements Boolean chain $\mathbf{2}$.

3 Sphere Models and Possibilistic Imaging

The previous results provide a valuable stepping stone, demonstrating that defining a conditional operator whose possibility aligns with the corresponding conditional possibility measure is only feasible when the possibility measure is trivial. This suggests that, similarly to the case of probability theory, also possibility theory may benefit from exploring alternative approaches to information update, particularly those that can effectively capture conditional relationships in non-trivial scenarios.

Therefore, a natural question arises: given a conditional operator $\rhd$, is it possible to characterize its possibility in terms of an updating procedure different from conditionalization? In other words, do conditionals encode a different notion of possibilistic updating?

The main result of this section is the characterization of the possibility of Lewis-Stalnaker conditionals in terms of imaged possibilities, which can be seen as an alternative updating procedure to conditionalization. To show this characterization result, we need first to review the semantic properties of those conditionals and their models.

Let us recall the definition of a sphere model from (Lewis 1973). In the following we will consider a classical finitely-generated language $\mathcal{L}$ expanded with a new binary connective $\Box\!\!\rightarrow$. As usual, $\varphi \Diamond\!\!\rightarrow \psi$ will be a shorthand for $\neg(\varphi \Box\!\!\rightarrow \neg\psi)$.

Definition 3.1. Given a finite set of possible worlds Ω, a *sphere model* on Ω for the language $\mathcal{L}$ is a system $\Sigma = (\Omega, \mathscr{S}, v)$ where $\mathscr{S}$ is a function $\mathscr{S} : \Omega \to \mathcal{P}(\mathcal{P}(\Omega))$ that assigns, to each $\alpha \in \Omega$, a set $\mathscr{S}_\alpha$ of subsets of Ω with the following properties: for all $\alpha \in \Omega$,

$(S1)$ $\mathscr{S}_\alpha$ is nested: for all $S_i, S_j \in \mathscr{S}_\alpha$, either $S_i \subseteq S_j$ or $S_j \subseteq S_i$;

$(S2)$ for all $S \in \mathscr{S}_\alpha$, $\alpha \in S$;

$(S3)$ either $\bigcup \mathscr{S}_\alpha = \emptyset$, or $\{\alpha\} \in \mathscr{S}_\alpha$.

For each propositional variable $p \in \mathcal{L}$, $v(p)$ is a subset of Ω. The map v is extended to compound formulas as follows:

- $v(\neg\Phi) = I \setminus v(\Phi)$;
- $v(\Phi \wedge \Psi) = v(\Phi) \cap v(\Psi)$;

- $v(\varphi \Box\!\!\rightarrow \psi) = \{\alpha \in \Omega : v(\varphi) \cap \bigcup \mathscr{S}_\alpha = \emptyset$, or $\exists S \in \mathscr{S}_\alpha\, (\emptyset \neq (v(\varphi) \cap S) \subseteq v(\psi))\}$.

Given a sphere model $\Sigma = (\Omega, \mathscr{S}, v)$, we write $\Sigma, \alpha \Vdash \Phi$, if $\alpha \in v(\Phi)$.

Conforming to a standard notation we will say that a sphere model $\Sigma = (\Omega, \mathscr{S}, v)$ is

- *Centered*, if for all $\alpha \in \Omega$, $\bigcap \mathscr{S}_\alpha = \{\alpha\}$;
- *Absolute*, if for all $\alpha \in \Omega$, $\bigcup \mathscr{S}_\alpha = \Omega$.

In absolute sphere models the evaluation of $\varphi \Box\!\!\rightarrow \psi$ and $\varphi \Diamond\!\!\rightarrow \psi$ simplifies as follows:

- $v(\varphi \Box\!\!\rightarrow \psi) = \{\alpha \in \Omega : \exists S \in \mathscr{S}_\alpha, v(\varphi) \cap S \subseteq v(\psi)\}$;
- $v(\varphi \Diamond\!\!\rightarrow \psi) = \{\alpha \in \Omega : \forall S \in \mathscr{S}_\alpha$, if $v(\varphi) \cap S \neq \emptyset$, then $v(\varphi \wedge \psi) \cap S \neq \emptyset\}$.

From now on we will always assume that the sphere models we consider are centered and absolute.

Remark 3.2. From now on, to be uniform with the algebraic, rather than logical, framework used in (Flaminio, Godo, and Hosni 2020) and (Rosella, Flaminio, and Bonzio 2023) on which the present paper builds upon, we will assume without loss of generality the set of possible worlds of every sphere model $\Sigma = (\Omega, \mathscr{S}, v)$ to be the set of atoms $\mathbb{at}(\mathbf{A})$ of a finite Boolean algebra $\mathbf{A}$. Under this assumption, for simplicity, we can forget about the logical language $\mathcal{L}$ and consider formulas as elements of $\mathbf{A}$, meaning that rather than looking at formulas in the syntax, we already consider interpreted formulas as elements of the dual Boolean algebra induced by what we call a *Lewis sphere model*, that is a sphere model $\Sigma = (\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$ where for every $a \in A$, $v(a) = \{\alpha \in \mathbb{at}(\mathbf{A}) \mid \alpha \leq a\} = \mathbb{at}_{\leq}(a)$. For example, the evaluation of a conditional $b \Box\!\!\rightarrow a$ becomes $v(b \Box\!\!\rightarrow a) = \{\alpha \in \mathbb{at}(\mathbf{A}) \mid \exists S \in \mathscr{S}_\alpha$ s.t. $\mathbb{at}_{\leq}(b) \cap S \subseteq \mathbb{at}_{\leq}(a)\}$. We will denote by $\mathbb{L}$ the class of centered and absolute Lewis sphere models.

We may refer to a conditional $b \Box\!\!\rightarrow a$ interpreted in a Lewis sphere model as a *Lewis conditional*, or a conditional in the logic C1. Indeed, the logic induced by the the logical consequence over Lewis models amounts to the logic C1 in (Lewis 1971).

It is convenient to display every (absolute) sphere system $\mathscr{S}_\alpha = \langle S_0, S_1, \ldots, S_k \rangle$ as partition of $\Omega = \mathbb{at}(\mathbf{A})$ by the following ordered sequence

$$\langle D_0^\alpha, D_1^\alpha, \ldots, D_k^\alpha \rangle$$

where for all $j = 0, \ldots, k$,

$$D_j^\alpha = S_j \setminus \textstyle\bigcup_{i<j} S_i.$$

Thus, the sets D_j^α can be seen as the disjoint rings around α that determine, in a Lewis model, a total pre-order on $\mathbb{at}(\mathbf{A})$.

Let us fix now an element $b \in A \setminus \{\bot\}$ and a centered sphere model $\Sigma = (\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$. Furthermore, let $i_0 = \min\{i \mid D_i^\beta \cap \mathbb{at}_{\leq}(b) \neq \emptyset\}$ the closest ring to β containing an atom below b according to Σ. Then the set of worlds/atoms below b that are closest to an atom β is

$$\Sigma_b(\beta) = D_{i_0}^\beta \cap \mathbb{at}_{\leq}(b). \tag{5}$$

Note that, if $\beta \leq b$, then $\Sigma_b(\beta) = \{\beta\}$. Also note that $\Sigma_b(\beta) \neq \emptyset$ for each $\beta \in \mathbb{at}(\mathbf{A})$.

Next lemma gathers different equivalent conditions for conditional formulas $b \Diamond\!\!\rightarrow a$ and $b \Box\!\!\rightarrow a$ to hold in a sphere model.

Lemma 3.3. *For every sphere model $\Sigma = (\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$, for all $\alpha \in \mathbb{at}(\mathbf{A})$ and for all $a \in A$ and $b \in A \setminus \{\bot\}$, the following are equivalent:*

(i) $\Sigma, \alpha \Vdash b \Diamond\!\!\rightarrow a$;
(ii) $\Sigma_b(\alpha) \cap \mathbb{at}_{\leq}(a) \neq \emptyset$.

Moreover, the following are equivalent as well:

(i′) $\Sigma, \alpha \Vdash b \Box\!\!\rightarrow a$;
(ii′) $\Sigma_b(\alpha) \subseteq \mathbb{at}_{\leq}(a)$.

Proof. The proof directly follows by the definitions given above. Let us sketch the equivalence between (i) and (ii) as an illustrative example. Suppose that $\Sigma, \alpha \Vdash b \Diamond\!\!\rightarrow a$ and let $S \in \mathscr{S}_\alpha$ be the first sphere, in the inclusion order, such that $v(b) \cap S \neq \emptyset$. Thus $v(b) \cap S = v(b) \cap D_{i_0}^\alpha$ and so $v(a \wedge b) \cap D_{i_0}^\alpha = v(a) \cap v(b) \cap D_{i_0}^\alpha \neq \emptyset$. By (5) $\Sigma_b(\alpha) \cap \mathbb{at}_{\leq}(a) = D_{i_0}^\alpha \cap \mathbb{at}_{\leq}(b) \cap \mathbb{at}_{\leq}(a)$. Thus the claim follows by what we said in Remark 3.2 according to which for all $d \in A$, $v(d)$ stands for $\mathbb{at}_{\leq}(d)$. □

Within the wide class $\mathbb{L}$ of Lewis sphere models, let us isolate the subclass of *Stalnaker* models that we will denote by $\mathbb{S}$ and that are defined, within $\mathbb{L}$, as follows.

Definition 3.4. A Lewis sphere model $\Sigma = (\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$ is *Stalnaker* if for all $\alpha \in \mathbb{at}(\mathbf{A})$, and displaying $\mathscr{S}_\alpha$ as the sequence $\langle D_0^\alpha = \{\alpha\}, D_1^\alpha, D_2^\alpha, \ldots, D_k^\alpha \rangle$ of disjoint rings, each set D_i^α, for all $i = 1, \ldots, k$, contains a single atom of $\mathbf{A}$, i.e. $|D_i^\alpha| = 1$.

The above definition reflects the original idea that the spheres of Stalnaker models are total orders on the set of worlds. We may refer to a conditional $b \Box\!\!\rightarrow a$ interpreted in a Stalnaker model as a *Stalnaker conditional*, or a conditional in the logic C2. Indeed, the logic induced by the the logical consequence over Stalnaker models amounts to the logic C2 in (Lewis 1971).

To recall how possibilistic imaging has been approached in (Dubois and Prade 1994)[1], let us fix a Lewis sphere model $\Sigma = (\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$ and an element $b \in A \setminus \{\bot\}$. Furthermore, recall how $\Sigma_b(\beta)$ is defined in (5) for all $\beta \in \mathbb{at}(\mathbf{A})$.

The following is our definition of possibilistic imaging inspired by the one given in (Dubois and Prade 1994).

Definition 3.5 (Possibilistic imaging). Let $\mathbf{A}$ be a finite Boolean algebra, π be a possibility distribution on $\mathbb{at}(\mathbf{A})$, and Σ a sphere model over $\mathbb{at}(\mathbf{A})$. For $b \in A \setminus \{\bot\}$, the *possibility imaged at* b is the map π_b such that, for every $\alpha \in \mathbb{at}(\mathbf{A})$,

$$\pi_b(\alpha) = \begin{cases} \pi(\alpha) \vee \bigvee\{\pi(\beta) : \beta \notin \mathbb{at}_{\leq}(b), \alpha \in \Sigma_b(\beta)\}, & \text{if } \alpha \leq b \\ 0, & \text{if } \alpha \not\leq b \end{cases}$$

[1] Although our definition is slightly different from the one given by Dubois and Prade, the inspiring idea is taken from their paper.

The intuition is that the weight of an atom β not below b is transferred to the atoms under b which are closest to β, according to Σ.

The following easy result shows that the above definition is sound, i.e., π_b is indeed a normalized possibility distribution on $\mathbb{at}(\mathbf{A})$ for all $b \neq \bot$.

Proposition 3.6. *For every normalized possibility distribution $\pi : \mathbb{at}(\mathbf{A}) \to [0,1]$ and for every $b \in A'$, $\pi_b : \mathbb{at}(\mathbf{A}) \to [0,1]$ is a normalized possibility distribution as well.*

Proof. Assume that $\alpha_0 \in \mathbb{at}(\mathbf{A})$ is such that $\pi(\alpha_0) = 1$. Then, if $\alpha_0 \leq b$, one has $\pi_b(\alpha_0) \geq \pi(\alpha_0) = 1$. Conversely, if $\alpha_0 \not\leq b$, consider $\Sigma_b(\alpha_0)$. Since the sphere systems we are dealing with are absolute, $\Sigma_b(\alpha_0) \neq \emptyset$ and, for all $\beta \in \Sigma_b(\alpha_0)$, $\pi_b(\beta) = 1$ by definition of π_b. Therefore, π_b is normalized. □

Since the sphere models we are considering in this paper are all centered, i.e. $D_0^\beta = \{\beta\}$ for each $\beta \in \mathbb{at}(A)$, for every $\alpha, \beta \in \mathbb{at}_{\leq}(b)$, $\alpha \notin \Sigma_b(\beta) = \{\beta\}$, and the above definition can hence be substantially simplified to

$$\pi_b(\alpha) = \bigvee\{\pi(\beta) : \beta \in \Omega, \alpha \in \Sigma_b(\beta)\}. \quad (6)$$

Notice that in both definitions if $\alpha \not\leq b$, $\pi_b(\alpha) = 0$.

Example 3.7. By way of example, consider an algebra $\mathbf{A}$, $b \in A \setminus \{\bot\}$ and a sphere model Σ such that for $\alpha \leq \neg b$, $\Sigma_b(\alpha) = \{\beta_1, \beta_2, \beta_3\}$ as in Figure 1. Let π be a (normalized) possibility distribution on $\mathbb{at}(\mathbf{A})$ such that

$$\pi(\beta_1) < \pi(\beta_2) < \pi(\alpha) < \pi(\beta_3).$$

Then one has that

$$\pi_b(\beta_1) = \pi_b(\beta_2) = \pi(\alpha) < \pi_b(\beta_3) = \pi(\beta_3),$$

while $\pi_b(\alpha) = 0$. Figure 1 provides a graphical representation for the redistribution of the possibilities of the atoms that do not stand below a fixed $b \in A \setminus \{\bot\}$.

Finally, from the imaged possibility distribution π_b one can define the corresponding imaged possibility and necessity measures on $\mathbf{A}$ as usual:

$$\Pi_b(a) = \bigvee_{\alpha \leq a} \pi_b(\alpha) \quad \text{and} \quad N_b(a) = \bigwedge_{\alpha \leq \neg a} 1 - \pi_b(\alpha).$$

Observe that for any distribution π on $\mathbb{at}(\mathbf{A})$ and any $b \in A'$, the imaged possibility at b is such that, for $c \in A$, $\Pi_b(c) = \Pi_b(c \wedge b)$.

4 Algebraic Models of Lewis Counterfactuals and Stalnaker Conditionals

Boolean algebras of conditionals have been defined in (Flaminio, Godo, and Hosni 2020) as algebraic models for conditional formulas. In a nutshell, given any Boolean algebra $\mathbf{A}$ where we represent plain events, its associated Boolean algebra of conditionals $\mathcal{C}(\mathbf{A})$ is defined as a suitable quotient of the free Boolean algebra $\text{Free}(A \times A')$ generated by the pair of elements (a, b) for $a \in A$ and $b \in A' = A \setminus \{\bot\}$. Pairs $(a, b) \in A \times A'$ are interpreted as *basic conditional events* $(a \mid b)$, and then these basic conditionals are closed under Boolean operations, yielding *compound conditionals*.

If $\mathbf{A}$ is a Boolean algebra with n atoms, i.e. $|\mathbb{at}(\mathbf{A})| = n$, it is shown in (Flaminio, Godo, and Hosni 2020) that the atoms of $\mathcal{C}(\mathbf{A})$ are in one-to-one correspondence with sequences $\overline{\alpha} = \langle \alpha_1, \ldots, \alpha_{n-1} \rangle$ of $n-1$ pairwise different atoms of $\mathbf{A}$, each of these sequences giving rise to an atom $\omega_{\overline{\alpha}}$ defined as the following conjunction of $n-1$ basic conditionals:

$$\omega_{\overline{\alpha}} = (\alpha_1 \mid \top) \sqcap (\alpha_2 \mid \neg\alpha_1) \sqcap \ldots \sqcap (\alpha_{n-1} \mid \neg\alpha_1 \wedge \ldots \wedge \neg\alpha_{n-2}), \quad (7)$$

It is then clear that $|\mathbb{at}(\mathcal{C}(\mathbf{A}))| = n!$.

Now, let Σ be a (centered and absolute) Lewis sphere model on $\mathbb{at}(\mathbf{A})$. As we did in Section 3, for every $\alpha \in \mathbb{at}(\mathbf{A})$, let us display the sphere system $\mathscr{S}_\alpha$ as

$$\mathscr{S}_\alpha = \langle D_0 = \{\alpha\}, D_1, D_2, \ldots, D_k \rangle$$

For all $\alpha \in \mathbb{at}(\mathbf{A})$, let Path_α be the set of *maximal paths* through $\mathscr{S}_\alpha$, that is to say, a string $\sigma = \langle \beta_0, \beta_1, \ldots, \beta_{n-1} \rangle$ of elements of $\mathbb{at}(\mathbf{A})$ belongs to Path_α iff

- $\beta_0 = \alpha$; and
- $\langle \beta_{|D_{i-1}|+1}, \ldots, \beta_{|D_i|} \rangle$ is a permutation of the elements in D_i, for each $i = 1, \ldots, k$,

Therefore, for every $\alpha \in \mathbb{at}(\mathbf{A})$, Path_α fixes the set of atoms of $\mathcal{C}(\mathbf{A})$ that corresponds to the lists being the maximal paths defined as above. Note that, by (7) every maximal path $\sigma \in \text{Path}_\alpha$ corresponds to one and only one atom of $\mathcal{C}(\mathbf{A})$. We can hence define the following accessibility relation on $\mathbb{at}(\mathcal{C}(\mathbf{A}))$:

$$R_\Sigma = \{(\omega, \omega') \mid \omega[1] = \alpha, \omega' \in \text{Path}_\alpha\}, \quad (8)$$

where $\omega[1]$ denotes the first element of the sequence associated to ω. In other words R_Σ is such that every atom ω of $\mathcal{C}(\mathbf{A})$ that stands below α accesses every atom ω' in Path_α. Notice that R_Σ is symmetric on Path_α and no atoms below α can access any other atom that does not stand below the same $\alpha \in \mathbb{at}(\mathbf{A})$.

The relation R_Σ hence defines a modal frame on $\mathbb{at}(\mathcal{C}(\mathbf{A}))$, $F_\Sigma = (\mathbb{at}(\mathcal{C}(\mathbf{A})), R_\Sigma)$. This gives a normal modal operator $\Box_\Sigma : \mathcal{C}(\mathbf{A}) \to \mathcal{C}(\mathbf{A})$ defined as usual: for all $c \in \mathcal{C}(\mathbf{A})$

$$\Box_\Sigma(c) = \bigvee\{\omega \in \mathbb{at}(\mathcal{C}(\mathbf{A})) \mid \bigvee_{(\omega,\omega') \in R_\Sigma} \omega' \leq c\}.$$

It is not difficult to see that $\Box_\Sigma$ is equivalently defined as

$$\Box_\Sigma(c) = \bigvee\{\alpha \in \mathbb{at}(\mathbf{A}) \mid \bigvee \text{Path}_\alpha \leq c\}. \quad (9)$$

Summing up, starting from a Lewis sphere model $\Sigma = (\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$, we have defined a unary operator $\Box_\Sigma$ on $\mathcal{C}(\mathbf{A})$, and hence the structure $(\mathcal{C}(\mathbf{A}), \Box_\Sigma)$ is a Boolean algebra with operator in the sense of (Blackburn, de Rijke, and Venema 2002)). By Proposition 5.7 in Rosella et al. (2023) the above is a *Lewis algebra* in the following sense.

Definition 4.1. (Rosella, Flaminio, and Bonzio 2023) For every Boolean algebra $\mathbf{A}$, a *Lewis algebra of* $\mathbf{A}$ is a pair $(\mathcal{C}(\mathbf{A}), \Box)$ where $\Box : A \to A$ satisfies the following conditions (and $c_1 \to c_2$ stands as usual for $\neg c_1 \vee c_2$ in $\mathcal{C}(\mathbf{A})$):

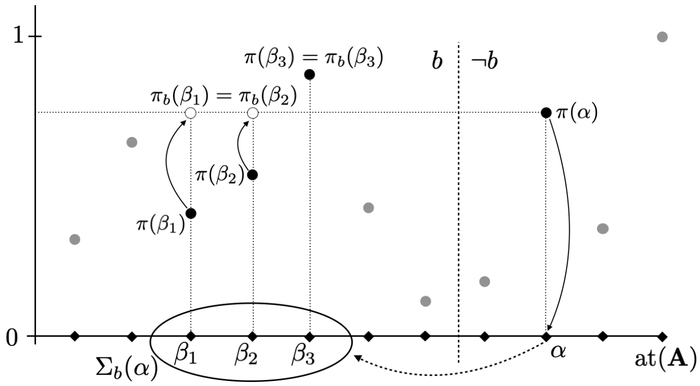

Figure 1: An illustration on the generalized imaging for possibility measures. The possibility $\pi(\alpha)$ of the atom α that lays below $\neg b$ (as it stands on the right-hand-side of the dotted line) is tranferred to the atoms $\beta_1, \beta_2, \beta_3$ that lay below b and are the closest to α according to Σ. The curved arrows show how $\pi(\beta_1)$ and $\pi(\beta_2)$ move to $\pi_b(\beta_1) = \pi_b(\beta_2) = \pi(\alpha)$. Conversely, $\pi_b(\beta_3) = \pi(\beta_3)$ because $\pi(\beta_3) > \pi(\alpha)$.

$(L1)$ $\Box(t \to t') \leq (\Box t \to \Box t')$ for all $t, t' \in \mathcal{C}(\mathbf{A})$;

$(L2)$ $\Box(a \mid \top) = (a \mid \top)$ for all $a \in A$;

$(L3)$ $\Box(a \mid a \vee b) \sqcup \Box(b \mid a \vee b) \sqcup [\Box(c \mid a \vee b) \to \Box((c \mid a) \sqcap (c \mid b))] = 1$ for all $a, b, c \in A$ with $a, b \neq \bot$.

In every Lewis algebra $(\mathcal{C}(\mathbf{A}), \Box)$, one can clearly define $\Diamond : \mathcal{C}(\mathbf{A}) \to \mathcal{C}(\mathbf{A})$ as usual: for all $c \in \mathcal{C}(\mathbf{A})$

$$\Diamond c = \neg \Box \neg c.$$

To fully appreciate the results we will show in the rest of the present paper, it is important to recall that in (Rosella, Flaminio, and Bonzio 2023) Lewis algebras have been shown to be algebraic models for *would* and *might* counterfactuals. Those are interpreted in a Lewis algebra $(\mathcal{C}(\mathbf{A}), \Box)$ as follows: for all $a \in A$ and $b \in A'$,

$$b \mathrel{\Box\!\!\to} a = \Box(a \mid b) \text{ and } b \mathrel{\Diamond\!\!\to} a = \Diamond(a \mid b). \qquad (10)$$

Although every sphere model Σ on $\mathbb{at}(\mathbf{A})$ determines a modal operator $\Box_\Sigma$ on $\mathcal{C}(\mathbf{A})$ and hence a Lewis algebra $(\mathcal{C}(\mathbf{A}), \Box_\Sigma)$ too, not every Lewis algebra arises in this way. Indeed, for instance, the identity operator $id : \mathcal{C}(\mathbf{A}) \to \mathcal{C}(\mathbf{A})$ satisfies (L1)–(L3) above, but no sphere model Σ in the sense of Definition 3.1 is such that $\Box_\Sigma = id$.

Definition 4.2. Let Σ be a Lewis model on $\mathbb{at}(\mathbf{A})$. A Lewis algebra $(\mathcal{C}(\mathbf{A}), \Box)$ is said to be Σ-*determined* if $\Box = \Box_\Sigma$.

The next result is a characterization of Stalnaker models within Lewis's. To prove it we first need to recall that, for every Lewis model Σ, the modal operator $\Box_\Sigma$ defined as in (9) is a normal necessity operator and, as such, it satisfies $\Box_\Sigma(c_1 \wedge c_2) = \Box_\Sigma(c_1) \wedge \Box_\Sigma(c_2)$, while in general it does not satisfy $\Box_\Sigma(c_1 \vee c_2) = \Box_\Sigma(c_1) \vee \Box_\Sigma(c_2)$.

Theorem 4.3. *A Lewis model is a Stalnaker model iff the the modal algebra* $(\mathcal{C}(\mathbf{A}), \Box_\Sigma)$ *satisfies*

($\vee$) *For all* $c_1, c_2 \in \mathcal{C}(\mathbf{A})$ *such that* $c_1 \wedge c_2 = \bot$, $\Box_\Sigma(c_1 \vee c_2) = \Box_\Sigma(c_1) \vee \Box_\Sigma(c_2)$.

Proof. Left-to-right. For every $\alpha \in \mathbb{at}(\mathbf{A})$, the unique atom Path_α stands either below c_1 or below c_2 because those are disjoint. Whence the claim directly follows from the definition of $\Box_\Lambda$ as in (9).

Right-to-left. Since Σ is not Stalnaker, there exists $\alpha \in \mathbb{at}(\mathbf{A})$ and an index i_0 such that $D_{i_0} \in \mathscr{S}_\alpha$ is such that $\omega_1, \omega_2 \in D_{i_0} \setminus \bigcup_{j<i_0} D_j$ and $\omega_1 \neq \omega_2$. In other words, $\omega_1, \omega_2 \in \mathrm{Path}_\alpha$. Let $c = \omega_1 \vee \omega_2$ so that $\omega_1 \wedge \omega_2 = \bot$ and, directly by (9), $\Box_\Sigma(c) = \alpha$ but $\Box_\Sigma(\omega_1) = \Box_\Sigma(\omega_2) = \bot$ whence ($\vee$) fails in Σ. $\Box$

In analogy to the above Theorem 4.3, also a form of *conditional excluded middle* (CEM) that can be expressed in terms of the modality $\Box_\Sigma$ characterizes Stalnaker's within Lewis's models.

Theorem 4.4. *A Lewis model* Σ *is a Stalnaker model iff the the modal algebra* $(\mathcal{C}(\mathbf{A}), \Box_\Sigma)$ *satisfies*

(CEM) $\Box_\Sigma(c) \vee \Box_\Sigma(\neg c) = \top$

Proof. Left-to-right. For all $c \in A$, $c \wedge \neg c = \bot$ and, $c \vee \neg c = \top$ and $\Box_\Lambda(\top) = \top$. Hence, by Theorem 4.3, one has $\Box_\Lambda(c) \vee \Box_\Lambda(\neg c) = \Box_\Lambda(c \vee \neg c) = \Box_\Lambda(\top) = \top$.

Right-to-left. Let us consider a Lewis model Σ and the two atoms $\omega_1, \omega_2 \in \mathbb{at}(\mathcal{C}(\mathbf{A}))$ as in the proof of Theorem 4.3. Remember that ω_1 and ω_2 are such that $\omega_1[1] = \omega_2[1] = \alpha$. Let $\omega^* \leq \alpha'$ with $\alpha' \neq \alpha$ and $\omega^* \neq \mathrm{Path}_{\alpha'}$ and define $c = \omega_1 \vee \omega^*$. Therefore, by (9), $\Box_\Sigma c = \bot$ because in particular $c \not\geq \bigvee \mathrm{Path}_\alpha$ and $c \not\geq \bigvee \mathrm{Path}_{\alpha'}$. Also, $\Box_\Sigma(\neg c) \neq \top$ and in particular $\Box_\Sigma(\neg c) = \bigvee \mathbb{at}(\mathbf{A}) \setminus \{\alpha\}$ because $\omega_2 \in \mathrm{Path}_\alpha$ and $\omega_2 \leq \neg c$, but $\omega_1 \in \mathrm{Path}_\alpha$ but $\omega_1 \not\leq \neg c$. Thus, $\bigvee \mathrm{Path}_\alpha \not\leq \neg c$ and hence $\alpha \not\leq \Box_\Sigma(\neg c)$. We hence conclude that

$$\Box_\Sigma(c) \vee \Box_\Sigma(\neg c) = \bot \vee (\bigvee \mathbb{at}(\mathbf{A}) \setminus \{\alpha\}) < \top.$$

Thus Σ does not satisfy (CEM). $\Box$

5 Canonical Extensions, Possibility of Couterfactuals and Generalized Imaging

Consider a possibility measure Π on a finite Boolean algebra $\mathbf{A}$ with n atoms with its associated normalized possibility distribution π and let $(*, \Rightarrow)$ be a residuated pair. We start by defining a mapping $\pi^* : \mathbb{at}(\mathcal{C}(\mathbf{A})) \to [0,1]$ as follows: for every atom ω (that for the moment we can think as associated to the complete list $\overline{\alpha} = \langle \alpha_1, \dots, \alpha_n \rangle$ without loss of generality) of the Boolean algebra of conditionals $\mathcal{C}(\mathbf{A})$ as in Equation (7), we define the function $\pi^* : \mathcal{C}(\mathbf{A}) \to [0,1]$ as follows:

$$\pi^*(\omega_{\overline{\alpha}}) = \Pi(\alpha_1) * (\Pi(\neg\alpha_1) \Rightarrow \Pi(\alpha_2)) * \\ * \dots * (\Pi(\textstyle\bigwedge_{j=1}^{n-2} \neg\alpha_i) \Rightarrow \Pi(\alpha_{n-1})).$$

For the sake of a lighter notation, for all $a, b \in A$ we will henceforth write $\frac{\Pi(a)}{\Pi(b)}\!\succ$ in place of $\Pi(b) \Rightarrow \Pi(a)$, so that the above expression becomes

$$\pi^*(\omega_{\overline{\alpha}}) = \Pi(\alpha_1) * \frac{\Pi(\alpha_2)}{\Pi(\neg\alpha_1)}\!\succ * \dots * \frac{\Pi(\alpha_{n-1})}{\Pi(\bigwedge_{j=1}^{n-2} \neg\alpha_i)}\!\succ. \quad (11)$$

It is proved in (Flaminio, Godo, and Ugolini 2021) that π^* is a normalized possibility distribution whenever so is π. We will henceforth denote by μ^*_Π the possibility measure on $\mathcal{C}(\mathbf{A})$ defined by the distribution π^* and we call it the $*$-*canonical extension* of Π. Thus, for all $c \in \mathcal{C}(\mathbf{A})$,

$$\mu^*_\Pi(c) = \bigvee_{\omega \in \mathbb{at}(\mathcal{C}(\mathbf{A})), \omega \leq c} \pi^*(\omega).$$

Its dual necessity measure is defined, for all $c \in \mathcal{C}(\mathbf{A})$, as

$$\mu^*_N(c) = 1 - \mu^*_\Pi(\neg c). \quad (12)$$

Let us remark once again that, given an initial possibility measure $\Pi : \mathbf{A} \to [0,1]$, the possibility distribution π^*, and hence μ^*_Π and μ^*_N consequently, depend on the fixed t-norm $*$ (and its residuum $\Rightarrow$) needed in the key (11) above. Importantly, the $*$-canonical extension $\mu^*_\Pi : \mathcal{C}(\mathbf{A}) \to [0,1]$ of any normalized possibility measure $\Pi : \mathbf{A} \to [0,1]$ can be seen as a $*$-conditional possibility in the sense that, for every basic conditional $(a \mid b) \in \mathcal{C}(\mathbf{A})$, it holds

$$\mu^*_\Pi(\text{“}(a \mid b)\text{”}) = \Pi_*(a \mid b),$$

see (Flaminio, Godo, and Ugolini 2021, Theorem 1 and Corollary1) for further details. Also, the next basic result has been proved in (Flaminio, Godo, and Ugolini 2021, Lemma 4) and it will be used later in this section.

Proposition 5.1. *For every finite Boolean algebra* $\mathbf{A}$*, every possibility distribution* $\pi : \mathbb{at}(\mathbf{A}) \to [0,1]$ *and every t-norm* $*$*, the* $*$*-canonical extension* π^* *of* π *to* $\mathbb{at}(\mathcal{C}(\mathbf{A}))$ *satisfies that, for every* $\alpha \in \mathbb{at}(\mathbf{A})$*,*

$$\bigvee_{\omega \in \mathbb{at}(\mathcal{C}(\mathbf{A})), \omega[1] = \alpha} \pi^*(\omega) = \pi(\alpha),$$

Let us now consider a possibility measure $\Pi : \mathbf{A} \to [0,1]$ and a sphere model $\Sigma = (\Omega, \mathscr{S}, v)$ where as usual $\Omega = \mathbb{at}(\mathbf{A})$. For every counterfactual $b \;\Box\!\!\rightarrow a$ and might counterfactual $b \;\Diamond\!\!\rightarrow a$ that one can define with elements of $\mathbf{A}$, it is natural to define their possibility values as follows.

Definition 5.2. Let $\pi : \mathbb{at}(\mathbf{A}) \to [0,1]$ be a possibility distribution and let $\Sigma = (\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$ be a sphere model. Then we define the following:

$$\Pi(b \;\Box\!\!\rightarrow a) = \bigvee \{\pi(\alpha) : \Sigma, \alpha \Vdash b \;\Box\!\!\rightarrow a\}$$

and

$$\Pi(b \;\Diamond\!\!\rightarrow a) = \bigvee \{\pi(\alpha) : \Sigma, \alpha \Vdash b \;\Diamond\!\!\rightarrow a\}.$$

In a completely analogous way the necessity of a (might) counterfactual is defined from the above and by duality.

The next result gives us a first representation for the possibility and the necessity of counterfactual formulas in terms of imaged possibility and imaged necessity measures as defined at the end of the previous section.

Proposition 5.3. *For every sphere model on* $\Omega = \mathbb{at}(\mathbf{A})$ *and for all* $a \in A$ *and* $b \in A'$, $\Pi(b \;\Diamond\!\!\rightarrow a) = \Pi_b(a)$ *and* $N(b \;\Box\!\!\rightarrow a) = N_b(a)$

Proof. Thanks to the equivalent claims of Lemma 3.3, by the equivalence between (i) and (ii), one has that for all $a \in A$ and $b \in A \setminus \bot$,

$$\Pi_b(a) = \bigvee_{\beta \leq a} \pi_b(\beta) = \bigvee_{\beta \leq a} \left(\bigvee \{\pi(\alpha) : \beta \in \Sigma_b(\alpha)\} \right) \quad (13)$$

Also, one from the same Lemma 3.3, the equivalence between (i) and (ii) gives

$$\begin{aligned} \Pi(b \;\Diamond\!\!\rightarrow a) &= \bigvee \{\pi(\alpha) : \alpha \Vdash b \;\Diamond\!\!\rightarrow a\} \\ &= \bigvee \{\pi(\alpha) : \Sigma_b(\alpha) \cap \mathbb{at}_{\leq}(a) \neq \emptyset\}. \end{aligned} \quad (14)$$

Thus, (13) and (14) implies $\Pi(b \;\Diamond\!\!\rightarrow a) = \Pi_b(a)$.

For the second claim, $N(b \;\Box\!\!\rightarrow a) = 1 - \Pi(\neg(b \;\Box\!\!\rightarrow a)) = 1 - \Pi(b \;\Diamond\!\!\rightarrow \neg a) = 1 - \Pi_b(\neg a) = N_b(a)$. □

Our next result finally considers canonical extensions of possibility and necessity measures on Boolean algebras of conditionals and the associated modal Lewis algebras that are Σ-determined for some fixed sphere model Σ. The following shows that the possibility and necessity of those (might) counterfactuals that can be expressed within the language of Lewis algebras as in (10), are indeed imaged possibility and necessity.

Theorem 5.4. *Given a finite Boolean algebra* $\mathbf{A}$ *and a sphere model* $\Sigma = (\mathbb{at}(\mathbf{A}), \mathscr{S}, v)$*, let* $(\mathcal{C}(\mathbf{A}), \Box)$ *be* Σ*-determined. Also, let* $\Pi : \mathbf{A} \to [0,1]$ *be a possibility measure and* $*$ *a t-norm. Then it holds that* $\mu^*_\Pi(\Box(a \mid b)) = \Pi(b \;\Box\!\!\rightarrow a)$ *and* $\mu^*_\Pi(\Diamond(a \mid b)) = \Pi(b \;\Diamond\!\!\rightarrow a)$.

Proof. In the following chains of equalities we will always assume that ω denotes a generic atom of $\mathcal{C}(\mathbf{A})$.

$$\begin{aligned} \mu^*_\Pi(\Box(a \mid b)) &= \textstyle\bigvee_{\omega \leq \Box(a|b)} \pi^*(\omega) = \bigvee_{\omega[1] \Vdash b \Box\!\rightarrow a} \pi^*(\omega) \\ &= \textstyle\bigvee_{\alpha \Vdash b \Box\!\rightarrow a} \left(\bigvee_{\omega[1] = \alpha} \pi^*(\omega) \right) \\ &= \textstyle\bigvee_{\alpha \Vdash b \Box\!\rightarrow a} \pi(\alpha) = \Pi(b \;\Box\!\!\rightarrow a). \end{aligned}$$

The fourth equality follows from Proposition 5.1.

In a similar way, we can prove the second equality where, again, we will use the result of Proposition 5.1 in the penultimate equation.

$$\begin{aligned}\mu^*_\Pi(\Diamond(a \mid b)) = \bigvee_{\omega \leq \Diamond(a|b)} \pi^*(\omega) &= \bigvee_{\omega \leq \neg\Box\neg(a|b)} \pi^*(\omega) = \\ \bigvee_{\omega \not\leq \Box(\neg a|b)} \pi^*(\omega) &= \bigvee_{\omega[1] \Vdash\!\!\!/\, b \Box\!\!\rightarrow \neg a} \pi^*(\omega) = \\ \bigvee_{\omega[1] \Vdash \neg(b \Box\!\!\rightarrow \neg a)} \pi^*(\omega) &= \bigvee_{\omega[1] \Vdash b \Diamond\!\!\rightarrow a} \pi^*(\omega) = \\ \bigvee_{\alpha \Vdash b \Diamond\!\!\rightarrow a} \left(\bigvee_{\omega[1] = \alpha} \pi^*(\omega) \right) &= \bigvee_{\alpha \Vdash b \Diamond\!\!\rightarrow a} \pi(\alpha) = \\ &= \Pi(b \Diamond\!\!\rightarrow a).\end{aligned}$$

The claim is hence settled. □

The claims of the above Proposition 5.3 and Theorem 5.4 and be put together as in the next result whose proof, thanks to (12), is easily obtained by direct computation.

Corollary 5.5. *For every finite Boolean algebra* $\mathbf{A}$*, sphere model* Σ *on* $\mathbb{at}(\mathbf{A})$ *and for every possibility measure* Π *on* $\mathbf{A}$*, and t-norm* $*$*, the following equations hold with respect to any* Σ*-determined Lewis algebra* $(\mathcal{C}(\mathbf{A}), \Box)$.

1. $\mu^*_\Pi(\Diamond(a \mid b)) = \Pi_b(a)$;
2. $\mu^*_N(\Box(a \mid b)) = N_b(a)$;
3. $\mu^*_N(\Diamond(a \mid b)) \leq \Pi_b(a)$ *and* $N_b(a) \leq \mu^*_\Pi(\Box(a \mid b))$.

Remark 5.6. It is interesting to observe that, by the identities in the above Corollary 5.5, imaged possibility and necessities do not depend on the t-norm $*$ that is needed to define the canonical extensions μ^*_Π and μ^*_N. Thus, take t-norms $*_1$ and $*_2$ and a basic conditional $(a \mid b) \in \mathcal{C}(\mathbf{A})$ such that

$$\begin{gathered}\mu^{*_1}_\Pi(a \mid b) = \Pi(b) \Rightarrow_{*_1} \Pi(a \wedge b) \neq \\ \neq \Pi(b) \Rightarrow_{*_2} \Pi(a \wedge b) = \mu^{*_2}_\Pi(a \mid b).\end{gathered}$$

However, for every Σ-determined Lewis algebra $(\mathcal{C}(\mathbf{A}), \Box)$,

$$\mu^{*_1}_\Pi(\Box(a \mid b)) = \mu^{*_2}_\Pi(\Box(a \mid b)) = \Pi_b(a).$$

In other words, the canonical extension μ^*_Π of a possibility function Π defined on a finite algebra $\mathbf{A}$, composed with the modal operator $\Box$, always results in a specific form of imaging that does not depend on the particular conditional possibility μ^*_Π we are dealing with.

Finally, let us analyze the case of Stalnaker models or equivalently, by Theorem 4.3, of those Lewis algebras $(\mathcal{C}(\mathbf{A}), \Box)$ such that $\Box$ satisfies ($\vee$) or equivalently (CEM), thanks to Theorem 4.4. By Proposition 3.3(4) in (Rosella, Flaminio, and Bonzio 2023), every Lewis algebra $(\mathcal{C}(\mathbf{A}), \Box)$ satisfies the typical axiom (D) from modal logic, i.e., for every $c \in \mathcal{C}(\mathbf{A})$, $\Box c \leq \Diamond c$. Now, if $(\mathcal{C}(\mathbf{A}), \Box)$ satisfies (CEM), i.e., $\Box c \vee \Box\neg c = \top$, one also has that for all $c \in \mathcal{C}(\mathbf{A})$, $\Box c \vee \neg\Diamond c = \top$, whence $\Box c \geq \Diamond c$. Thus, $\Box c = \Diamond c$ holds in every $(\mathcal{C}(\mathbf{A}), \Box)$ being a model for Stalnaker conditionals. Hence, under (CEM), $\Box$ is an endomorphism of $\mathcal{C}(\mathbf{A})$. In this framework the previous Corollary 5.5 simplifies.

Proposition 5.7. *For every finite Boolean algebra* $\mathbf{A}$*, Stalnaker sphere model* Λ *on* $\mathbb{at}(\mathbf{A})$ *and for every possibility measure* Π *on* $\mathbf{A}$*, the following hold with respect to any* Λ*-determined Lewis algebra:*

$$\begin{gathered}\mu^\star_\Pi(\Box(a \mid b)) = \Pi(b \Box\!\!\rightarrow a) = \Pi(b \Diamond\!\!\rightarrow a) = \\ \mu^\star_\Pi(\Diamond(a \mid b)) = \Pi_b(a);\end{gathered}$$

$$\begin{gathered}\mu^\star_N(\Box(a \mid b)) = N(b \Box\!\!\rightarrow a) = N(b \Diamond\!\!\rightarrow a) \\ = \mu^\star_N(\Diamond(a \mid b)) = N_b(a).\end{gathered}$$

6 Discussion and Conclusion

This paper revisits the fundamental question of how conditional probabilities relate to probabilities of conditionals within the framework of possibility theory. Investigating whether a conditional possibilistic measure can be represented as a non-conditional possibility of the corresponding conditional statement holds significant technical and conceptual value. Indeed, by establishing a connection between conditional possibilistic measures and non-conditional possibilities, we gain a clearer understanding of the meaning and significance of values produced by conditional possibility measures. Furthermore, if a conditional possibilistic measure can be represented as a non-conditional possibility, we can leverage existing semantic devices for calculating and representing possibilities to handle conditional possibilities as well. This would significantly advance our ability to work with and reason about conditional possibilities.

Our work establishes several key findings. First, we demonstrate a "limitative triviality result" proving that, within a Boolean setting, no conditional operator can make its possibility coincide with the corresponding conditional possibility. This highlights limitations in directly translating conditional probabilities to possibilistic logic. Second, we show how to characterize the possibility of well-known Lewis-Stalnaker conditionals using the concept of "imaged possibilistic measures". We then unify these results through an algebraic perspective based on Lewis algebras. This framework allows for a general study of the interplay between conditional possibility measures, imaged possibilities, and possibilities of conditionals.

Our work opens new avenues for research at the intersection of philosophical foundations of knowledge representation, logic, algebra, and uncertainty theory, all centered around the theme of conditional knowledge and inference. Here, we outline some promising directions for further exploration:

1. **Possibility Measures and Plausibility Orders**. While some authors, e.g. (Dubois 1986; Dubois and Prade 1988; Boutilier 1994), have discovered a link between Lewis's (1973) comparative possibility operator, $\preceq$, and the plausibility ranking of events induced by possibility measures, a deep and systematic logical investigation between Lewis's different logics for $\preceq$ and possibility theory is still lacking. Some initial results in this direction can be found in (Fariñas del Cerro and Herzig 1991), where they show how the logic of qualitative possibility coincides with Lewis's (1973) logic VN. Building on these results, we aim to discover new representation theorems connecting other logics in Lewis's (1973) hierarchy with logics underlying different possibility measures.
2. **Possibilisitic Imaging and Its Logic**. While the logic underlying possibilistic theory and conditional possibility

has been explored, e.g. (Dubois and Prade 2014), a logical investigation of imaged possibility measures is still lacking. Building on existing work, this research would clarify the connection between imaged possibility measures and preferential models, ultimately revealing the underlying logic behind possibilistic imaging.

3. **Belief Revision and Belief Updating.** An interesting direction for future research would be to explore the potential impact of the present paper's findings on belief change and belief revision theory. It is widely accepted that revision and update are distinct belief change procedures, e.g. (Katsuno and Mendelzon 1992), (Bonanno 2023), (Grahne 1998), with the former addressing belief changes in response to new information about a static world and the latter accommodating belief updating in response to changes in a dynamic world. Furthermore, Gärdenfors's probabilistic imaging is typically considered an instantiation of belief updating, while probabilistic conditionalization is often associated with belief revision, see (Katsuno and Mendelzon 1992) and (Dubois and Prade 1994) for more discussion on this. It would be intriguing to investigate whether this dichotomy extends to conditional and imaged possibility measures and how possibilistic imaging captures a specific notion of belief updating. In this context, it would be fruitful to investigate the relationships between our triviality results and Gärdenfors's (1986) triviality theorem on belief revision methods and conditionals, as well as our representation theorem of possibilistic imaging in terms of counterfactuals and its connections to the logic(s) of belief updating as developed, for instance, in (Bonanno 2023) and (Grahne 1998).

4. **Beyond the Possibilisitic Triviality Result**. Our triviality result relies on some specific general assumptions, for instance, that our underlying residuated pair $(*, \Rightarrow)$ is such that $*$ has no zero-divisors. Exploring whether a triviality result still holds under more general conditions and for different lower conditional possibility measures within the class $\mathcal{CP}(\Pi)$ would provide valuable insights into the fundamental reasons why a possibility of a conditional might not correspond to a conditional possibility. Also, since possibility measures offer a suitable framework to represent and reasoning about (conditional) preferences (Benferhat, Dubois, and Prade 2001; Dubois, Fargier, and Vantaggi 2007), it can be interesting to explore the interpretation of this triviality result from this perspective.

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