Proceedings of the 21st International Conference on Principles of Knowledge Representation and Reasoning
Main Track

Non-monotone Fixpoint Theory Based on the Structure of Weak Bilattices

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Abstract

We extend the well-known representation theorem for interlaced bilattices to the broader class of *weak interlaced bilattices*. Based on this new theorem, we develop a fixpoint theory for non-monotone functions over weak infinitarily interlaced bilattices. Our theory generalizes classical fixpoint constructions introduced by Fitting, as-well-as recent results in the area of approximation fixpoint theory. We argue that the proposed theory has direct practical applications: we develop the semantics of higher-order logic programming with negation under an arbitrary weak infinitarily interlaced bilattice with negation, generalizing in this way recent work on the three-valued semantics of this formalism. We consider a line of research, initiated by Fitting, which investigates the structure of the consistent parts of bilattices in order to obtain natural generalizations of Kleene's three-valued logic. We demonstrate that the consistent parts of bilattices are closely connected to weak bilattices, generalizing previous results of Fitting and Kondo.

1 Introduction

The purpose of this paper is to contribute to the research area of *non-monotone fixpoint theory*. More specifically, we extend some classical results of the fixpoint theory of functions over bilattices (Fitting 2002; Denecker, Marek, and Truszczyński 2000) to the broader class of functions over weak bilattices. For this to be achieved, we develop the first, to our knowledge, representation theorem for weak bilattices, which forms the basis of our fixpoint theory. Before stating in detail the contributions of the paper, we outline the rich and interesting history of non-monotone fixpoint theory, so as that we present our results in the proper context.

1.1 Non-Monotone Fixpoint Theory

The use of fixpoint theorems for *monotone* functions over partially ordered sets, has played a pivotal role in the foundations of programming languages. For example, the semantics of functional languages is usually presented (Abramsky and Jung 1995; Gunter 1993; Tennent 1991) using Kleene's fixpoint theorem for continuous functions over pointed complete partial orders, while the semantics of (negationless) logic programming has been developed (van Emden and Kowalski 1976) using the Knaster-Tarski fixpoint theorem (Tarski 1955) for monotone functions over complete lattices.

The situation becomes less clear-cut when there arise applications that involve some form of non-monotonicity. For example, the addition of negation in logic programming, triggered certain fixpoint constructions (van Gelder, Ross, and Schlipf 1991; van Gelder 1993) that could not be formalized with the standard fixpoint machinery for monotone functions. Similar concerns have arisen due to the introduction of non-monotonic logics in artificial intelligence such as, for example, default logic (Reiter 1980) and autoepistemic logic (Moore 1985). The fixpoint constructions used in the above applications, were initially ad-hoc and made apparent the need for an abstract fixpoint theory for non-monotone functions over partially ordered sets. The pioneer in the development of such a theory was Melvin Fitting, who was the first to create an abstract framework on lattices and operators on lattices in order to capture the various semantic approaches that had been proposed for non-monotonic logic programming (Fitting 2002).

Fitting remarked (Fitting 1986) that non-monotone fixpoint constructions already existed in Kripke's seminal work on a theory of truth (Kripke 1975), well-before they emerged in computer science. More specifically, Fitting observed that in Kripke's work there actually exist two distinct orderings, namely the *truth ordering* and the *information ordering*. As a result, Fitting re-formalized Kripke's work through mathematical structures that embody these two orderings. An appropriate such structure is that of a *bilattice*, initially introduced in (Ginsberg 1988). Intuitively, a bilattice $\mathcal{B}$ is a structure $\langle B, \leq_t, \leq_k \rangle$, where B is a non-empty set equipped with two partial orders $\leq_t$ and $\leq_k$, each giving B the structure of a lattice. The relation $\leq_t$ corresponds to the truth ordering and $\leq_k$ to the information ordering. Kripke's operators can be thought of as functions $F\colon B \to B$. Actually, these functions turn out to be monotone in the $\leq_k$ ordering and (possibly) non-monotone in the $\leq_t$ ordering[1]. Therefore, Fitting's work implicitly introduced the following quite general and interesting research question, setting the basis for the development of non-monotone fixpoint theory:

> Let $\mathcal{B} = \langle B, \leq_t, \leq_k \rangle$ be a bilattice and let $F\colon B \to B$ be a function that is monotone with respect to $\leq_k$ and non-monotone with respect to $\leq_t$. Can we characterize some class of interesting fixpoints of F that are minimal with respect to $\leq_t$?

Fitting also considered the same question for operators that arise in the theory of non-monotonic logic programming. He

[1]As Fitting remarked, one of the reasons that self-reference in natural languages is such a problem "is that the presence of negation in the language makes truth revision operators non-monotone with respect to the $\leq_t$ ordering".

used the above ideas to characterize the most well-known existing semantic approaches for logic programming with negation, as fixpoints of an operator over a bilattice consisting of four truth values (Fitting 2002). Fitting's ideas have generated a line of research regarding the interplay between bilattices and logic programming (see, for example, (Loyer and Straccia 2005; Loyer and Straccia 2006; Straccia 2006)). Fitting's work, although groundbreaking, only answers the above question for specific operators that arise in the context of Kripke's theory of truth and in non-monotonic logic programming.

The above problem was considered in a more abstract context in (Denecker, Marek, and Truszczyński 2000), in which the authors study the special case of bilattices $\mathcal{B} = \langle B, \leq_t, \leq_k \rangle$ where $B = L \times L$, and $\mathcal{L} = \langle L, \leq \rangle$ is a complete lattice. They develop a general theory for characterizing an important class of $\leq_t$-minimal fixpoints of certain $\leq_k$-monotone functions of the form $F\colon B \to B$. This theory is known as *approximation fixpoint theory (AFT)* and it has proven quite influential both in logic programming (Pelov, Denecker, and Bruynooghe 2007; Antic, Eiter, and Fink 2013; Heyninck, Arieli, and Bogaerts 2024) and artificial intelligence (Vennekens, Gilis, and Denecker 2006; Strass and Wallner 2015; Liu and You 2022). AFT was later modified in (Denecker, Marek, and Truszczynski 2004) to apply to $\leq_k$-monotone functions of the form $F\colon W \to W$, where W is the *consistent part* of a bilattice of the form $L \times L$ (a notion that will be defined in Section 2). Actually, W in this case is weaker than a bilattice because it is not necessarily a lattice with respect to the $\leq_k$ ordering; therefore the work in (Denecker, Marek, and Truszczynski 2004) gave a new interesting twist to the aforementioned research problem. It was subsequently extended (Charalambidis, Rondogiannis, and Symeonidou 2018) to functions of the form $F\colon W \to W$, where W is a restriction of the product $L_1 \times L_2$, where $\mathcal{L}_1 = \langle L_1, \leq \rangle$ and $\mathcal{L}_2 = \langle L_2, \leq \rangle$ are complete lattices that have a common ordering relation $\leq$ and satisfy some order-theoretic restrictions. This extension was used in (Charalambidis, Rondogiannis, and Symeonidou 2018) to provide the first well-founded semantics for higher-order logic programming with negation.

1.2 Contributions

The main contribution of the paper is that we solve the aforementioned problem for functions of the form $F\colon W \to W$, where $\mathcal{W} = \langle W, \leq_t, \leq_k \rangle$ is an arbitrary *weak interlaced bilattice* (WIBL). WIBLs are a non-trivial extension of bilattices, and their importance in KR was first observed in (Font and Moussavi 1993) who showed that they can facilitate reasoning in time-intervals. Fitting (1991) used WIBLs to generate novel extensions of Kleene's strong three-valued logic that allow various levels of inconsistency and incompleteness. Therefore, WIBLs have important applications beyond traditional bilattices.

Our results generalize all the existing fixpoint constructions mentioned in the previous subsection. The more detailed contributions of the paper, are as follows:

- We extend the well-known representation theorem for interlaced bilattices (Avron 1996) to the class of weak interlaced bilattices. Since weak interlaced bilattices generalize interlaced bilattices, this result has an interest in its own right.
- We use the new representation theorem to develop a novel fixpoint theory for non-monotone functions, over weak infinitarily interlaced bilattices. Our theory generalizes classical fixpoint constructions for non-monotone functions introduced by Fitting (2002), and recent results of approximation fixpoint theory (Denecker, Marek, and Truszczyński 2000; Denecker, Marek, and Truszczynski 2004; Charalambidis, Rondogiannis, and Symeonidou 2018).
- We argue that the proposed theory has direct applications: we develop the semantics of higher-order logic programming with negation under an arbitrary weak infinitarily interlaced bilattice with negation, generalizing in this way recent work (Charalambidis, Rondogiannis, and Symeonidou 2018) on the three-valued semantics of this formalism. Our results also extend Fitting's work (2002, Section 9), who demonstrated that we can define the semantics of *first-order* logic programming with negation under any infinitarily distributive bilattice with negation.
- We consider a line of research, initiated by Fitting (1991), which investigates the structure of the consistent parts of bilattices in order to obtain natural generalizations of Kleene's three-valued logic. We demonstrate that the consistent parts of bilattices are closely connected to weak bilattices, generalizing previous results of Fitting (1991, Section 6) and Kondo (2002).

The rest of the paper is organized as follows. Section 2 contains background on bilattices. Section 3 develops the new representation theorem and Section 4 uses it in order to develop the fixpoint theory over weak infinitarily interlaced bilattices. Section 5 develops a generalized semantics for higher-order logic programming with negation. Section 6 generalizes existing results on the structure of the consistent parts of bilattices, and Section 7 gives pointers to future work.

2 Interlaced Bilattices

A partially ordered set (or poset) $\mathcal{L} = \langle L, \leq \rangle$ is a *join-semilattice* (respectively, *meet-semilattice*) if for all $x, y \in L$ there exists a least upper bound (respectively, greatest lower bound) in L. $\mathcal{L}$ is called a *lattice* if it is both a join-semilattice and a meet-semilattice. A lattice is *bounded* if it has a least element and a greatest element, denoted by 0_L and 1_L respectively. A lattice $\langle L, \leq \rangle$ is called *complete* if for all $S \subseteq L$, there exists a least upper bound and a greatest lower bound in L, denoted by $\bigvee_L S$ and $\bigwedge_L S$ respectively. Therefore, every complete lattice is bounded. A join-semilattice is called *complete* if for all non-empty $S \subseteq L$, there exists a least upper bound in L. A *complete meet-semilattice* is defined similarly.

Given a partially ordered set $\langle P, \leq \rangle$, every linearly ordered subset S of P will be called a *chain*. A partially ordered set is *chain-complete* if it has a least element 0_P and every chain $S \subseteq P$ has a least upper bound.

Definition 1. *Let $\mathcal{P} = \langle P, \leq \rangle$ and $\mathcal{P}' = \langle P', \leq' \rangle$ be partially ordered sets and let a function $f\colon P \to P'$. Then f is* *called* monotone *(or order-preserving) if for any $x, y \in P$ such that $x \leq y$, $f(x) \leq' f(y)$.*

Two partial orders $\mathcal{P} = \langle P, \leq \rangle$ and $\mathcal{P}' = \langle P', \leq' \rangle$ are called *isomorphic* if there exists an order-preserving bijection (called an *isomorphism*) $\theta \colon P \to P'$. For brevity reasons we will often write $\mathcal{P} \overset{\theta}{\cong} \mathcal{P}'$, or simply $\mathcal{P} \cong \mathcal{P}'$.

Definition 2. *An* interlaced bilattice *(IBL) is a structure $\mathcal{B} = \langle B, \leq_t, \leq_k \rangle$ such that:*

1. *$\langle B, \leq_t \rangle$ is a bounded lattice (we denote by $\wedge_t$ and $\vee_t$ the meet and join operations, and by $\mathbf{t}$ and $\mathbf{f}$ the greatest and least elements, respectively).*
2. *$\langle B, \leq_k \rangle$ is a bounded lattice (we denote by $\wedge_k$ and $\vee_k$ the meet and join operations, and by $\top$ and $\bot$ the greatest and least elements, respectively).*
3. *Each of the four operations $\vee_t, \wedge_t, \vee_k, \wedge_k$ is order-preserving with respect to both $\leq_t$ and $\leq_k$.*

If $\langle B, \leq_t \rangle$ and $\langle B, \leq_k \rangle$ are complete lattices and the infinitary operations $\bigvee_t$, $\bigwedge_t$, $\bigvee_k$, and $\bigwedge_k$ are order-preserving with respect to both $\leq_t$ and $\leq_k$, then $\mathcal{B}$ is called an infinitarily interlaced bilattice *(or simply,* infinitary IBL*).*

Remark 1. *To demonstrate the notion of order-preservation of an operation with respect to an ordering, consider, for example, the operation $\vee_t$ and the ordering $\leq_k$. If for some a_1, a_2, b_1, b_2 we have $a_1 \leq_k b_1$ and $a_2 \leq_k b_2$, then $a_1 \vee_t a_2 \leq_k b_1 \vee_t b_2$ must hold. For the infinitary case, given two indexed sets $\{a_i\}_{i \in I}$ and $\{b_i\}_{i \in I}$ with a common indexing set I, if $a_i \leq_k b_i$ for every $i \in I$, then $\bigvee_t \{a_i\}_{i \in I} \leq_k \bigvee_t \{b_i\}_{i \in I}$ (Fitting 2020, Definition 9.4).*

Definition 3. *Let $\mathcal{L}_1 = \langle L_1, \leq_{L_1} \rangle$, $\mathcal{L}_2 = \langle L_2, \leq_{L_2} \rangle$ be bounded lattices. Their* bilattice product *$\mathcal{L}_1 \otimes \mathcal{L}_2$, is the tuple $\langle L_1 \times L_2, \leq_t, \leq_k \rangle$ where for any $x_1, y_1 \in L_1$ and $x_2, y_2 \in L_2$:*

- *$(x_1, x_2) \leq_t (y_1, y_2)$ iff $x_1 \leq_{L_1} y_1$ and $x_2 \leq_{L_2} y_2$,*
- *$(x_1, x_2) \leq_k (y_1, y_2)$ iff $x_1 \leq_{L_1} y_1$ and $y_2 \leq_{L_2} x_2$.*

For any bounded lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, $\mathcal{L}_1 \otimes \mathcal{L}_2$ is equal to $\mathcal{L}_1 \odot \mathcal{L}_2^{\text{op}}$ (Fitting 2020, Definition 5.1) where $\mathcal{L}_2^{\text{op}}$ is the *opposite* of $\mathcal{L}_2$, ie., $\mathcal{P}^{\text{op}} = \langle P, \geq \rangle$ is the opposite partial order of $\mathcal{P} = \langle P, \leq \rangle$. Therefore, based on Theorem 5.2 from (Fitting 2020), we have the following proposition:

Proposition 1. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be bounded lattices. Then, $\mathcal{L}_1 \otimes \mathcal{L}_2$ is an IBL. In particular, for any $(x_1, x_2), (y_1, y_2) \in L_1 \times L_2$:*

1. *$(x_1, x_2) \wedge_t (y_1, y_2) = (x_1 \wedge_{L_1} y_1, x_2 \wedge_{L_2} y_2)$,*
2. *$(x_1, x_2) \vee_t (y_1, y_2) = (x_1 \vee_{L_1} y_1, x_2 \vee_{L_2} y_2)$,*
3. *$(x_1, x_2) \wedge_k (y_1, y_2) = (x_1 \wedge_{L_1} y_1, x_2 \vee_{L_2} y_2)$,*
4. *$(x_1, x_2) \vee_k (y_1, y_2) = (x_1 \vee_{L_1} y_1, x_2 \wedge_{L_2} y_2)$.*

Also, the pairs $(0_{L_1}, 0_{L_2})$, $(1_{L_1}, 1_{L_2})$, $(0_{L_1}, 1_{L_2})$ and $(1_{L_1}, 0_{L_2})$ are, respectively, the $\leq_t$-least, $\leq_t$-greatest, $\leq_k$-least and $\leq_k$-greatest elements of $L_1 \times L_2$.

We also have the infinitary version of the previous proposition. For the next proposition and the rest of the paper we will denote the *first* and *second* selection functions by $[\cdot]_1$ and $[\cdot]_2$: given any pair (x, y), it is $[(x, y)]_1 = x$ and $[(x, y)]_2 = y$. Moreover, for any set of pairs S, we will write $[S]_1$ for $\{x \mid (x, y) \in S\}$ and $[S]_2$ for $\{y \mid (x, y) \in S\}$.

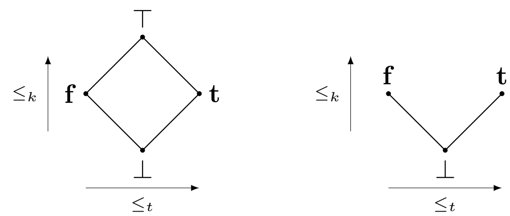

Figure 1: The IBL $\mathcal{FOUR}$ and the WIBL $\mathcal{THREE}$

Proposition 2. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be complete lattices. Then, $\mathcal{L}_1 \otimes \mathcal{L}_2$ is an infinitary IBL. In particular, for any $S \subseteq L_1 \times L_2$:*

1. *$\bigwedge_t S = (\bigwedge_{L_1} [S]_1, \bigwedge_{L_2} [S]_2)$,*
2. *$\bigvee_t S = (\bigvee_{L_1} [S]_1, \bigvee_{L_2} [S]_2)$,*
3. *$\bigwedge_k S = (\bigwedge_{L_1} [S]_1, \bigvee_{L_2} [S]_2)$,*
4. *$\bigvee_k S = (\bigvee_{L_1} [S]_1, \bigwedge_{L_2} [S]_2)$.*

The following is a central result in the theory of bilattices, which is known as the *Representation Theorem* for IBLs (Avron 1996, Theorem 4.3).

Theorem 1. *Let $\mathcal{B}$ be an (infinitary) IBL. Then, there exist (complete) bounded lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, unique up to isomorphism, such that $\mathcal{B}$ is isomorphic to $\mathcal{L}_1 \otimes \mathcal{L}_2$.*

In the following we introduce weak interlaced bilattices (Font and Moussavi 1993), a generalization of interlaced bilattices.

Definition 4. *A* weak interlaced bilattice *(WIBL) is a structure $\mathcal{W} = \langle W, \leq_t, \leq_k \rangle$ such that:*

1. *$\langle W, \leq_t \rangle$ is a bounded lattice (we denote by $\wedge_t$ and $\vee_t$ the meet and join operations, and, $\mathbf{t}, \mathbf{f}$ the greatest and least elements, respectively).*
2. *$\langle W, \leq_k \rangle$ is a meet-semilattice equipped with a least element (we denote by $\wedge_k$ the meet operation, and $\bot$ the least element. Also, we denote by $\vee_k$ the join operation, which may not always be defined).*
3. *Each of the operations $\vee_t, \wedge_t, \wedge_k$ is order-preserving with respect to both $\leq_t$ and $\leq_k$.*
4. *The operation $\vee_k$ is order-preserving if the corresponding join elements exist.*

If $\langle W, \leq_t \rangle$ is a complete lattice and $\langle W, \leq_k \rangle$ is a complete meet-semilattice, and the infinitary operations $\bigvee_t$, $\bigwedge_t$, $\bigvee_k$, and $\bigwedge_k$ are order-preserving with respect to both $\leq_t$ and $\leq_k$, then $\mathcal{W}$ is called a weak infinitarily interlaced bilattice *(or simply,* infinitary WIBL*).*

It is apparent that any (infinitary) IBL is an (infinitary) WIBL. Examples of WIBLs are the well-known $\mathcal{FOUR}$ and $\mathcal{THREE}$ depicted in Figure 1. Note that $\mathcal{FOUR}$ is also an (infinitary) IBL.

Two WIBLs $\mathcal{W} = \langle W, \leq_t, \leq_k \rangle$ and $\mathcal{W}' = \langle W', \leq_t', \leq_k' \rangle$ are called isomorphic if there exists a bijection (called an isomorphism) $\theta \colon W \to W'$ that preserves both orderings. For brevity reasons we will often write $\mathcal{W} \overset{\theta}{\cong} \mathcal{W}'$, or simply $\mathcal{W} \cong \mathcal{W}'$.

Definition 5. *Let $\mathcal{W} = \langle W, \leq_t, \leq_k \rangle$ be a WIBL. $\mathcal{W}$ has a* negation *if there exists a mapping $\neg\colon W \to W$ such that*

1. *if $x \leq_t y$ then $\neg y \leq_t \neg x$,*
2. *if $x \leq_k y$ then $\neg x \leq_k \neg y$,*
3. *$\neg\neg x = x$.*

Definition 6. *Let $\mathcal{W} = \langle W, \leq_t, \leq_k \rangle$ be a WIBL. $\mathcal{W}$ has a* conflation *if there exists a mapping $-\colon W \to W$ such that*

1. *if $x \leq_t y$ then $-x \leq_t -y$,*
2. *if $x \leq_k y$ then $-y \leq_k -x$,*
3. *$- - x = x$.*

The mapping $-\colon W \to W$ will be called a weak conflation *if it satisfies conditions (1) and (2) but not necessarily (3).*

Definition 7. *Let $\mathcal{W} = \langle W, \leq_t, \leq_k \rangle$ be a WIBL with weak conflation and let $x \in W$. We will say that x is* consistent *if $x \leq_k -x$. The set of all consistent elements of $\mathcal{W}$ is denoted by* $\mathsf{cons}(\mathcal{W})$.

Remark 2. *It is easy to see that given a bounded lattice $\mathcal{L} = \langle L, \leq \rangle$, the IBL $\mathcal{L} \otimes \mathcal{L}$ has a conflation defined as $-(x, y) = (y, x)$. Moreover, an element $(x, y) \in L \times L$ is consistent if and only if $x \leq y$. In the rest of the paper, when writing $\mathsf{cons}(\mathcal{L} \otimes \mathcal{L})$ we will mean the consistent part of $\mathcal{L} \otimes \mathcal{L}$ under this conflation.*

3 Representation Theorem for WIBLs

Before presenting the Representation Theorem for WIBLs, we give an intuitive motivation for its construction. The classical Representation Theorem (Theorem 1) implies that every IBL $\mathcal{B}$ is isomorphic to $\mathcal{L}_1 \otimes \mathcal{L}_2$ for bounded lattices $\mathcal{L}_1 = \langle L_1, \leq_1 \rangle, \mathcal{L}_2 = \langle L_2, \leq_2 \rangle$. The partial orders $\leq_t$ and $\leq_k$ of $\mathcal{L}_1 \otimes \mathcal{L}_2$ are defined over the *entire* Cartesian product $L_1 \times L_2$. On the other hand, when we consider a WIBL, we can intuitively view it as "an IBL that is (possibly) missing some elements" and we must somehow capture this "(possible) non-existence of elements". Our Representation Theorem states that every WIBL $\mathcal{W}$ is isomorphic to the *restricted product* of two bounded lattices $\mathcal{L}_1 = \langle L_1, \leq_1 \rangle, \mathcal{L}_2 = \langle L_2, \leq_2 \rangle$, in the sense that the relations $\leq_t$ and $\leq_k$ of this product are defined on a relation $\leq$ that is a *subset* of $L_1 \times L_2$. Therefore, this relation must become a vital part of our Representation Theorem (in the case of IBLs, this relation remains "hidden" because it coincides with the Cartesian product $L_1 \times L_2$).

We now introduce the concepts of *interlattice structure* and *restricted bilattice product*, which capture the above intuition.

Definition 8. *Given a partially ordered set $\langle P, \leq \rangle$ and $L_1, L_2 \subseteq P$, an* infinitary (resp., finitary) interlattice structure *is a tuple $\mathcal{T} = \langle P, L_1, L_2, \leq \rangle$ such that:*

1. *$\langle L_1, \leq \rangle$ and $\langle L_2, \leq \rangle$ are complete (resp., bounded) lattices.*
2. *For any $b \in L_2$ and any (resp., any finite) $S \subseteq L_1$ such that $x \leq b$ for every $x \in S$, $\bigvee_{L_1} S \leq b$.*
3. *For any $a \in L_1$ and any (resp., any finite) $S \subseteq L_2$ such that $a \leq x$ for every $x \in S$, $a \leq \bigwedge_{L_2} S$.*

We will refer to the properties (2) and (3) of Definition 8 as the *interlattice lub* and *interlattice glb* properties, respectively. By applying these properties for $S = \emptyset$, we have that for any $b \in L_2$ and $a \in L_1$, $0_{L_1} \leq b$ and $a \leq 1_{L_2}$. In other words, 0_{L_1} and 1_{L_2} are the least and greatest elements in $L_1 \cup L_2$, respectively.

Notice that interlattice structures also generalize the notion of lattices: for any (complete) bounded lattice $\mathcal{L} = \langle L, \leq \rangle$, the structure $\langle L, L, L, \leq \rangle$ is an (infinitary) interlattice structure. As a result, in the rest of the paper, given a (complete) bounded lattice $\mathcal{L}$, we will assume that it is also an (infinitary) interlattice structure.

We now define a form of restricted product of an interlattice structure.

Definition 9. *Let $\mathcal{T} = \langle P, L_1, L_2, \leq \rangle$ be an interlattice structure. Its $\leq$-restricted bilattice product $RBP(\mathcal{T})$ is defined as a structure $\langle L_1 \times_{\leq} L_2, \leq_t, \leq_k \rangle$, where $L_1 \times_{\leq} L_2 = \{(x_1, x_2) \mid x_1 \in L_1, x_2 \in L_2, x_1 \leq x_2\}$ and for any $(x_1, x_2), (y_1, y_2) \in L_1 \times_{\leq} L_2$:*

- *$(x_1, x_2) \leq_t (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$,*
- *$(x_1, x_2) \leq_k (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $y_2 \leq x_2$.*

We will now show that $RBP(\mathcal{T})$ is a WIBL. Our theorem relies on the following lemma on the properties of $\mathcal{T}$.

Lemma 3. *Let $\mathcal{T} = \langle P, L_1, L_2, \leq \rangle$ be an infinitary (resp., finitary) interlattice structure. Then for any (resp., any finite) $S \subseteq (L_1 \times_{\leq} L_2)$:*

1. *$\bigvee_{L_1}[S]_1 \leq \bigvee_{L_2}[S]_2$ and $\bigvee_t S = (\bigvee_{L_1}[S]_1, \bigvee_{L_2}[S]_2)$.*
2. *$\bigwedge_{L_1}[S]_1 \leq \bigwedge_{L_2}[S]_2$ and $\bigwedge_t S = (\bigwedge_{L_1}[S]_1, \bigwedge_{L_2}[S]_2)$.*
3. *If S is not empty, then $\bigwedge_{L_1}[S]_1 \leq \bigvee_{L_2}[S]_2$ and $\bigwedge_k S = (\bigwedge_{L_1}[S]_1, \bigvee_{L_2}[S]_2)$.*
4. *If $\bigvee_{L_1}[S]_1 \leq \bigwedge_{L_2}[S]_2$, then $\bigvee_k S = (\bigvee_{L_1}[S]_1, \bigwedge_{L_2}[S]_2)$.*

Theorem 2. *Let $\mathcal{T}$ be an (infinitary) interlattice structure. Then, $RBP(\mathcal{T})$ is an (infinitary) WIBL.*

We can now state the Representation Theorem, which shows that the above way of constructing weak interlaced bilattices is completely general, in the sense that any WIBL can be constructed as the restricted bilattice product of an interlattice structure.

Theorem 3 (Representation Theorem for WIBLs)**.** *Let $\mathcal{W}$ be an (infinitary) WIBL. Then, there exists a, unique up to isomorphism, (infinitary) interlattice structure $\mathcal{T} = \langle P, L_1, L_2, \leq \rangle$ such that $L_1 \cup L_2 = P$, $(\leq) \cap (L_2 \times L_1) = \emptyset$, and $\mathcal{W} \cong RBP(\mathcal{T})$.*

Note that the properties $L_1 \cup L_2 = P$ and $(\leq) \cap (L_2 \times L_1) = \emptyset$ are stated just to ensure the uniqueness up to isomorphism.

We are now going to illustrate the key ideas behind the representation theorem by providing a sketch of the construction. Given a WIBL $\mathcal{W} = \langle W, \leq_t, \leq_k \rangle$, we define $L_1 = \{(1, x) \mid x \in W, \perp \leq_t x\}$ and $L_2 = \{(2, y) \mid y \in W, y \leq_t \perp\}$; the labels 1 and 2 are used to ensure the disjointness of L_1 and L_2. We also introduce the following ordering within $L_1 \cup L_2$:

- $(1, x) \leq (1, y)$ if and only if $x \leq_t y$.

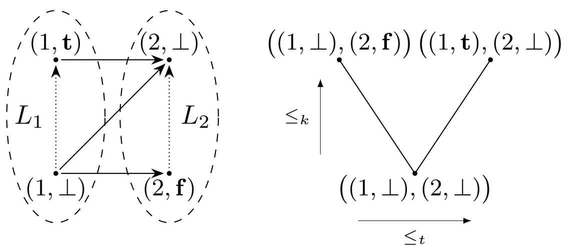

Figure 2: An interlattice structure $\mathcal{T}$ and the corresponding $RBP(\mathcal{T})$ that is isomorphic to WIBL $\mathcal{THREE}$. Arrows in $\mathcal{T}$ correspond to the relation $\leq$ and the nodes of $RBP(\mathcal{T})$ are the solid arrows between L_1 and L_2.

- $(2,x) \leq (2,y)$ if and only if $x \leq_t y$.
- $(1,x) \leq (2,y)$ if and only if there is a $w \in W$ such that $w \vee_t \perp = x$ and $w \wedge_t \perp = y$.

Based on the above construction, it is then proved that $\mathcal{T} = \langle L_1 \cup L_2, L_1, L_2, \leq\rangle$ is an interlattice structure and $RBP(\mathcal{T})$ is isomorphic to $\mathcal{W}$.

For example, for the weak bilattice $\mathcal{THREE}$, depicted in Figure 1, it is $L_1 = \{(1,\perp),(1,\mathbf{t})\}$, $L_2 = \{(2,\perp),(2,\mathbf{f})\}$ and the ordering as depicted in Figure 2.

Remark 3. *When Theorem 3 is used for some infinitary IBL $\mathcal{B} = \langle B, \leq_t, \leq_k\rangle$, we would have $a \leq b$ for every $a \in L_1$ and $b \in L_2$. In this case, the restricted bilattice product is the same as the bilattice product. Thus, $\mathcal{B}$ is isomorphic to $\langle L_1, \leq\rangle \otimes \langle L_2, \leq\rangle$. Therefore, we get Theorem 1 as a special case.*

4 A Fixpoint Theory for WIBLs

In this section we develop a fixpoint theory for functions of the form $F\colon W \to W$, where $\mathcal{W} = \langle W, \leq_t, \leq_k\rangle$ is an arbitrary infinitary WIBL. We assume that F is $\leq_k$-monotone but not necessarily $\leq_t$-monotone. Our goal is to find a special class of fixpoints of F that are $\leq_t$-minimal. In particular, we target at those fixpoints that in the bibliography (Fitting 2002; Denecker, Marek, and Truszczyński 2000) are usually termed *stable fixpoints*, and the $\leq_k$-minimum among them is called the *well-founded fixpoint*. Such fixpoints have a long and interesting history that has its roots in the non-monotonic extensions of logic programming (van Gelder, Ross, and Schlipf 1991; van Gelder 1993).

The research in this area was initially centered around functions $F\colon B \to B$, where $\mathcal{B} = \langle B, \leq_t, \leq_k\rangle$ is an infinitary IBL (Fitting 2002; Denecker, Marek, and Truszczyński 2000). Actually, the work in (Denecker, Marek, and Truszczyński 2000) considers the special case of infinitary IBLs of the form $\mathcal{L} \otimes \mathcal{L}$, where $\mathcal{L}$ is a complete lattice; it turns out that such bilattices have a special intuitive appeal. In (Denecker, Marek, and Truszczynski 2004), the development of *consistent approximation fixpoint theory* was initiated, which aimed at finding the fixpoints of functions of the form $F\colon \mathsf{cons}(\mathcal{L} \otimes \mathcal{L}) \to \mathsf{cons}(\mathcal{L} \otimes \mathcal{L})$, where $\mathcal{L}$ is a complete lattice. Using the notation of the present paper (Remark 2), the theory of (Denecker, Marek, and Truszczynski 2004) actually aims at finding the fixpoints of functions of the form $F\colon L \times_{\leq} L \to L \times_{\leq} L$ where $L \times_{\leq} L$ is the underlying set of $RBP(\mathcal{L})$. Similarly, the work in (Charalambidis, Rondogiannis, and Symeonidou 2018) considers the fixpoints of functions of the form $F\colon L_1 \times_{\leq} L_2 \to L_1 \times_{\leq} L_2$ for some special form of interlattice structure $\mathcal{T} = \langle P, L_1, L_2, \leq\rangle$.

Our present work genuinely generalizes the existing works in several respects. It is, to our knowledge, the only work that considers the general case of functions over the whole class of infinitary WIBLs. Notice that the domains of the approximating functions that are studied in (Denecker, Marek, and Truszczynski 2004) and (Charalambidis, Rondogiannis, and Symeonidou 2018), actually are infinitary WIBLs of special forms (see also our forthcoming discussion in Section 6).

Certain definitions and results stated in this section are generalizations of corresponding definitions and results in (Charalambidis, Rondogiannis, and Symeonidou 2018; Denecker, Marek, and Truszczynski 2004). These generalizations are necessary because the proposed fixpoint theory applies to arbitrary WIBLs. Moreover, Proposition 9, Theorem 6, Corollary 12, Proposition 14, and Proposition 13, are novel and reflect the fact that our fixpoint theory concerns functions defined over general WIBLs. In particular, we draw attention to the significance of Theorem 6 and Corollary 12, which ensure that our fixpoint construction is well-defined. At the heart of our development, we use our representation theorem (Theorem 3) to represent a given infinitary WIBL $\mathcal{W}$ as $RBP(\mathcal{T})$ for some infinitary interlattice structure $\mathcal{T}$. Theorem 6 and Corollary 12 ensure that if we choose any other isomorphic representation $RBP(\mathcal{T}')$ of $\mathcal{W}$ for a different interlattice structure $\mathcal{T}'$, our fixpoint theory will produce the same fixpoints as the fixpoints produced under the interlattice structure $\mathcal{T}$.

We now proceed to the technical presentation of the proposed fixpoint theory. Let $\mathcal{W} = \langle W, \leq_t, \leq_k\rangle$ be an infinitary WIBL and let $F\colon W \to W$ be a $\leq_k$-monotone function. By Theorem 3, there exists a, unique up to isomorphism, interlattice structure $\mathcal{T} = \langle P, L_1, L_2, \leq\rangle$ such that $L_1 \cup L_2 = P$, $(\leq) \cap (L_2 \times L_1) = \emptyset$, and $\mathcal{W} \overset{\theta}{\cong} RBP(\mathcal{T}) = \langle L_1 \times_{\leq} L_2, \leq_t, \leq_k\rangle$. The first step in the development of our fixpoint theory, is to "transform" $F\colon W \to W$ to a function $f\colon (L_1 \times_{\leq} L_2) \to (L_1 \times_{\leq} L_2)$. More specifically, we define $f = \theta \circ F \circ \theta^{-1}$. It can easily be verified, using the fact that θ is an isomorphism, that f is indeed a function in $(L_1 \times_{\leq} L_2) \to (L_1 \times_{\leq} L_2)$. Moreover, it is trivial to verify that f is also $\leq_k$-monotone. In other words, f is intended to "behave in the same way as F" in the domain $L_1 \times_{\leq} L_2$ which is isomorphic to our initial domain W.

Using the interlattice properties of the infinitary interlattice structure $\mathcal{T}$, it follows that for all $x \in L_1$ and $y \in L_2$, $0_{L_1} \leq y$ and $x \leq 1_{L_2}$. In particular, $L_1 \times_{\leq} L_2$ is non-empty since $(0_{L_1}, 1_{L_2}) \in L_1 \times_{\leq} L_2$. Given $a \in L_1$ and $b \in L_2$, we write $[a,b]_{L_1} = \{x \in L_1 \mid a \leq x \leq b\}$. Symmetrically, $[a,b]_{L_2} = \{x \in L_2 \mid a \leq x \leq b\}$.

Proposition 4. *For all $a \in L_1$ and $b \in L_2$, $\langle [0_{L_1}, b]_{L_1}, \leq\rangle$ and $\langle [a, 1_{L_2}]_{L_2}, \leq\rangle$ are complete lattices.*

In the rest of this section, we allow the given function f to appear free in most definitions and results.

The notion of f-*reliability* gives an initial restriction on the pairs that will participate in the chains that will lead to the well-founded fixpoint of f. This notion is an extension of the corresponding notion introduced in (Denecker, Marek, and Truszczynski 2004).

Definition 10. *The pair $(a, b) \in L_1 \times_{\leq} L_2$ will be called f-reliable if $(a, b) \leq_k f(a, b)$.*

Proposition 5. *Let $(a, b) \in L_1 \times_{\leq} L_2$ and assume that (a, b) is f-reliable. Then, for every $x \in [0_{L_1}, b]_{L_1}$, it holds $0_{L_1} \leq [f(x, b)]_1 \leq b$. Moreover, for every $x \in [a, 1_{L_2}]_{L_2}$, it holds $a \leq [f(a, x)]_2 \leq 1_{L_2}$.*

The above proposition implies that for every f-reliable pair (a, b), the restriction of $[f(\cdot, b)]_1$ to $[0_{L_1}, b]_{L_1}$ and the restriction of $[f(a, \cdot)]_2$ to $[a, 1_{L_2}]_{L_2}$ are $\leq$-monotone functions $[0_{L_1}, b]_{L_1} \to [0_{L_1}, b]_{L_1}$ and $[a, 1_{L_2}]_{L_2} \to [a, 1_{L_2}]_{L_2}$ on these intervals. Since by Proposition 4 we know that $\langle [0_{L_1}, b]_{L_1}, \leq \rangle$ and $\langle [a, 1_{L_2}]_{L_2}, \leq \rangle$ are complete lattices, the functions $[f(\cdot, b)]_1$ and $[f(a, \cdot)]_2$ have least fixpoints in these lattices. We define $b^{\downarrow} = \mathit{lfp}([f(\cdot, b)]_1)$ and $a^{\uparrow} = \mathit{lfp}([f(a, \cdot)]_2)$. In the following, we will call the function mapping the f-reliable pair (a, b) to $(b^{\downarrow}, a^{\uparrow})$, the *stable revision operator* of f. We will denote this mapping by $\mathcal{C}_f$, namely:

$$\mathcal{C}_f(x, y) = (y^{\downarrow}, x^{\uparrow}) = (\mathit{lfp}([f(\cdot, y)]_1), \mathit{lfp}([f(x, \cdot)]_2))$$

We would like $\mathcal{C}_f$ to return elements of $L_1 \times_{\leq} L_2$ when applied to reliable pairs of $L_1 \times_{\leq} L_2$. This is ensured by the following proposition:

Proposition 6. *For every f-reliable pair (a, b), $b^{\downarrow} \leq b$, $a \leq a^{\uparrow} \leq b$, and $(b^{\downarrow}, a^{\uparrow}) \in L_1 \times_{\leq} L_2$.*

In order to obtain the $\leq_k$-least fixpoint of $\mathcal{C}_f$, we will create a chain of pairs from $L_1 \times_{\leq} L_2$ ordered with respect to $\leq_k$. In order to ensure that at each step of the iteration the chain indeed increases with respect to $\leq_k$, we need the notion of f-*prudent* pairs initially introduced in (Denecker, Marek, and Truszczynski 2004).

Definition 11. *An f-reliable pair (a, b) is f-prudent if $a \leq b^{\downarrow}$.*

The following proposition establishes the fact that by iterating $\mathcal{C}_f$ over an f-prudent pair, we obtain an f-prudent pair.

Proposition 7. *Let $(a, b) \in L_1 \times_{\leq} L_2$ be f-prudent. Then, $(a, b) \leq_k (b^{\downarrow}, a^{\uparrow})$ and $(b^{\downarrow}, a^{\uparrow})$ is f-prudent.*

The following proposition ensures that $\mathcal{C}_f$ is $\leq_k$-monotone over the set of prudent elements.

Proposition 8. *Let $f : (L_1 \times_{\leq} L_2) \to (L_1 \times_{\leq} L_2)$ be a $\leq_k$-monotone function and let $(a, b), (c, d) \in L_1 \times_{\leq} L_2$. If (a, b) is f-reliable, (c, d) is f-prudent and if $(a, b) \leq_k (c, d)$, then $(b^{\downarrow}, a^{\uparrow}) \leq_k (d^{\downarrow}, c^{\uparrow})$.*

We are now almost ready to create the increasing $\leq_k$-chain that will lead us to the $\leq_k$-least fixpoint of $\mathcal{C}_f$. We must have some guarantee that such a chain has a limit. In previous work, this was ensured by (Denecker, Marek, and Truszczynski 2004, Proposition 2.3) and (Charalambidis, Rondogiannis, and Symeonidou 2018, Proposition 12), respectively; both of these propositions worked for restricted cases of infinitary WIBLs. Since we are dealing with arbitrary infinitary WIBLs, we need a more general result.

Proposition 9. *Let $\mathcal{W} = \langle W, \leq_k, \leq_t \rangle$ be an infinitary WIBL. Then $\langle W, \leq_k \rangle$ is a chain-complete partial order.*

The following result ensures that f-prudence is preserved at the limit of a chain of f-prudent pairs.

Proposition 10. *Let $f : (L_1 \times_{\leq} L_2) \to (L_1 \times_{\leq} L_2)$ be a $\leq_k$-monotone function and let $\{(a_\kappa, b_\kappa)\}_{\kappa < \lambda}$, where λ is an ordinal, be a chain of f-prudent pairs from $L_1 \times_{\leq} L_2$. Then, $\bigvee_k \{(a_\kappa, b_\kappa)\}_{\kappa < \lambda}$, is f-prudent.*

All the previous results imply the following theorem.

Theorem 4. *Let $f : (L_1 \times_{\leq} L_2) \to (L_1 \times_{\leq} L_2)$ be a $\leq_k$-monotone function. The set of f-prudent elements of $L_1 \times_{\leq} L_2$ is a chain-complete poset under $\leq_k$ with least element $(0_{L_1}, 1_{L_2})$. The stable revision operator is a well-defined, increasing and monotone operator in this poset, and therefore it has a least fixpoint which is f-prudent and can be obtained as the limit of the following sequence:*

$$\begin{array}{lcl} (a_0, b_0) & = & (0_{L_1}, 1_{L_2}) \\ (a_{\lambda+1}, b_{\lambda+1}) & = & \mathcal{C}_f(a_\lambda, b_\lambda) \\ (a_\lambda, b_\lambda) & = & \bigvee_k \{(a_\kappa, b_\kappa) : \kappa < \lambda\}, \\ & & \quad \textit{for limit ordinal } \lambda \end{array}$$

Every fixpoint of $\mathcal{C}_f$ is also a fixpoint of f as the following proposition suggests. These fixpoints will be called the *stable fixpoints of* f.

Proposition 11. *If (x, y) is a fixpoint of $\mathcal{C}_f$ then it is a fixpoint of f.*

In particular, the fixpoint of $\mathcal{C}_f$ specified by the iterative procedure of Theorem 4 will be called the *well-founded fixpoint of* f. Actually, every stable fixpoint of f is a $\leq_t$-minimal pre-fixpoint of f as the following theorem suggests and therefore it is a $\leq_t$-minimal fixpoint by the previous proposition.

Theorem 5. *Every fixpoint of the stable revision operator $\mathcal{C}_f$ is a $\leq_t$-minimal pre-fixpoint of f.*

The first step that we followed in the development of our fixpoint theory, was to transform $F : W \to W$ to a function $f : (L_1 \times_{\leq} L_2) \to (L_1 \times_{\leq} L_2)$, where L_1, L_2 are the lattices implied by the interlattice structure $\mathcal{T} = \langle P, L_1, L_2, \leq \rangle$ provided by the representation theorem. However, it is not immediately obvious what would happen if we transformed $F : W \to W$ to a function $f' : (L'_1 \times_{\leq} L'_2) \to (L'_1 \times_{\leq} L'_2)$, for some interlattice structure $\mathcal{T}' = \langle P', L'_1, L'_2, \leq' \rangle$, where $RBP(\mathcal{T}')$ is isomorphic to $RBP(\mathcal{T})$. The following theorem and corollary suggest that there is a bijection between the fixpoints of $\mathcal{C}_f$ and $\mathcal{C}_{f'}$ and corresponding fixpoints are mapped to the same element of W.

Theorem 6. *Let $\mathcal{T} = \langle P, L_1, L_2, \leq \rangle$ and $\mathcal{T}' = \langle P', L'_1, L'_2, \leq' \rangle$ be infinitary interlattice structures such that $RBP(\mathcal{T}) \overset{\eta}{\cong} RBP(\mathcal{T}')$ and let $f : L_1 \times_{\leq} L_2 \to L_1 \times_{\leq} L_2$, $f' : L'_1 \times_{\leq} L'_2 \to L'_1 \times_{\leq} L'_2$ be $\leq_k$-monotone operators. If $\eta \circ f = f' \circ \eta$ then $\eta \circ \mathcal{C}_f = \mathcal{C}_{f'} \circ \eta$.*

Corollary 12. *Let $\mathcal{W} = \langle W, \leq_t, \leq_k \rangle$ be an infinitary WIBL and $F : W \to W$ be a $\leq_k$-monotone function. Let $\mathcal{T}$ and $\mathcal{T}'$ be infinitary interlattice structures such that $\mathcal{W} \overset{\theta}{\cong} RBP(\mathcal{T})$ and $\mathcal{W} \overset{\theta'}{\cong} RBP(\mathcal{T}')$ and let $\eta = \theta' \circ \theta^{-1}$ an isomorphism between $RBP(\mathcal{T})$ and $RBP(\mathcal{T}')$. Let $f = \theta \circ F \circ \theta^{-1}$ and $f' = \theta' \circ F \circ \theta'^{-1}$. Then, (x, y) is a fixpoint of $\mathcal{C}_f$ if and only if $\eta(x, y)$ is a fixpoint of $\mathcal{C}_{f'}$.*

The above corollary suggests that if we start with a given $F : W \to W$, the stable and well-founded fixpoints are independent of the choice of f. Therefore, we can talk unambiguously about the *stable fixpoints of F* and the *well-founded fixpoint of F*, respectively.

We now examine how the theory developed in this section generalizes the fixpoint theories developed in (Denecker, Marek, and Truszczyński 2000; Denecker, Marek, and Truszczynski 2004; Charalambidis, Rondogiannis, and Symeonidou 2018). First, notice that the construction of (Charalambidis, Rondogiannis, and Symeonidou 2018) generalizes the construction of (Denecker, Marek, and Truszczynski 2004). Therefore, it suffices to compare our approach with (Denecker, Marek, and Truszczyński 2000) and (Charalambidis, Rondogiannis, and Symeonidou 2018).

Proposition 13. *Let F be a consistent approximating operator in the sense of (Charalambidis, Rondogiannis, and Symeonidou 2018, Definition 22). Then, (x, y) is a fixpoint of $\hat{\mathcal{C}}_F$ under the theory developed in (Charalambidis, Rondogiannis, and Symeonidou 2018, Appendix C), if and only if (x, y) is a fixpoint of $\mathcal{C}_F$ under the proposed approach.*

Finally, the following proposition demonstrates that the proposed approach also generalizes the fixpoint technique of (Denecker, Marek, and Truszczyński 2000).

Proposition 14. *Let $\mathcal{L} = \langle L, \leq \rangle$ be a complete lattice and let $F : (L \times L) \to (L \times L)$ be an approximating operator in the sense of (Denecker, Marek, and Truszczyński 2000, Definition 13). Then, (x, y) is a fixpoint of $\hat{\mathcal{C}}_F$ under the approximation fixpoint theory developed in (Denecker, Marek, and Truszczyński 2000), if and only if (x, y) is a fixpoint of $\mathcal{C}_F$ under the proposed approach.*

5 WIBLs and Higher-order Logic Programming

In this section we demonstrate that we can use the theory developed in this paper to give a general semantics to higher-order logic programming with negation. More specifically, we develop the semantics of higher-order logic programming with negation under an arbitrary infinitary WIBL with negation, generalizing in this way recent work (Charalambidis, Rondogiannis, and Symeonidou 2018) on the three-valued semantics of this formalism. Our results also extend Fitting's work (Fitting 2002, Section 9), who demonstrated that we can define the semantics of *first-order* logic programming with negation under any infinitarily distributive bilattice with negation.

Before we discuss about higher-order logic programming, we define some standard ways for constructing WIBLs. The results that follow will be used in the recursive construction of the domain of types and the semantics of higher-order logic programming.

Our first result, Proposition 15, suggests that the Cartesian product of WIBLs is also a WIBL. The latter result, Corollary 16, states that given a non-empty set I and a WIBL $\mathcal{W}$, the function space $I \to \mathcal{W}$ is also a WIBL.

Proposition 15. *Let I be a non-empty set and $\{\mathcal{W}_i\}_{i \in I}$ be a set of (infinitary) WIBLs indexed by I and $\mathcal{W}_i = \langle W_i, \leq_{t,i}, \leq_{k,i} \rangle$. Then, $\mathcal{W} = \Pi_{i \in I} \mathcal{W}_i = \langle \Pi_{i \in I} W_i, \leq_t, \leq_k \rangle$ where $\leq_t$ and $\leq_k$ are defined pointwisely, is also an (infinitary) WIBL. Moreover, if every $\mathcal{W}_i$ has negation then $\Pi_{i \in I} \mathcal{W}_i$ also has negation defined pointwisely.*

Corollary 16. *Let I be a non-empty set and $\mathcal{W}$ be an (infinitary) WIBL. The function space $I \to \mathcal{W}$ is an (infinitary) WIBL. Moreover, if $\mathcal{W}$ has negation, $I \to \mathcal{W}$ also has negation.*

Let $\mathcal{W}$ and $\mathcal{W}'$ be WIBLs. We denote by $[\mathcal{W} \to \mathcal{W}']$ the set consisting of all $\leq_k$-monotone functions from the underlying set of $\mathcal{W}$ to the underlying set of $\mathcal{W}'$, equipped with the pointwise orderings.

Proposition 17. *Let $\mathcal{W}$ and $\mathcal{W}'$ be (infinitary) WIBLs. Then, $[\mathcal{W} \to \mathcal{W}']$ is also an (infinitary) WIBL. Moreover, if $\mathcal{W}'$ has negation, $[\mathcal{W} \to \mathcal{W}']$ also has negation.*

We now proceed to the semantics of higher-order logic programming with negation. Our starting point is (Charalambidis, Rondogiannis, and Symeonidou 2018), where the higher-order logic $\mathcal{HOL}$ was defined and a fragment of clauses of $\mathcal{HOL}$ was considered as a higher-order logic programming language. In the rest of this section we demonstrate how the semantics of (Charalambidis, Rondogiannis, and Symeonidou 2018) can be extended in the wider context of infinitary WIBLs. We assume that the reader has some familiarity with the material in (Charalambidis, Rondogiannis, and Symeonidou 2018): although we outline some of the basic notions from the aforementioned paper, our exposition is restricted due to space limitations.

We start by defining the types of $\mathcal{HOL}$.

Definition 12. *The types of $\mathcal{HOL}$ are defined as:*

$$\pi := o \mid \rho \to \pi$$
$$\rho := \iota \mid \pi$$

where o is the base Boolean type*, ι is the* base individual type*, π is a* predicate type*, and ρ is an* argument type.

The semantics of types as defined in (Charalambidis, Rondogiannis, and Symeonidou 2018) interprets the base type o as the set $\{\mathit{false}, \mathit{true}, \perp\}$ and ι as a non-empty set D of individuals. Our framework allows us to define the semantics of types of $\mathcal{HOL}$ with respect to an arbitrary infinitary WIBL with negation.

Definition 13. *Let D be a non-empty set and let $\mathcal{W}$ be a WIBL with negation. For every type π, the set of meanings of type π with respect to $\mathcal{W}$ and D, is denoted by $[\![\pi]\!]_{\mathcal{W},D}$ and defined as follows:*

- $[\![o]\!]_{\mathcal{W},D} = \mathcal{W}$

- $[\![\iota \to \pi]\!]_{\mathcal{W},D} = D \to [\![\pi]\!]_{\mathcal{W},D}$
- $[\![\pi_1 \to \pi_2]\!]_{\mathcal{W},D} = [[\![\pi_1]\!]_{\mathcal{W},D} \to [\![\pi_2]\!]_{\mathcal{W},D}]$

Notice that since $\mathcal{THREE}$ is actually an infinitary WIBL with negation, we can take $\mathcal{W} = \mathcal{THREE}$ and an non-empty D in the above definition. As an example, the domain $[\![o \to o]\!]_{\mathcal{THREE},D}$ is exactly the $\leq_k$-monotone functions from $\mathcal{THREE}$ to $\mathcal{THREE}$. Therefore, the definition of the semantics of types of (Charalambidis, Rondogiannis, and Symeonidou 2018) is a special case of Definition 13.

The following proposition, which generalizes Proposition 1 of (Charalambidis, Rondogiannis, and Symeonidou 2018), suggests that the meaning of every predicate type is an infinitary WIBL with negation.

Proposition 18. *Let D be a non-empty set and $\mathcal{W}$ an infinitary WIBL with negation. Then, for every type π, $[\![\pi]\!]_{\mathcal{W},D}$ is an infinitary WIBL with negation.*

We will denote by $\leq_{t,\pi}$, $\leq_{k,\pi}$ the orderings of $[\![\pi]\!]_{\mathcal{W},D}$.

Definition 14. *The set of* expressions *of $\mathcal{HOL}$ is defined as follows:*

- *Every variable (respectively, constant) of type ρ is an expression of type ρ; the constants* `false` *and* `true` *are expressions of type o.*
- *If E_1 is an expression of type $\rho \to \pi$ and E_2 is an expression of type ρ, then $(\mathsf{E}_1\ \mathsf{E}_2)$ is an expression of type π. If R is an argument variable of type ρ and E is an expression of type π, then $(\lambda\mathsf{R}.\mathsf{E})$ is an expression of type $\rho \to \pi$. If E is an expression of type o and R is a variable of type ρ then $(\exists_\rho \mathsf{R}\ \mathsf{E})$ is an expression of type o.*
- *If $\mathsf{E}_1, \mathsf{E}_2$ are expressions of type π, then $(\mathsf{E}_1 \bigwedge_\pi \mathsf{E}_2)$ and $(\mathsf{E}_1 \bigvee_\pi \mathsf{E}_2)$ are expressions of type π. If E is an expression of type π, then $(\neg_\pi \mathsf{E})$ is an expression of type π. If $\mathsf{E}_1, \mathsf{E}_2$ are expressions of type ι, then $(\mathsf{E}_1 \approx \mathsf{E}_2)$ is an expression of type o.*

The notions of *free* and *bound* variables of an expression are defined as usual. An expression is called *closed* if it does not contain any free variables.

The notions of an *interpretation* and a *state* of $\mathcal{HOL}$, can be defined in an analogous way as in (Charalambidis, Rondogiannis, and Symeonidou 2018, Definitions 6 and 7). The semantics of expressions of $\mathcal{HOL}$ is straightforward (Charalambidis, Rondogiannis, and Symeonidou 2018, see Definition 8). The semantics of the logical constant symbols $\neg_\pi$, $\vee_\pi$, and $\wedge_\pi$ can be defined directly using the corresponding operations in the WIBL with negation: $\neg_\pi$ is interpreted by the negation operation of the WIBL, $\vee_\pi$ by the operation $\vee_{t,\pi}$, and $\wedge_\pi$ by the operation $\wedge_{t,\pi}$ that correspond to the WIBL of the elements of type π (see Corollary 16 and Proposition 17).

Based on the above remarks, the following lemma generalizes Lemma 1 of (Charalambidis, Rondogiannis, and Symeonidou 2018) and its proof is analogous. It implies that the meanings of expressions of $\mathcal{HOL}$, are elements of the corresponding WIBL.

Lemma 19. *Let D be a non-empty set and $\mathcal{W}$ be an infinitary WIBL with negation. Let E be an expression of $\mathcal{HOL}$ of type π. Moreover, let s be a state and let $\mathcal{I}$ be an interpretation. Then, $[\![\mathsf{E}]\!]_s(\mathcal{I}) \in [\![\pi]\!]_{\mathcal{W},D}$.*

The authors of (Charalambidis, Rondogiannis, and Symeonidou 2018) consider a fragment of $\mathcal{HOL}$ and study it as a higher-order logic programming language. A *$\mathcal{HOL}$ program* is a finite set of clauses of the form $\mathsf{p} \leftarrow_\pi \mathsf{E}$, where p is a predicate constant of type π and E is a closed expression also of type π. The semantics of such programs is defined using *Herbrand interpretations*, (Charalambidis, Rondogiannis, and Symeonidou 2018, see Definition 16). The notion of Herbrand interpretation easily extends in our setting: since our semantics is defined relatively to a WIBL with negation, say $\mathcal{W}$, a Herbrand interpretation $\mathcal{I}$ of a program P assigns to each predicate constant p of type π, an element of $[\![\pi]\!]_{\mathcal{W},U_\mathsf{P}}$, where U_P is the Herbrand universe of program P. Let us denote by $\mathcal{H}_{\mathsf{P},\mathcal{W}}$ the set of Herbrand interpretations of a program P relatively to $\mathcal{W}$. The notion of a *Herbrand model* of a program is defined as follows.

Definition 15. *Let P be a program and $\mathcal{W}$ an infinitary WIBL with negation. Then, a Herbrand interpretation $\mathcal{I} \in \mathcal{H}_{P,\mathcal{W}}$ will be called a* Herbrand model *of P if for all clauses $\mathsf{p} \leftarrow_\pi \mathsf{E}$ of P, it holds $[\![\mathsf{E}]\!](\mathcal{I}) \leq_{t,\pi} \mathcal{I}(\mathsf{p})$.*

We then have the following generalization of Proposition 6 of (Charalambidis, Rondogiannis, and Symeonidou 2018).

Proposition 20. *Let P be a program and $\mathcal{W}$ an infinitary WIBL with negation. Then, $\mathcal{H}_{\mathsf{P},\mathcal{W}}$ is an infinitary WIBL with negation.*

We can now apply the fixpoint theory of Section 4. The function on which the theory will be applied, is the *immediate consequence operator* of a given program.

Definition 16. *Let P be a program and $\mathcal{W}$ an infinitary WIBL with negation. The* immediate consequence operator of P with respect to $\mathcal{W}$ *is the function $\Psi_{\mathsf{P},\mathcal{W}} \colon \mathcal{H}_{\mathsf{P},\mathcal{W}} \to \mathcal{H}_{\mathsf{P},\mathcal{W}}$ that is defined for every p of type π of P as: $\Psi_{\mathsf{P},\mathcal{W}}(\mathcal{I})(\mathsf{p}) = \bigvee_{t,\pi}\{[\![\mathsf{E}]\!](\mathcal{I}) \mid (\mathsf{p} \leftarrow_\pi \mathsf{E}) \in \mathsf{P}\}$.*

The following proposition, which is well-known for classical logic programs (Lloyd 1987, Proposition 6.4), now extends to our more general framework.

Proposition 21. *Let P be a program and $\mathcal{W}$ be an infinitary WIBL with negation. A Herbrand interpretation $\mathcal{I} \in \mathcal{H}_{\mathsf{P},\mathcal{W}}$ is a Herbrand model of P iff $\Psi_{\mathsf{P},\mathcal{W}}(\mathcal{I}) \leq_t \mathcal{I}$.*

The immediate consequence operator is $\leq_k$-monotone over the set of Herbrand interpretations.

Proposition 22. *Let P be a program and $\mathcal{W}$ an infinitary WIBL with negation. Then, $\Psi_{\mathsf{P},\mathcal{W}} \colon \mathcal{H}_{\mathsf{P},\mathcal{W}} \to \mathcal{H}_{\mathsf{P},\mathcal{W}}$ is $\leq_k$-monotone.*

By Theorem 5, $\Psi_{\mathsf{P},\mathcal{W}}$ has a well-founded fixpoint, which is a $\leq_t$-minimal fixpoint of $\Psi_{\mathsf{P},\mathcal{W}}$. By Proposition 21, this is also a Herbrand model of program P, which we can take as its intended model.

The above discussion generalizes the material in (Charalambidis, Rondogiannis, and Symeonidou 2018): the main results of the aforementioned paper can be obtained if we consider $\mathcal{W}$ to be the infinitary WIBL $\mathcal{THREE}$. Moreover, in the special case where the infinitary WIBL $\mathcal{W}$ coincides with the IBL $\mathcal{FOUR}$, our approach provides a four-valued semantics for higher-order logic programming with negation, which is of interest in its own right.

6 WIBLs as Consistent Parts of IBLs

In this section we extend a line of research that was initiated in (Fitting 1991, Section 6). Given a bilattice $\mathcal{B} = \langle B, \leq_t, \leq_k \rangle$ with (weak) conflation, we write $\mathsf{CONS}(\mathcal{B}) = \langle \mathsf{cons}(\mathcal{B}), \leq_t, \leq_k \rangle$, where $\leq_t$ and $\leq_k$ are the restrictions of the corresponding relations of $\mathcal{B}$. Fitting considered the question of characterizing the structure of $\mathsf{CONS}(\mathcal{B})$. The motivation behind Fitting's investigation was to obtain natural generalizations of Kleene's multiple-valued logics (Kleene 1952). In particular, Fitting wondered which such generalizations could result as the consistent parts of bilattices. This is an important question because it can lead to new and possibly unexpected extensions of Kleene's logics.

To achieve that goal, Fitting introduced in (Fitting 1991) an interval construction over a lattice $\mathcal{L} = \langle L, \leq \rangle$, which he defined as $\mathcal{K}(\mathcal{L}) = \{[a, b] \mid a, b \in L, a \leq b\}$ where $[a, b] = \{x \mid a \leq x \leq b\}$. In (Kondo 2002, Section 4), an isomorphic formulation of Fitting's interval construction is given, based on a restricted subset of the Cartesian product $L \times L$. In our terminology, Kondo's formulation corresponds to the following:

Definition 17. *Let $\mathcal{L}$ be a complete lattice. The* interval construction *of $\mathcal{L}$ is the structure $RBP(\mathcal{L})$.*

Fitting established the following two theorems in (Fitting 1991, Theorem 6.1, Theorem 6.2):

Theorem 7. *Let $\mathcal{L}$ be a complete lattice with an order reversing involution. There exists an IBL $\mathcal{B}$ with negation and conflation such that $RBP(\mathcal{L}) \cong \mathsf{CONS}(\mathcal{B})$.*

Theorem 8. *Let $\mathcal{B}$ be a distributive bilattice with a negation and conflation that commute. Then there exists a complete and distributive lattice $\mathcal{L}$ such that $RBP(\mathcal{L}) \cong \mathsf{CONS}(\mathcal{B})$.*

The above results suggest that there exists a strong connection between the consistent parts of bilattices with conflation and weak bilattices of the form $RBP(\mathcal{L})$. This connection was additionally strengthened by Kondo, with the following two results (Kondo 2002, Theorem 3, Theorem 4) (adapted in our terminology):

Theorem 9. *Let $\mathcal{L}$ be a bounded lattice. Then, there exists an IBL $\mathcal{B}$ with conflation such that $RBP(\mathcal{L}) \cong \mathsf{CONS}(\mathcal{B})$.*

Theorem 10. *Let $\mathcal{B}$ be an IBL with conflation. Then, there exists a lattice $\mathcal{L}$ such that $RBP(\mathcal{L}) \cong \mathsf{CONS}(\mathcal{B})$.*

Remark 4. *Notice that in all the above theorems, $\mathsf{CONS}(\mathcal{B})$ is a WIBL of the form $RBP(\mathcal{L})$. As proven by Kondo (2002, page 37), there exist WIBLs that are not of the form $RBP(\mathcal{L})$.*

In the following, we extend the above results to the case where $\mathsf{CONS}(\mathcal{B})$ is a weak bilattice of the form $RBP(\mathcal{T})$, for some interlattice structure $\mathcal{T}$. Since, by Theorem 3, every WIBL can be written in the form $RBP(\mathcal{T})$, our results address the whole class of WIBLs. It turns out that in order for $\mathsf{CONS}(\mathcal{B})$ to have this more general form, $\mathcal{B}$ must have weak conflation (instead of standard conflation).

Lemma 23. *Let $\mathcal{B} = \langle B, \leq_t, \leq_k \rangle$ be an (infinitary) IBL with weak conflation such that $\mathsf{cons}(\mathcal{B})$ is closed under (infinitary) $\bigvee_t$ and $\bigwedge_t$. Then, $\mathsf{CONS}(\mathcal{B})$ is an (infinitary) WIBL.*

Theorem 11. *Let $\mathcal{T}$ be an infinitary interlattice structure. There exists an infinitary IBL $\mathcal{B}$ with weak conflation such that $\mathsf{cons}(\mathcal{B})$ is closed under infinitary $\bigvee_t$ and $\bigwedge_t$ and $RBP(\mathcal{T}) \cong \mathsf{CONS}(\mathcal{B})$.*

Our second result is the converse of Theorem 11.

Theorem 12. *Let $\mathcal{B}$ be an (infinitary) IBL with weak conflation such that $\mathsf{cons}(\mathcal{B})$ is closed under (infinitary) $\bigvee_t$ and $\bigwedge_t$. Then, there exists an (infinitary) interlattice structure $\mathcal{T}$ such that $RBP(\mathcal{T}) \cong \mathsf{CONS}(\mathcal{B})$.*

Notice that if $\mathcal{B}$ is an (infinitary) IBL with *standard* conflation, then $\mathsf{cons}(\mathcal{B})$ is closed under (infinitary) $\bigvee_t$ and $\bigwedge_t$ (see for example (Fitting 2006)). Therefore, the above proposition is valid without the closure assumption in the case of IBLs with standard conflation (and therefore Theorem 12 applies to a broader class of IBLs compared to Theorem 10).

Our results strengthen the connections between weak bilattices and the consistent parts of bilattices, and contribute to Fitting's program whose initial motivation (Fitting 1991) was the discovery of novel extensions of Kleene's logics.

7 Future Work

The results of the present paper provide a general tool that can be used in several non-monotonic applications. Since our framework generalizes the results of (Denecker, Marek, and Truszczyński 2000; Denecker, Marek, and Truszczynski 2004; Charalambidis, Rondogiannis, and Symeonidou 2018), it can potentially be used in contexts where the existing results fall short. For example, as shown in Section 5, a new semantics can be defined for higher-order logic programming with negation based on different infinitary WIBLs. One such example is the case of four-valued semantics for higher-order logic programming with negation, which deserves special attention as it generalizes the classical four-valued semantics of first-order logic programming with negation (Fitting 2002; Denecker, Marek, and Truszczyński 2000).

An alternative fixpoint theory for non-monotone functions, was developed in (Ésik and Rondogiannis 2015; Charalambidis, Chatziagapis, and Rondogiannis 2020): functions defined over *lexicographic lattice structures* are considered, and if they satisfy some natural properties, they are shown to possess a least fixpoint. This theory has been used to provide an infinite-valued semantics for higher-order logic programming with negation (Charalambidis, Ésik, and Rondogiannis 2014). It would be interesting to investigate whether there exist connections between WIBLs and lexicographic lattice structures as-well-as between the two corresponding non-monotone fixpoint theories.

Acknowledgements

This work is funded by a research project which is implemented in the framework of H.F.R.I call "Basic research Financing (Horizontal support of all Sciences)" under the National Recovery and Resilience Plan "Greece 2.0" funded by the European Union – NextGenerationEU (H.F.R.I. Project Number: 16116).

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