Belief Change on Rational Rankings

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No man ever steps in the same river twice, for it's not the same river and he's not the same man.

Heraclitus

Abstract

We introduce a new epistemic space: the space of rational rankings. This space is very useful for understanding some aspects of belief dynamics, particularly issues concerning the improvement of new information. Thus, we define, in a very clear and succinct way, a class of operators that capture the fact that new information is improved. An interesting feature of this space is that the behavior of these operators can be characterized through a few simple equations and inequalities whose meaning is transparent. We prove that these operators are indeed improvement operators. Moreover, we show that these operators exhibit good behavior when they undergo a sufficient number of iterations. In such cases, they become Darwiche and Pearl revision operators.

1 Introduction

Unlike Heraclitus' quote, in many models of belief revision, the new information does not affect the epistemic state (Alchourrón, Gärdenfors, and Makinson 1985; Darwiche and Pearl 1997). Thus, the epistemic state remains the same even if one piece of information arrives multiple times. However, this is not always true, and we would like to model situations in which new information always produces a change. Another behavior observed in most models of belief revision is that, in general, after revision, there is a loss of plausibility of the negation of the input. However, one might want the plausibility of the negation of the input to remain the same before and after revision.

The following example shows a situation in which permanent change is desirable and, at the same time, illustrates why we don't want the plausibility of the negation of the new piece of information to decrease after the change:

Example 1. *Dr. Roberts, MD, has been practicing medicine for 20 years. Since he began his practice, he has applied treatment A to the common cold. Treatment A always works, so he has great confidence in this treatment, and this confidence increases with every successful case even when its effectiveness against the common cold was already part of his beliefs. One day he receives information about a case of the common cold where treatment A did not work. This information comes from a well-respected colleague, so he gives it some credit, but, backed by all his experience, his confidence in treatment A does not decrease.*

The epistemic states and operators we introduce in this work allow modeling this kind of situation where the degree of confidence in treatment A does not change in numerical terms after receiving the information that the treatment does not work. However, the degree of confidence in the new information increases. If this new information is repeated a sufficient number of times, the doctor must change his opinion about treatment A.

Another feature that one would like to model is that the changes are produced little by little (very little by little!) as in the following example:

Example 2. *Tom is learning to play the guitar; he is an absolute beginner. He practices daily, which increases his confidence in his playing. After some practice, he plays in front of a small audience. As he receives positive feedback, he continues building his confidence in his abilities step by step. Each time he gets a bad reaction from the audience, he thinks he just had a bad day, but his confidence does not decrease because it has been reinforced many times. After some years of receiving positive feedback, he concludes that he is no longer a beginner but an intermediate player. Nevertheless, he is aware that there is still room for improvement and he still has a lot to learn about guitar playing.*

Having given two examples to illustrate our motivations, we will now establish the context of our work in detail and where it fits within the current lines of research.

Our starting point is the logical model for belief change proposed by Alchourrón, Gärdenfors, and Makinson (1985), known as the AGM framework. This framework adheres to three main principles: coherence, success, and minimal change. Specifically, when dealing with the incorporation of new information into a corpus of beliefs (belief revision), these principles form the foundation of the revision process (revision operators). These operators are functions that map the epistemic state of an agent and the new information into a new epistemic state that is coherent (i.e., free of contradictions), incorporates the new information, and remains as close as possible to the original epistemic state. In the logical AGM framework, epistemic states are represented as logical theories (sets of sentences closed under logical deduction), and the new information is represented by a logical formula. Typically, propositional logic serves as the underlying logic. When the logic is finite propositional logic, Katsuno and Mendelzon (1991) proposed a simplified model (the KM framework) where the epistemic state corresponds to a propositional formula, and the new information is also a propositional formula. The KM framework provides an easily understandable set of rationality principles and includes a useful representation theorem: revision operators are characterized as functions mapping a formula (an epistemic state) into a total preorder over the set of interpretations.[1]

Both the AGM and KM models are effective for explaining a single step of revision. However, the KM representation theorem led Darwiche and Pearl (1997) to demonstrate that these frameworks are inadequate for iterated revision processes. Consequently, they proposed a new model for iterated revision, referred to as the DP framework. The core concept in the DP framework is the notion of a (complex) epistemic state. These epistemic states differ from formulas and theories; they are objects associated with certain beliefs. For instance, the simplest examples of epistemic states are total preorders (TPOs) over interpretations, where the beliefs associated with a TPO are represented by a formula whose models are precisely the minimal models of the TPO.

While TPOs provide a natural and straightforward setting for studying DP revision operators, more general settings exist where DP revision operators can be defined. For example, ordinal conditional functions (OCFs) proposed by Spohn (1988) offer such a general setting. Indeed, there are DP revision operators defined on OCFs that cannot be described in terms of TPOs (see (Aravanis, Peppas, and Williams 2019) and (Schwind, Konieczny, and Pino Pérez 2022)). It is crucial to recognize that a belief change operator, such as a DP revision operator, must be defined within a specific "epistemic space," and numerous epistemic spaces, including some quite uncommon ones, exist (Schwind, Konieczny, and Pino Pérez 2022).

Another important area of research in belief change, particularly in belief revision, involves non-prioritized change operators (Hansson 1997; Hansson 1999). In these operators, the success postulate is relaxed. This category includes credibility-limited revision operators (Hansson et al. 2001; Booth et al. 2012), where only information from a credible set is incorporated, and information not in this set is rejected with no impact on the current belief state. Another family of non-prioritized revision operators aims to capture the idea that the plausibility of new information increases. These improvement operators (Konieczny and Pino Pérez 2008; Konieczny, Medina Grespan, and Pino Pérez 2010; Medina Grespan and Pino Pérez 2013) uniquely achieve success after several iterations of the same information, contrasting with credibility-limited revision operators, where some pieces of information are never incorporated even after multiple iterations. However, these two approaches have been successfully merged in credibility-limited improvement operators (Booth et al. 2014). Notably, research on these operator families has been primarily conducted within the framework of total preorders as the epistemic space, though some improvement operators have also defined in the space of OCFs (Konieczny and Pino Pérez 2008).

In this paper, we introduce a novel complex epistemic space called *rational rankings*, where rankings are represented by rational numbers. This exploration is motivated by the idea that studying structures beyond TPOs and OCFs can offer deeper insights into the properties that improvement operators should possess in broader contexts. The framework of rational rankings is both rich and intuitive, allowing us to define operators on this space that capture the increase in plausibility of new information. This behavior is characterized by a set of simple equations and inequalities with clear, intuitive meanings. We show that these operators fit within the class of improvement operators proposed in the literature and demonstrate that, after sufficient iterations, they function as Darwiche and Pearl's revision operators.

2 Preliminaries

We consider a finite set of propositional variables $P = \{x_1, x_2, \ldots, x_n\}$ and the set of propositional formulas, $\mathcal{L}_P$ built up from P and the usual connectives. $\mathcal{L}_P^*$ denotes the set of consistent formulas from $\mathcal{L}_P$. The symbols $\top$ and $\bot$ denote respectively the tautology and the contradiction. The set $\mathcal{W} = \{w_1, w_2, \ldots, w_{2^n}\}$ is the set of all interpretations (or worlds) i.e. the classical truth functions on P. Propositional formulas will be denoted in general by lower case Greek letters like α and their set of models is denoted by $[\![\alpha]\!]$.

Let $\leq$ be a a total pre-order, i.e. a reflexive ($x \leq x$), transitive ($(x \leq y \wedge y \leq z) \rightarrow x \leq z$) and total ($x \leq y \vee y \leq x$) relation over $\mathcal{W}$. The corresponding strict relation $<$ is defined as $x < y$ iff $x \leq y$ and $y \not\leq x$, and the corresponding equivalence relation $\simeq$ is defined as $x \simeq y$ iff $x \leq y$ and $y \leq x$. We denote $w \lll w'$ when $w < w'$ and there is no w'' such that $w < w'' < w'$.

Operators of change adequate to iteration (Darwiche and Pearl 1997) have to be defined in epistemic spaces (Schwind, Konieczny, and Pino Pérez 2022). Let us recall their definition more precisely.

Definition 1. *An* epistemic space *is an ordered pair* $\mathcal{E} = \langle U, \mathcal{B} \rangle$ *where* U *is a set whose elements are called* epistemic states *and* $\mathcal{B}$ *is a* projection *function that maps each* $\Psi \in U$ *to a consistent propositional formula that represents the belief set associated with* Ψ.

Thus an epistemic state is just an element of an epistemic space, very much as a vector is just an element of a vector space.

The two more typical examples of epistemic spaces are the TPO epistemic space, $\mathcal{E}_{tpo}$ (TPO is the abbreviation of total preorder) and the OCF epistemic space, $\mathcal{E}_{ocf}$ (OCF is the abbreviation Ordinal conditional Function).[2] More

[1] A similar representation theorem in terms of sphere systems was provided by Grove (1988).

[2] An OCF (Spohn 1988) κ is a function associating each world with a non-negative integer such that there is a world w with $\kappa(w) = 0$. In some works, OCFs are called *rankings* as well

precisely, $\mathcal{E}_{tpo} = \langle U_{tpo}, B_{tpo}\rangle$ where U_{tpo} is the set of all total preorders over $\mathcal{W}$ and B_{tpo} maps each total preorder $\preceq$ in U_{tpo} to a consistent formula $\psi \in \mathcal{L}_P^*$ such that $[\![\psi]\!] = \min(\mathcal{W}, \preceq)$. And $\mathcal{E}_{ocf} = \langle U_{ocf}, B_{ocf}\rangle$ where U_{ocf} is the set of all OCFs over $\mathcal{W}$ and B_{ocf} maps an OCF κ to a consistent formula ψ such that $[\![\psi]\!] = \{w : \kappa(w) = 0\}$.

Given an epistemic space $\mathcal{E} = \langle U, \mathcal{B}\rangle$, a *change operator* $\circ$ is a function of the form $\circ : U \times \mathcal{L}_P \longrightarrow U$. General epistemic states are noted by upper case Greek letters like Ψ. The output of an operator is noted in infix notation. Thus, $\Psi \circ \alpha$ is the epistemic state obtained from Ψ with the change produced by the information α.

Given a change operator $\circ$ we define for every formula α:

$$\Psi \circ^0 \alpha := \Psi$$
$$\Psi \circ^{k+1} \alpha := (\Psi \circ^k \alpha) \circ \alpha \qquad k \geqslant 0$$

The improvement operators proposed in (Konieczny and Pino Pérez 2008) have an important property, the success at iteration (I1), which allows stating postulates in terms of a corresponding revision operator. More precisely, given a change operator $\circ$ of the form $\circ : U \times \mathcal{L}_P \longrightarrow U$ defined in an epistemic space $\mathcal{E} = \langle U, \mathcal{B}\rangle$, which satisfies (I1): for all Ψ and α, there exists n such that $\mathcal{B}(\Psi \circ^n \alpha) \vdash \alpha$, we can define the operator $\star : U \times \mathcal{L} \longrightarrow U$ associated to the operator $\circ$ in the following way:

$$\Psi \star \alpha = \Psi \circ^k \alpha \text{ with } k = \min\{n \in \mathbb{N} : \Psi \circ^n \alpha \vdash \alpha\} \quad (1)$$

Then, improvement operators are defined as follows:

Definition 2 ((Konieczny and Pino Pérez 2008))**.** *The operator $\circ : U \times \mathcal{L}_P \longrightarrow U$ is an improvement operator if it satisfies the following postulates:*

(I1) *There exists n such that $\mathcal{B}(\Psi \circ^n \alpha) \vdash \alpha$*
(I2) *If $\mathcal{B}(\Psi) \wedge \alpha \nvdash \bot$, then $\mathcal{B}(\Psi \star \alpha) \equiv \mathcal{B}(\Psi) \wedge \alpha$*
(I3) *If $\alpha \nvdash \bot$, then $\mathcal{B}(\Psi \circ \alpha) \nvdash \bot$*
(I4) *For any positive integer n if $\alpha_i \equiv \beta_i$ for all $i \leqslant n$ then $\mathcal{B}(\Psi \circ \alpha_1 \circ \cdots \circ \alpha_n) \equiv \mathcal{B}(\Psi \circ \beta_1 \circ \cdots \circ \beta_n)$*
(I5) *$\mathcal{B}(\Psi \star \alpha) \wedge \beta \vdash \mathcal{B}(\Psi \star (\alpha \wedge \beta))$*
(I6) *If $\mathcal{B}(\Psi \star \alpha) \wedge \beta \nvdash \bot$, then $\mathcal{B}(\Psi \star (\alpha \wedge \beta)) \vdash \mathcal{B}(\Psi \star \alpha) \wedge \beta$*
(I7) *If $\alpha \vdash \mu$ then $\mathcal{B}((\Psi \circ \mu) \star \alpha) \equiv \mathcal{B}(\Psi \star \alpha)$*
(I8) *If $\alpha \vdash \neg\mu$ then $\mathcal{B}((\Psi \circ \mu) \star \alpha) \equiv \mathcal{B}(\Psi \star \alpha)$*
(I9) *If $\mathcal{B}(\Psi \star \alpha) \nvdash \neg\mu$ then $\mathcal{B}((\Psi \circ \mu) \star \alpha) \vdash \mu$*

In (Konieczny and Pino Pérez 2008), a representation theorem of improvement operators in terms of total preorders over worlds is provided.

It is worth remarking that postulate (I4) is stronger than the DP postulate (R4), that is, $\mathcal{B}(\Psi \circ \alpha) \equiv \mathcal{B}(\Psi \circ \beta)$ whenever $\alpha \equiv \beta$; for a discussion on this, see (Konieczny and Pino Pérez 2008). Actually, (I4) aims to code at the level of beliefs the fact that if $\alpha \equiv \beta$, then $\Psi \circ \alpha$ and $\Psi \circ \beta$ are exactly the same epistemic state, which is clearly stronger than (R4) (Darwiche and Pearl 1997).

and they can take values on several ordered sets like natural or real numbers (Spohn 2009; Kern-Isberner, Skovgaard-Olsen, and Spohn 2021) or even truly transfinite ordinals (Konieczny 2009).

Definition 3 (Gradual assignment)**.** *Given an epistemic space $\mathcal{E} = \langle E, \mathcal{B}\rangle$ and an operator over this space, a mapping $\Psi \mapsto \preceq_\Psi$ associating each epistemic state $\Psi \in E$ with a total preorder over worlds $\preceq_\Psi$ is called a* gradual assignment *for $\circ$ if the following conditions are satisfied:*

1. *If $w, w' \in [\![\mathcal{B}(\Psi)]\!]$, then $w \simeq_\Psi w'$*
2. *If $w \in [\![\mathcal{B}(\Psi)]\!]$ and $w' \notin [\![\mathcal{B}(\Psi)]\!]$, then $w \prec_\Psi w'$*
3. *For any positive integer n, if $\alpha_i \equiv \beta_i$ for any $i \leqslant n$, then $\preceq_{\Psi\circ\alpha_1\circ\ldots\circ\alpha_n} = \preceq_{\Psi\circ\beta_1\circ\ldots\circ\beta_n}$*
4. *If $w, w' \in [\![\alpha]\!]$ then $w \preceq_\Psi w' \Leftrightarrow w \preceq_{\Psi\circ\alpha} w'$*
5. *If $w, w' \in [\![\neg\alpha]\!]$ then $w \preceq_\Psi w' \Leftrightarrow w \preceq_{\Psi\circ\alpha} w'$*
6. *If $w \in [\![\alpha]\!], w' \in [\![\neg\alpha]\!]$ then $w \preceq_\Psi w' \Rightarrow w \prec_{\Psi\circ\alpha} w'$*

We have the following representation theorem for the class of improvement operators:

Theorem 1. *A change operator $\circ$ is an improvement operator if and only if there exists a gradual assignment such that for each epistemic state and each formula α, $[\![B(\Psi \star \alpha)]\!] = \min([\![\alpha]\!], \leqslant_\Psi)$.*

Note that conditions 4 and 5 align with Darwiche and Pearl's conditions (CR1) and (CR2) respectively (Darwiche and Pearl 1997). Condition 6 corresponds to condition (P) introduced in (Booth and Meyer 2006; Jin and Thielscher 2007) for defining *admissible* DP revision operators. Consequently, the operator $\star$ associated with an improvement operator $\circ$ is an admissible DP revision operator.

Let us give some examples of operators defined on $\mathcal{E}_{tpo}$, the space of total preorders: the natural revision operator (Boutilier 1996), the lexicographic revision operator (Nayak 1994) and the one-improvement operator (Konieczny and Pino Pérez 2008; Konieczny, Medina Grespan, and Pino Pérez 2010).

Example 3. *The natural revision operator $\circ_N$ is defined on the epistemic space $\mathcal{E}_{tpo}$ mapping each* TPO *$\Psi \in \mathcal{E}_{tpo}$ and each formula α into the* TPO *$\Psi \circ_N \alpha$ that satisfies $\min(\Psi \circ_N \alpha) = \min([\![\alpha]\!], \preceq_\Psi)$ along with the following condition:*

(N) *If $w, w' \notin \min([\![\alpha]\!], \Psi)$, then $w \preceq_\Psi w' \Leftrightarrow w \preceq_{\Psi\circ_N\alpha} w'$,*

where $\preceq_\Psi$ denotes Ψ and $\preceq_{\Psi\circ_N\alpha}$ denotes $\Psi \circ_N \alpha$.

That is, Boutilier's natural revision operator on $\mathcal{E}_{tpo}$ selects the set of all minimal models of μ according to an input TPO and defines this set as the first level of the revised TPO, leaving the rest of the TPO unchanged.

Example 4. *Nayak's lexicographic operator $\circ_L$ is defined also in the epistemic space $\mathcal{E}_{tpo}$ by putting $\min(\Psi \circ_L \alpha) = \min([\![\alpha]\!], \Psi)$, plus conditions 4 and 5 of a gradual assignment, and:*

(L) *If $w \models \alpha$ and $w' \not\models \alpha$, then $w \prec_{\Psi\circ_L\alpha} w'$*

Lexicographic revision reorders all models of α to be higher than all models of $\neg\alpha$, while preserving the internal relationships within the worlds of α and within the worlds of $\neg\alpha$.

Example 5. *The one-improvement operator $\odot$ is defined also in the epistemic space $\mathcal{E}_{tpo}$ by mapping each* TPO *$\preceq_\Psi \in \mathcal{E}_{tpo}$ and each formula μ into the* TPO *$(\preceq_\Psi \odot \mu)$ that satisfies conditions 4, 5 and 6 of a gradual assignment, and the following additional conditions:*

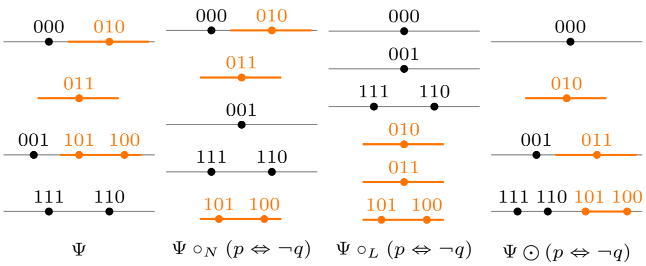

Figure 1: The natural revision operator $\circ_N$, the lexicographic revision operator $\circ_L$ and the one-improvement operator $\odot$ at work.

- *If $w \in [\![\alpha]\!], w' \in [\![\neg\alpha]\!]$ then $w' <_\Psi w \Rightarrow w' \preceq_{\Psi \odot \alpha} w$*
- *If $w \in [\![\alpha]\!], w' \in [\![\neg\alpha]\!]$ then $w' \lll_\Psi w \Rightarrow w \preceq_{\Psi \odot \alpha} w'$*

These additional conditions characterize the behavior of $\odot$. They state that the models of α at a level just above a level with models of $\neg\alpha$ move down to this level.

Let us now illustrate how the behavior of these operators differs from each other.

Example 6 (adapted from (Schwind, Konieczny, and Pino Pérez 2022))**.** *Let $P = \{p, q, r\}$. Figure 1 shows a* TPO *Ψ over worlds*[3] *, and the revised* TPOs *$\Psi \circ_N (p \Leftrightarrow \neg q)$, $\Psi \circ_L (p \Leftrightarrow \neg q)$ and $\Psi \odot (p \Leftrightarrow \neg q)$. We have $Bel(\Psi) \equiv p \wedge q$, $Bel(\Psi \circ_N (p \Leftrightarrow \neg q)) \equiv Bel(\Psi \circ_L (p \Leftrightarrow \neg q)) \equiv p \wedge \neg q$, but $Bel(\Psi \odot (p \Leftrightarrow \neg q)) \equiv p$ and the three associated* TPOs *are very different. Therefore, it is easy to identify scenarios (sequences of input formulas) that result in different beliefs for each* TPO*.*

Note that there are other improvement operators defined in $\mathcal{E}_{ocf}$, the space of ordinal conditional functions, which can not be represented in the space of TPOs (see (Schwind, Konieczny, and Pino Pérez 2022)).

3 Generalized Improvement Operators

We would like to consider operators that improve only a portion of the new information (for instance, only the most plausible models of the new information). This approach could lead to a "natural improvement" operator reminiscent of Boutilier's natural revision. To clarify this idea, we will define the operator $\circ_{NI}$ on the space $\mathcal{E}_{tpo}$, which emulates this behavior. First, given a TPO $\preceq_\Psi$ and a formula μ, let μ_Ψ be a formula such that $[\![\mu_\Psi]\!] = \min([\![\mu]\!], \preceq_\Psi)$. Then define $\preceq_\Psi \circ_{NI} \mu = \preceq_\Psi \odot \mu_\Psi$. Figure 2 illustrates the behavior of this operator at work.

It is easy to see that the operator $\circ_{NI}$ does not satisfy (I9), which corresponds to condition 6 of a gradual assignment. This is evident from Figure 2 where the world 010 remains at the same level as the world 000 after the change. However, the operator $\circ_{NI}$ satisfies postulate (I1), which is central to the improvement principle. In fact, it is clear that the operator $\star$ associated with $\circ_{NI}$ is indeed the natural revision operator $\circ_N$. Thus, it is reasonable to replace postulate (I9) with "non-worsening postulates", those corresponding to postulates (C3) and (C4) of Darwiche and Pearl. This leads to the following definition, which extends the class of improvement operators:

[3] A world w is at the same or at a lower level than a world w' iff $w \leqslant_\Psi w'$. Thus, minimal (i.e., most plausible) worlds are at the lowest levels.

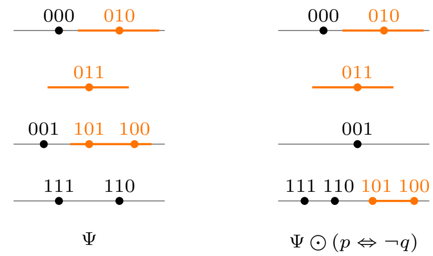

Figure 2: The natural improvement operator $\circ_{NI}$ at work.

Definition 4 (Generalized Improvement Operator)**.** *The operator $\circ : U \times \mathcal{L}_P \longrightarrow U$ is a generalized improvement operator if it satisfies the postulates (I1-I8) plus the following ones:*

(I9a) *If $\mathcal{B}(\Psi \star \alpha) \vdash \mu$ then $\mathcal{B}((\Psi \circ \mu) \star \alpha) \vdash \mu$*
(I9b) *If $\mathcal{B}(\Psi \star \alpha) \nvdash \neg\mu$, then $\mathcal{B}((\Psi \circ \mu) \star \alpha) \nvdash \neg\mu$*

Obviously enough, if an operator sastisfies (I9), then it satisfies (I9a) and (I9b). As a consequence, every improvement operator is a generalized improvement operator.

From the semantic perspective, we have the following:

Definition 5. *Given an epistemic space $\mathcal{E} = \langle U, \mathcal{B} \rangle$ and an operator over this space, a mapping $\Psi \mapsto \preceq_\Psi$ associating each epistemic state $\Psi \in E$ with a total preorder over worlds $\preceq_\Psi$ is called a* generalized gradual assignment *if it satisfies conditions 1-5 of a gradual assignment and the following conditions:*

6a. If $\omega \in [\![\alpha]\!]$ and $\omega' \in [\![\neg\alpha]\!]$ then $\omega <_\Psi \omega' \Rightarrow \omega <_{\Psi \circ \alpha} \omega'$
6b. If $\omega \in [\![\alpha]\!]$ and $\omega' \in [\![\neg\alpha]\!]$ then $\omega \preceq_\Psi \omega' \Rightarrow \omega \preceq_{\Psi \circ \alpha} \omega'$

Conditions 6a and 6b above correspond to Darwiche and Pearl's conditions (CR3) and (CR4) (Darwiche and Pearl 1997). Then, we can prove with a technique similar to the proof of Theorem 1 the following result for the class of operators satisfying (I1):

Theorem 2. *A change operator $\circ$ is a generalized improvement operator if and only if there exists a generalized gradual assignment such that $[\![B(\Psi \star \alpha)]\!] = \min([\![\alpha]\!], \preceq_\Psi)$.*

It is easy to see that if $\circ$ is a generalized improvement operator, then its corresponding operator $\star$ is a DP revision operator. More interestingly, we obtain the following correspondence:

Theorem 3. *An operator is a generalized improvement operator satisfying (R1) if and only if it is a DP revision operator satisfying (I4).*

4 Rational Improvement Operators

The goal of this section is to better understand the behavior of various improvement operators within a specific epistemic space: the space of rational rankings. Given that the order on $\mathbb{Q}$ is dense and unbounded, it more naturally captures the dynamics of improvement operators. The density of $\mathbb{Q}$ is particularly advantageous because it enables the insertion of an intermediate layer between two existing layers without altering their ranks. This capability is not possible with ordinal conditional functions (OCFs) when the layers are consecutive. This approach aligns with the principle of improving without changing the rank of models that contradict the new information.

Although rankings into real numbers have been studied in (Spohn 2009), there are key differences, which we will explore in Section 6.

A *rational ranking* is a function $r : \mathcal{W} \to \mathbb{Q}$. The set of all rational rankings is denoted by $\mathcal{R}(\mathbb{Q})$. For a given rational ranking r, we denote r_k as the preimage $r^{-1}[k]$ for $k \in \mathbb{Q}$, *i.e.*, $r_k = r^{-1}[k] = \{w \in \mathcal{W} : r(w) = k\}$. The (possibly empty) set r_k represents the *k-th level* of the ranking r.

Given any consistent formula α and a ranking r, we define the rank of α in r, denoted by $r(\alpha)$, as:

$$r(\alpha) = \max\{k \in \mathbb{Q} : r_k \cap [\![\alpha]\!] \neq \varnothing\} \quad (2)$$

The following notation, $top(\alpha)$, which denotes the set of most plausible models of α in r, will be useful:

$$top(\alpha) := r_{r(\alpha)} \cap [\![\alpha]\!]$$

Definition 6. *The epistemic space of rational rankings, $\mathcal{E}_{\mathbb{Q}} = \langle U, \mathcal{B}\rangle$ is defined by taking U as the set of rational rankings and $\mathcal{B}$ as the function from U to $\mathcal{L}_P^*$ defined as:*

$$w \in [\![\mathcal{B}(r)]\!] \iff r(w) = r(\top) \quad (3)$$

The intended meaning of a rational ranking r as an epistemic state is that $r(w') < r(w)$ indicates that the world w is more plausible than the world w'. The beliefs associated with r correspond to a formula whose models are precisely those with the highest rank.

In this work, we adopt the perspective that the value assigned to a world in a rational ranking directly represents its plausibility, in contrast to OCFs. We find it natural to associate higher ranks with greater plausibility.

A significant difference from OCFs is that rational rankings are not normalized, meaning that a rational ranking can have beliefs with arbitrarily high ranks. This characteristic allows for the plausibility of models representing new information to be increased without altering the plausibility of models representing the negation of that information.

Definition 7. *Given a rational ranking r, its* underlying preorder *$\preceq_r$ is defined by:*

$$w \preceq_r w' \iff r(w') \leqslant r(w) \quad (4)$$

Observation 1. *Since $\mathcal{B}(r) = r(\top)$, we have that*

$$\mathcal{B}(r) = \max([\![\top]\!], \preceq_r)$$

Definition 8 (Rational improvement operators)**.** *$\circ$ is a* rational improvement operator *if there is a rational number $c > 0$ such that the following properties are satisfied for all rational ranking r, all consistent formulae α and all worlds w, w':*

(RI1) *If $w \in [\![\alpha]\!]$ then $r(w) \leqslant (r \circ \alpha)(w)$*

(RI2) *there is a $w'' \in [\![\alpha]\!]$ such that $r(w'') + c < (r \circ \alpha)(w'')$*

(RI3) *if $w \notin [\![\alpha]\!]$, then $r(w) = (r \circ \alpha)(w)$*

(RI4) *if $w, w' \in [\![\alpha]\!]$ then $r(w) \leqslant r(w') \iff (r \circ \alpha)(w) \leqslant (r \circ \alpha)(w')$*

(RI5) *If $\alpha \equiv \beta$ and $w \in [\![\alpha]\!]$ then $(r \circ \alpha)(w) = (r \circ \beta)(w)$*

Postulate (RI1) states that for any ranking function r, the plausibility of the models of α does not decrease after a rational improvement by α. (RI2) asserts that at least one model of α sees its plausibility strictly increased after a rational improvement by α. Moreover, this increase has a lower bound for any formula α in any rational ranking r. (RI3) requires that a rational improvement by α does not alter the plausibility of the models of $\neg\alpha$. This can be seen as a strengthening of condition 5 of a gradual assignment, which requires that the *relative* plausibility between the models of $\neg\alpha$ is not altered after revision by α. (RI4) ensures that the relative order of the models of α should remain unchanged after a rational improvement by α. Lastly, (RI5), together with (RI3), ensures the independence of syntax: a rational improvement by a formula or an equivalent one should result in the same ranking function.

It is important to note that postulates (RI1-RI5) are both natural and flexible, allowing for the definition of a wide variety of rational improvement operators, including those with a uniform behavior. But, one could also define an operator $\circ$ such that, for some models of α, the plausibility increases only from the second time α is encountered in a series of consecutive inputs of α: $r(w) = (r \circ \alpha)(w) < ((r \circ \alpha) \circ \alpha)(w)$.

The next result is crucial in demonstrating that rational improvement operators are indeed generalized improvement operators.

Theorem 4. *If $\circ$ is a rational improvement operator, then it satisfies (I1).*

Proof. Let $\Delta(\alpha) = \lceil r(\top) - \min\{r(w) \in \mathbb{Q} : w \in [\![\alpha]\!]\}\rceil$. Since each element in $[\![\alpha]\!]$ is a binary sequence, we can well-order this set with lexicographic order. Now define the function:

$$f : \mathbb{N} \longrightarrow [\![\alpha]\!]$$
$$k \mapsto \min_{<_{\text{lex}}}\left\{w \in [\![\alpha]\!] : (r \circ^{k+1} \alpha)(w) > (r \circ^{k} \alpha)(w) + c\right\}$$

This function is well defined since (RI2) implies that the set $\left\{w \in [\![\alpha]\!] : (r \circ^{k+1} \alpha)(w) > (r \circ^{k} \alpha)(w) + c\right\}$ is non-empty. Since $[\![\alpha]\!]$ is finite, the pigeonhole principle implies there is some $w_0 \in [\![\alpha]\!]$ with $|f^{-1}(w_0)| = \aleph_0$. Hence

$$f^{-1}(w_0) = \{k_1, k_2, k_3, \ldots\}$$

Now, using (RI1), it is easy to that

$$(r \circ^{k_n} \alpha)(w_0) > r(w_0) + nc$$

By the Archimedean property, there is a natural number N such that $Nc > \Delta(\alpha)$, hence

$$(r \circ^{k_N} \alpha)(w_0) > r(w_0) + Nc > r(w_0) + \Delta(\alpha) > r(\top)$$

As $(r \circ^{k_N} \alpha)(w_0) \geqslant (r \circ^{k_N} \alpha)(w_0)$, we have that $(r \circ^{k_N} \alpha)(w_0) > r(\top)$. Since $(r \circ^{n} \alpha)(w) = r(w)$ for every $w \notin$

$[\![\alpha]\!]$ and every $n \in \mathbb{N}$, we have that $w \notin [\![\alpha]\!] \implies (r \circ^n \alpha)(w) \leqslant r(\top)$. Thus, the models in $(r \circ^{k_N} \alpha)(\top)$ are a subset of $[\![\alpha]\!]$ hence $\mathcal{B}(r \circ^{k_N} \alpha) \vdash \alpha$. That is, $\circ$ satisfies (I1).

□

We can now state that:

Theorem 5. *Every rational improvement operator is a generalized improvement operator.*

The idea of the proof is to show that the assignment $r \mapsto \preceq_r$, where $\preceq_r$ is the underlying preorder of r, is a generalized gradual assignment. This, along with Theorem 4, allows us to apply Theorem 2 to reach our conclusion.

Proof. Let $\circ$ be a rational improvement operator. Theorem 4 implies that the associated revision operator $\star$ is well-defined. Hence $\mathcal{B}(r \star \alpha) \vdash \alpha$, thus $[\![\mathcal{B}(r \star \alpha)]\!] \subseteq [\![\alpha]\!]$. Since $[\![\mathcal{B}(r \star \alpha)]\!] = r_{r(\top)}$, it follows from the definition of $\preceq_r$ that

$$[\![\mathcal{B}(r \star \alpha)]\!] = \min([\![\top]\!], \preceq_r)$$

We now want to prove that $r \mapsto \preceq_r$ is a generalized gradual assignment for $\circ$. Properties (1.) and (2.) are immediate from the definition of $\preceq_r$ and the definition of $\mathcal{B}(r)$. To prove (3.), it is enough to prove it for $n = 1$ and then apply an inductive argument. To do so, suppose $\alpha \equiv \beta$. Then, (RI3) along with (RI5) implies that $r \circ \alpha = r \circ \beta$, thus $\preceq_{(r\circ\alpha)} = \preceq_{(r\circ\beta)}$. (4.) follows directly from (RI4). (5.) is an immediate consequence of (RI3). (6a.) and (6b.) follow from (RI1) and (RI3). Therefore, $r \mapsto \preceq_r$ is a generalized gradual assignment for $\circ$ and Theorem 2 implies that $\circ$ is a generalized improvement operator. □

Observation 2. *Since every rational improvement operator $\circ$ is a generalized improvement operator, it follows that its associated operator $\star$ is a DP revision operator.*

5 Examples and Properties

In this section, we introduce several families of rational improvement operators, discuss some of their properties, and provide characterizations for some of them.

Example 7. *Let $t \in \mathbb{Q}^+$. The t-*translation *is the operator*

$$\oplus_t : U \times \mathcal{L} \longrightarrow U$$

such that, if r is a ranking function and α is a sentence, the operator $\oplus_t$ is defined as

$$(r \oplus_t \alpha)(w) = \begin{cases} r(w) & \text{if } w \notin [\![\alpha]\!] \\ r(w) + t & \text{if } w \in [\![\alpha]\!] \end{cases}$$

Satisfaction of (RI1), (RI3) and (RI4) is a direct consequence of the definition of $\oplus_t$. Taking any $w \in [\![\alpha]\!]$ and $c = t/2$, we get that $r(w) + c < r(w) + t = (r \oplus_t \alpha)(w)$, hence $\oplus_t$ satisfies (RI2). Lastly, if $\alpha \equiv \beta$ we have that $(r \oplus_t \alpha)(w) = (r \oplus_t \beta)(w) = r(w) + t$ for every $[\![\alpha]\!]$ since $[\![\alpha]\!] = [\![\beta]\!]$, thus (RI5) is satisfied. Therefore, $\oplus_t$ is a rational improvement operator.

These operators are named t-translations because they uniformly move the entire set of models of α "upwards". Translation operators are reminiscent of $(A{\to}x)$-conditionalization (Spohn 2009). However, while conditionalizations are revision operators, translations are not.

These operators satisfy the following interesting commutativity property:

(Com) $\quad (r \circ \alpha) \circ \beta = (r \circ \beta) \circ \alpha$

This property a two-step adaptation to rational rankings of the commutativity property introduced in (Schwind and Konieczny 2020).

When an operator satisfies **(Com)** for every ranking r and formulas α and β, it is called a commutative operator.

Proposition 1. *The operator $\oplus_t$ is a commutative operator.*

Proof. Let $w \in \mathcal{W}$. The proof is direct by (RI3) in the case when $w \in [\![\neg\alpha]\!] \cup [\![\neg\beta]\!]$. If $w \in [\![\alpha]\!]\backslash[\![\beta]\!]$, then

$$\begin{aligned} ((r \oplus_t \alpha) \oplus_t \beta)(w) &= (r \oplus_t \alpha)(w) && \text{(RI3)} \\ &= r(w) + t \end{aligned}$$

and

$$\begin{aligned} ((r \oplus_t \beta) \oplus_t \alpha)(w) &= (r \oplus_t \beta)(w) + t \\ &= r(w) + t && \text{(RI3)} \end{aligned}$$

Hence,

$$((r \oplus_t \alpha) \oplus_t \beta)(w) = ((r \oplus_t \beta) \oplus_t \alpha)(w) = r(w) + t$$

The proof is similar for the case when $w \in [\![\beta]\!]\backslash[\![\alpha]\!]$. The remaining case is when $w \in [\![\alpha]\!] \cap [\![\beta]\!]$:

$$\begin{aligned} ((r \oplus_t \alpha) \oplus_t \beta)(w) &= (r \oplus_t \alpha)(w) + t && w \in [\![\beta]\!] \\ &= (r(w) + t) + t && w \in [\![\alpha]\!] \\ &= r(w) + 2t \\ ((r \oplus_t \beta) \oplus_t \alpha)(w) &= (r \oplus_t \beta)(w) + t && w \in [\![\alpha]\!] \\ &= (r(w) + t) + t && w \in [\![\beta]\!] \\ &= r(w) + 2t \end{aligned}$$

Thus for every $w \in \mathcal{W}$:

$$((r \oplus_t \alpha) \oplus_t \beta)(w) = ((r \oplus_t \beta) \oplus_t \alpha)(w)$$

This shows that $\oplus_t$ satisfies **(Com)**. □

Observation 3. *By the previous proposition and the results in (Schwind and Konieczny 2020) (Proposition 9 therein) the t-translation operators can not be representable as improvement operators in the space of* TPO*s. And the impossibility also remains for the revision operators corresponding to the translations.*

An interesting behavior of this family of operators is captured by the following definition:

Definition 9. *A rational improvement operator $\circ$ is a* Nayak operator *iff for each formula α, there is some $N \in \mathbb{N}$ such that:*

$$w \in [\![\alpha]\!], w' \notin [\![\alpha]\!] \implies (r \circ^N \alpha)(w') < (r \circ^N \alpha)(w)$$

These operators are named as such because, with a sufficient number of iterations, the underlying preorders behave similarly to the lexicographical revision proposed by Nayak (1994).

Observation 4. *Notice that* $\{\oplus_t : t \in \mathbb{Q}^+\}$ *is a family of Nayak operators.*

However, not all Nayak operators belong to this family. For instance, in Example 11, one can see operators that fall outside this classification.

Another interesting family of rational operators exhibits behavior inspired by Boutilier's natural revision operator:

Example 8. *Given* $t > 0$ *with* $t \in \mathbb{Q}$*, the* top-t-translation *is the operator*

$$*_{top-t} : U \times \mathcal{L} \longrightarrow U$$

assigning to each ranking r *and each sentence* α *the ranking* $*_{top-t}$ *given by*

$$(r *_{top-t} \alpha)(w) = \begin{cases} r(w) + t & \text{if } w \in r_{r(\alpha)} \cap [\![\alpha]\!] \\ r(w) & \text{otherwise} \end{cases}$$

It is easy to see from this definition that $*_{top-t}$ *is a rational improvement operator.*

Proposition 2. *The operator* $*_{top-t}$ *is not commutative.*

Proof. Consider a ranking r and formulas α, β such that $r(\alpha) = r(\beta)$. Let $n = r(\alpha)$, and let $w \in [\![\alpha]\!] \backslash [\![\beta]\!]$, $w' \in [\![\alpha]\!] \cap [\![\beta]\!]$ such that $r(w) = r(w') = n$. Now

$$\begin{aligned}(r *_{\text{top}-t} \alpha)(w) &= (r *_{\text{top}-t} \alpha)(w') \\ &= n + t\end{aligned}$$

Now since $w' \in [\![\alpha]\!] \cap [\![\beta]\!]$ we have

$$(r *_{\text{top}-t} \alpha)(\alpha) = (r *_{\text{top}-t} \alpha)(\beta) = n + t$$

and $((r *_{\text{top}-t} \alpha) *_{\text{top}-t} \beta)(w) = n + t$ since $w \notin [\![\beta]\!]$. However, $(r *_{\text{top}-t} \beta)(w) = n$ and $(r *_{\text{top}-t} \beta)(w') = n + t$. As $w' \in [\![\alpha]\!]$, this means that $(r *_{\text{top}-t} \beta)(\alpha) = n + t$, hence $(r *_{\text{top}-t} \beta)(w) \neq (r *_{\text{top}-t} \beta)(\alpha)$ and $((r *_{\text{top}-t} \beta) *_{\text{top}-t} (\alpha))(w) = n$. Therefore

$$((r *_{\text{top}-t} \alpha) *_{\text{top}-t} (\beta))(w) \neq ((r *_{\text{top}-t} \beta) *_{\text{top}-t} (\alpha))(w)$$

□

Definition 10. *A rational improvement operator* $\circ$ *is a* Boutillier *operator if the underlying orders of* r *and* $r \circ \alpha$ *restricted to* $\mathcal{W} \backslash top(\alpha)$ *coincide.*

Observation 5. $\{*_{top-t} : t \in \mathbb{Q}^+\}$ *is a family of Boutillier operators.*

Not all Boutilier operators belong to this family (see Example 10).

It is important to note that simply increasing the plausibility of models of new information does not necessarily result in a rational improvement operator. The manner in which this increase is implemented is crucial. The following example illustrates this point.

Example 9. *Consider the operator* $\circ_Z$ *that associates each pair* (r, α) *with a new ranking* $r \circ_Z \alpha$ *defined as follows: if* $r(\alpha) \neq r(\top)$*, then*

$$(r \circ_Z \alpha)(w) = \begin{cases} r(w) & \text{if } w \notin [\![\alpha]\!] \\ r(w) + \dfrac{r(\top) - r(\alpha)}{2} & \text{if } w \in [\![\alpha]\!] \end{cases}$$

and if $r(\alpha) = r(\top)$*:*

$$(r \circ_Z \alpha)(w) = \begin{cases} r(w) & \text{if } w \notin [\![\alpha]\!] \\ r(w) + 1 & \text{if } w \in [\![\alpha]\!] \end{cases}$$

Notice there is no $c \in \mathbb{Q}^+$ *such that this operator satisfies (RI2), so it is not a rational improvement operator. Clearly, when* $r(\alpha) < r(\top)$*, the models of* α *are "improving" with repeated applications of* $\circ$*. However, this operator does not generally satisfy (I1), meaning there is no associated* $\star$ *revision operator. Therefore,* $\circ$ *is not an improvement operator in the usual sense. This demonstrates that simply increasing the ranks of models of new information is insufficient to define an improvement operator.*

Operators exhibiting this behavior can be referred to as *Zeno operators*, as they resemble the Zeno Paradox. There is a goal that is approached but never fully achieved: if $r(\alpha) < r(\top)$ and $w \in [\![\alpha]\!]$ then for every $n \in \mathbb{N}$, $(r \circ^n \alpha)(w) < r(\top)$.

We can use the concept from Example 9 to define a Boutilier operator that is not part of the family described in Example 8. In Example 10, the operator is designed so that the highest-ranked models of α always improve by t, while some lower-ranked models of α move in a Zeno-like manner.

Example 10. *The idea of the operator* $*_t$*, which we are going to define, is very simple: the top models of* α *improve by t. The "isolated" models of* α *(those in a layer with only models of* α*) improve by half of the distance to the consecutive upward rank, while the rest of the models don't move. More precisely, let* $q_1, q_2, \ldots, q_m$ *be the image of the ranking* r *where* $q_i > qi + 1$*. Define* $t_i = (q_i - q_{i+1})/2$*. Then, fix t as a positive rational number and define* $*_t$ *as follows:*

1. $(r *_t \alpha)(w) = r(w) + t$ *if* $w \in top(\alpha)$*,*
2. $(r *_t \alpha)(w) = r(w) + t_i$ *if* $r(w) = q_{i+1}, w \in [\![\alpha]\!]$ *and there is no* $w' \notin [\![\alpha]\!]$ *with* $r(w') = q_{i+1}$*,*
3. $(r *_t \alpha)(w) = r(w)$ *in any other case.*

It is easy to see that $*_t$ *is a rational improvement operator. Moreover, it is clear from the definition of* $*_t$ *that it is a Boutillier operator because the underlying order of* r *and* $r *_t \alpha$ *restricted to* $\mathcal{W} \backslash top(\alpha)$ *is the same.*

With a similar idea we can define a Nayak operator which is not a t-translation for any $t \in \mathbb{Q}^+$.

Example 11. *The idea of the operator* $\oplus_{tt}$ *we are about to define is very simple: the top models of* α *improve by t. The other models of* α *improve by* $t/2$*. More precisely,*

$$(r \oplus_{tt} \alpha)(w) = \begin{cases} r(w) + t & \text{if } w \in top(\alpha) \\ r(w) + t/2 & \text{if } w \in [\![\alpha]\!] \backslash top(\alpha) \\ r(w) & \text{if } w \notin [\![\alpha]\!] \end{cases}$$

It is easy to see that $\oplus_{tt}$ is a rational improvement operator. And it is quite simple to verify that it is a Nayak operator different from $\oplus_t$ for any $t \in \mathbb{Q}^+$.

Although epistemic states always change with applications of rational improvement operators, the underlying preorders have a fixed point.

Theorem 6. *There is some $m \in \mathbb{N}$ such that*

$$\preceq_{r \circ^n \alpha} = \preceq_{r \circ^{n+1} \alpha}$$

for every $n \geqslant m$.

Proof. Let U_{tpo} be the set of all total preorders on $\mathcal{W}$. Since $\mathcal{W}$ is a finite set, U_{tpo} is also a finite set. Now, define a function

$$f : \mathbb{N} \longrightarrow U_{tpo}$$
$$k \mapsto \preceq_{r \circ^k \alpha}$$

Hence, there is some preorder $\preceq$ on $\mathcal{W}$ that has an infinite preimage $S \subseteq \mathbb{N}$. Suppose m is the least element of S. We are going to prove, using induction, that $S = \{n \in \mathbb{N} : m \leqslant n\}$, i.e., S is a final segment of $\mathbb{N}$ thus $\preceq_{r \circ^m \alpha}$ is the desired fixed point.

Suppose $k \in S$, then $\preceq = \preceq_{r \circ^k \alpha}$. Given any two w, w' we have to prove that $w \preceq_{r \circ^k \alpha} w' \iff w \preceq_{r \circ^{k+1} \alpha} w'$. This is immediate if $w, w' \notin [\![\alpha]\!]$ by (RI3) and when $w, w' \in [\![\alpha]\!]$ by (RI4).

If $w \in [\![\alpha]\!]$ and $w' \notin [\![\alpha]\!]$:

$$w \preceq_{r \circ^k \alpha} w' \iff (r \circ^k \alpha)(w') \leqslant (r \circ^k \alpha)(w) \quad (5)$$

Since $w' \notin [\![\alpha]\!]$ we have that $(r \circ^k \alpha)(w') = (r \circ^{k+1} \alpha)(w')$ and since $w \in [\![\alpha]\!]$ we have that $(r \circ^k \alpha)(w) \leqslant (r \circ^{k+1} \alpha)(w)$, thus $(r \circ^k \alpha)(w') \leqslant (r \circ^k \alpha)(w)$ implies that $(r \circ^{k+1} \alpha)(w') \leqslant (r \circ^{k+1} \alpha)(w)$ hence $w \preceq_{r \circ^k \alpha} w' \implies w \preceq_{r \circ^{k+1} \alpha} w'$.

If $w \notin [\![\alpha]\!]$, $w' \in [\![\alpha]\!]$ and $w \preceq_{r \circ^k \alpha} w'$, suppose $w \npreceq_{r \circ^{k+1} \alpha} w'$. Then $(r \circ^{k+1} \alpha)(w) < (r \circ^{k+1} \alpha)(w')$. Since S is infinite, there is some $p > k+1$ such that $\preceq_{r \circ^p \alpha} = \preceq_{r \circ^k \alpha}$ i.e. such that $(r \circ^p \alpha)(w') \leqslant (r \circ^p \alpha)(w)$.

Since $w \notin [\![\alpha]\!]$, we have that $(r \circ^k \alpha)(w) = (r \circ^n \alpha)(w)$ for every $n \in \mathbb{N}$ thus

$$\begin{aligned} (r \circ^p \alpha)(w') &\leqslant (r \circ^p \alpha)(w) \\ &= (r \circ^{k+1} \alpha)(w) \\ &< (r \circ^{k+1} \alpha)(w') \end{aligned}$$

thus $(r \circ^p \alpha)(w') < (r \circ^{k+1} \alpha)(w')$. Since $p > k+1$, there is some $\ell \in \mathbb{N}$, $n > 1$, such that $p = k + \ell$, therefore

$$(r \circ^{k+\ell} \alpha)(w') < (r \circ^{k+1} \alpha)(w')$$

which is impossible because of (RI1).

Hence, for every pair $w, w' \in \mathcal{W}$ and every $k \in S$:

$$w \preceq_{r \circ^k \alpha} w' \implies w \preceq_{r \circ^{k+1} \alpha} w'$$

In order to prove the reciprocal statement, suppose $w \npreceq_{r \circ^k \alpha} w'$. Since $\preceq_{r \circ^k \alpha}$ is a total preorder, this implies that $w' \prec_{r \circ^k \alpha} w$, but by the previous discussion this implies $w' \prec_{r \circ^{k+1} \alpha} w$, hence $w \npreceq_{r \circ^{k+1} \alpha} w'$. Thus we have proved $w \preceq_{r \circ^{k+1} \alpha} w' \implies w \preceq_{r \circ^k \alpha} w'$ by contraposition.

Therefore, for every $k > m$ we have that $\preceq_{r \circ^k \alpha} = \preceq_{r \circ^{k+1} \alpha}$, i.e., S is a final segment of $\mathbb{N}$ as we wanted to prove. ☐

Let us call m_0 the minimum m satisfying Theorem 6. Then the following observation is easy to see:

Observation 6. *$m_0 \geqslant k$ where*

$$k = \min \{n \in \mathbb{N} : r \circ^n \alpha \vdash \alpha\}$$

When $\circ$ is a Boutillier operator $m_0 = k$. If $\circ$ is Nayak, then generally $k < m_0$.

5.1 A characterization of Translations

The translation operators are the operators introduced in Example 7. More formally we have the following definition:

Definition 11. *A rational improvement operator is a* translation *if it is $\oplus_t$ for some $t \in \mathbb{Q}^+$.*

There are two additional postulates we can include in order to capture all the homogeneity behind the translations: For any rankings r, r', any formulas α, β and any worlds w and w'

(T1) If $w \in [\![\alpha]\!]$ and $w' \in [\![\beta]\!]$ then $(r \circ \alpha)(w) - r(w) = (r \circ \beta)(w') - r(w')$.

(T2) $w \in [\![\alpha]\!] \implies (r \circ \alpha)(w) - r(w) = (r' \circ \alpha)(w) - r'(w)$.

(T1) says that the increment of the rank of a given model does not depend on the formula and (T2) says that said increment does not depend of the ranking.

Lemma 1. *Postulate (T1) implies*

$$w, w' \in [\![\alpha]\!] \implies r(w) - r(w') = (r \circ \alpha)(w) - (r \circ \alpha)(w') \quad \text{(T1W)}$$

Proof. First, notice that replacing β by α in (T1) we obtain precisely (T1W), thus (T1) $\implies$ (T1W). ☐

Theorem 7. *$\circ$ is a translation iff it is a rational improvement operator satisfying properties (T1) and (T2).*

Proof. It is easy to check that every translation satisfies (T1) and (T2). On the other hand, suppose $\circ$ is a rational improvement operator satisfying (T1) and (T2). We prove it is a translation. First, take any ranking r, any consistent formula α, and take w a model of α with $(r \circ \alpha)(w) - r(w) > 0$ (which exists by (RI2)). Define

$$t = (r \circ \alpha)(w) - r(w)$$

Postulate (T2) implies that t is independent of r. (T1W) implies that t is also independent of w and (T1) implies that t does not depend on α. Hence t depends only on $\circ$. (RI3) entails that $(r \circ \alpha)(w) = r(w)$ for every $w \notin [\![\alpha]\!]$. By the very same definition of t, we have that $(r \circ \alpha)(w) = r(w) + t$ for every $w \in [\![\alpha]\!]$. Therefore $\circ = \oplus_t$ for some $t > 0$ and we can conclude that postulates (T1) and (T2), together with the postulates for rational improvement, characterize translations. ☐

Note that Postulate (T1W) is strictly weaker than (T1). To prove this, enumerate the (finite) set of equivalence classes of $\mathcal{L}_P$ under the relation $\equiv$ given by $\alpha \equiv \beta \iff [\![\alpha]\!] = [\![\beta]\!]$. We have classes $C_1, \dots, C_N$ with $N = 2^{2^n}$. Define $\circ_\equiv$:

$$r \circ_\equiv \alpha = r \oplus_k \alpha \iff \alpha \in C_k$$

It is easy to see that this is a rational improvement operator that satisfies (T1W) but not (T1), hence (T1W) $\not\Rightarrow$ (T1).

One can wonder whether postulates (T1) and (T2) are independent of each other or not. The answer is positive:

Proposition 3. *Postulates (T1) and (T2) are independent.*

Proof. Notice that $\circ_\equiv$ satisfies (T2) thus (T2) $\not\Rightarrow$ (T1).

Finally, given an enumeration $r^1, r^2, \dots$ of rankings, consider the function $*_k$ defined as $r *_k \alpha = r \oplus_k \alpha$ iff $r = r^k$. This is a rational improvement operator satisfying (T1) but not (T2), therefore (T1) $\not\Rightarrow$ (T2). □

Let us finish this section about translations with one observation about them.

Note that $r \oplus_t \top$ is the the ranking r' obtained from r by shifting the rank of all the models by t. Thus, clearly, $r' \neq r$ but the underlying preorder of both rankings is the same. Moreover, we have the following:

Observation 7. *Given r and α, if $r' = (r \oplus_t \alpha) \oplus_t \neg\alpha$ we have that $r' = r \oplus_t \top$. Thus, r and r' have the same underlying preorder but $r \neq r'$. In other words, the operator $\oplus_t$ has a cancellation property at the level of underlying orders.*

6 Related Work

Soft improvement. The notion of soft improvement operators introduced in (Konieczny, Medina Grespan, and Pino Pérez 2010) can be translated to the context of rational rankings as follows:

(r-Soft) If $w \in [\![\alpha]\!]$, $w' \notin [\![\alpha]\!]$ and $r(w') > r(w)$, then $r(w') \geqslant (r \circ \alpha)(w)$

This means that the increase in plausibility cannot result in the rank of a model being strictly greater than that of a counter-model which was more plausible in the initial ranking.

Unfortunately, our rational improvement operators cannot satisfy (r-Soft). The reason is that if an operator satisfies (RI2), it necessarily violates (r-Soft). Specifically, we can find a rational ranking r where the underlying order is linear and the difference between $r(\top)$ and the minimum rank in the image of r is less than the constant c given by property (RI2). If we choose an α such that $top(\top) \cap [\![\alpha]\!] = \varnothing$, then it is clear that if w is the model of α given by (RI2), it violates (r-Soft) because $(r \circ \alpha)(w) > r(\top)$.

Nevertheless, for certain inputs, rational improvement operators can exhibit soft behavior. For instance, if r is a rational ranking such that the image of r consists of the set $\{q_1, \dots, q_k\}$ with $q_i - 1 = q_{i+1}$, then $\oplus_{\frac{1}{2}}$ displays soft behavior. In such cases, its behavior resembles that of the half improvement operator (Konieczny, Medina Grespan, and Pino Pérez 2010).

Decrement operators. The decrement operators introduced by Sauerwald et al. (2019) are related to soft changes aimed at achieving contraction. The idea is to decrease the plausibility of the input until it is no longer believed. Related to this, we can define in the space of rational rankings operators that we call regressions. They are defined similarly to translations but use a negative rational number t.

These operators will generally not be soft, but for certain inputs, they behave like decrement operators. For example, if r is a rational ranking where the ranks in the image have a difference of 1 between consecutive ranks, then the regression operator $\oplus_{-\frac{1}{2}}$ behaves as a decrement operator.

General properties for iteration. In (Schwind, Konieczny, and Pino Pérez 2023), some generalized DP postulates were proposed for revision. These postulates aimed to ensure homogeneity in the behavior of operators. In the space of TPOs, these postulates characterized a family reduced to only three operators: Nayak lexicographic revision, Boutilier natural revision, and the restrained revision of Booth and Meyer (2006). The semantic aspect of these postulates is satisfied by the translations. A natural question is whether these (or similar) postulates can characterize the family of translations.

Expressive power of rankings. Rational rankings are not more expressive than OCFs, under standard hypotheses (postulate (G) in (Schwind, Konieczny, and Pino Pérez 2022)). However, in this paper we aim to work with a meaningful, unnormalized structure and consider a framework with rationality postulates directly applied to these structures, rather than just at the level of beliefs. Within this context, OCF-based operators are clearly not rational improvement operators in the sense of Definition 8.

Spohn's work on rankings. There are some important differences between our work and Spohn's work (2009): First, Spohn defines normalized rankings into the positive real numbers, while our rational rankings are not normalized. Additionally, he defines rankings into the set $\mathbb{R} \cup \{-\infty, \infty\}$. In Spohn's framework, negative ranks model degrees of disbelief, which is not the intention in our work. Moreover, the density of $\mathbb{Q}$ is a key property for our development, which is overlooked in Spohn's work.

7 Conclusions and Future Work

A first contribution of this work is the introduction and characterization of the class of generalized improvement operators. We have defined the epistemic space of rational rankings and the associated rational improvement operators. These operators consistently produce changes, and notably, improving new information does not require decreasing the plausibility of models that do not satisfy this new information. This framework effectively models scenarios such as the evolution of Dr. Roberts' beliefs in Example 1.

By working within the space of rational rankings, we can define t-translations with very small values of t. This flexibility allows us to model Tom's evolving beliefs in Example 2, demonstrating the need for many iterations to achieve a goal.

Another notable feature of using rational rankings is the ability to define regressions simply with negative numbers.

Through a few equations and inequalities with clear meanings, we have established a class of operators that capture the notion of improving the plausibility of the input: the rational improvement operators. Additionally, we have provided two independent postulates that characterize the translation operators.

The space of rational rankings can be very useful as a field of experimentation in order to find the properties characterizing well behaved operators. For instance, we can define operators as the Zeno operators (Example 9) which show that it is not enough to increase the plausibility of certain models of the input, for being a real improvement operator (to achieve success iterating the input enough number of times).

Some concepts, such as the softness of operators, are intrinsically contextual. As demonstrated, this is incompatible with (RI2). Future work should explore more relaxed notions than (RI2) that accommodate contextual operators, particularly postulates compatible with (r-Soft).

An important direction for future research is to determine the syntactic characterization of the classes of operators defined in this work. Additionally, investigating whether rankings on real numbers would provide a richer framework, potentially incorporating properties such as completeness, could be a valuable avenue for exploration.

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