Conditional Splittings of Belief Bases and Nonmonotonic Inference with c-Representations

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Abstract

The concept of conditional syntax splitting for inductive inference from conditional belief bases has been proposed as a generalization of syntax splitting which also covers cases where the conditionals in the subbases share some atoms. p-Entailment and system Z fail to satisfy conditional syntax splitting, and up to now, only two inductive inference operators, lexicographic inference and system W, have been shown to satisfy this property. In this paper, we introduce the concept of conditional semantic splitting. We show that c-representations satisfy a core postulate relating conditional splittings on the syntax and the semantic level. Based on these findings, we investigate conditional syntax splitting for nonmonotonic inference with c-representations. Regarding single c-representations, we utilize the concept of selection strategies, and show that a straightforward property of the selection strategy leads to inference operators satisfying conditional syntax splittings. Furthermore, we show that c-inference taking all c-representations of a belief base into account also fully complies with conditional syntax splitting.

1 Introduction

The concept of syntax splitting was developed by Parikh (1999) for belief sets in order to formulate postulates for belief revision, and was later transferred to other structures and applications, e.g., (Peppas et al. 2015; Kern-Isberner and Brewka 2017); a related concept was introduced by Weydert (1998) as minimum irrelevance. Syntax splitting for nonmonotonic reasoning from conditional belief bases (Kern-Isberner, Beierle, and Brewka 2020) is a combination of the postulates *relevance* and *independence*, stating that only conditionals from the considered part of the syntax splitting of a belief base are relevant for corresponding inferences, and that inferences using only atoms from one part of the syntax splitting should be independent of additional information on the other parts.

From a theoretical point of view, these splitting properties are interesting because they implement a notion of (ir)relevance in inferences. But splitting techniques have also consequences for applications: They allow for breaking down conditional reasoning to the subbases relevant for a query, hence usually reducing the relevant subsignature significantly. From a cognitive point of view, splitting techniques bring in the concept of local reasoning which accounts for the limited resources of humans. Moreover, local reasoning is also fundamental to all works on probabilistic networks (Pearl 1988). Full splittings in the sense of (Kern-Isberner, Beierle, and Brewka 2020), however, are quite rare in real-world applications.

The concept of conditional syntax splitting for inference from conditional belief bases (Heyninck et al. 2023) is a generalization of syntax splitting which also allows the subbases to overlap syntactically, while, in the case of safe splittings, semantic (conditional) independence holds given the joint atoms, providing much more realistic application scenarios. The relevance of conditional syntax splitting is further underpinned by the fact that the so called drowning-effect (Pearl 1990; Benferhat, Dubois, and Prade 1993) was formalized as a violation of conditional syntax splitting (Heyninck et al. 2023). The drowning effect is a phenomenon where, for some inductive inference relations, special subclasses will not properly inherit properties of their superclass. For example, given the knowledge that birds usually fly, penguins are usually birds, penguins usually do not fly, and birds usually have wings, an inductive inference relation suffering from the drowning effect would not be able to conclude that penguins usually have wings. This means that showing conditional syntax splitting is satisfied also implies that the drowning effect is avoided in general, and not just for the canonical example.

p-Entailment, characterized by the axioms of system P (Adams 1965; Kraus, Lehmann, and Magidor 1990), and system Z (Goldszmidt and Pearl 1996) do not satisfy conditional syntax splitting, and so far, only two inductive inference operators, lexicographic inference (Lehmann 1995) and system W (Komo and Beierle 2020; Komo and Beierle 2022) have been shown to satisfy it (Heyninck et al. 2023).

In this paper, we extend the study of conditional splittings of belief bases. As a basis for our investigations we use ranking functions (OCFs) (Spohn 1988) as a well established and popular semantics for conditional belief bases. In addition to their popularity, another reason for using OCFs is that the work on conditional syntax splitting is inspired by probabilistic techniques. Since OCFs can be understood as qualitative abstractions of logarithmic probabilities, they provide a perfect mediating framework to realize such probabilistic ideas for qualitative nonmonotonic reasoning.

This paper provides the following four main contributions.

1. We introduce the concept of conditional semantic splitting for semantics based on Spohn's ranking functions (Spohn 1988), and a postulate (**CSemSplit**) relating conditional splittings on the syntax and on the semantic level.
2. We show that c-representations, which are special ranking functions obtained by summing up impacts assigned to falsified conditionals (Kern-Isberner 2001; Kern-Isberner 2004), satisfy conditional semantic splitting.
3. Regarding single c-representations, we utilize the concept of selection strategies (Kern-Isberner, Beierle, and Brewka 2020; Beierle and Kern-Isberner 2021) and show that the property of preserving the impacts chosen for certain subbases for inferences on the full belief base leads to inference operators satisfying conditional syntax splitting.
4. Without any requirement regarding a selection strategy, we prove that c-inference, which is the skeptical inference taking all c-representations of a belief base into account (Beierle et al. 2018; Beierle et al. 2021), fully complies with conditional syntax splitting.

The rest of this paper is organized as follows. In Section 2, we present the basics of conditional logic needed here, and in Section 3, we recall the concept of conditional syntax splitting. In Section 4, we introduce the notion of conditional semantic splitting and show that it is satisfied by c-representations. In Section 5, we show that inference operators for single c-representations with a selection strategy satisfies conditional syntax splitting. In Section 6, we prove that c-inference taking all c-representations into account fully complies with condition syntax splitting. In Section 7, we conclude and point out further work.

2 Formal Basics

Let $\mathcal{L}$ be a finitely generated propositional language over a signature Σ with atoms $a, b, c, \ldots$, and with formulas $A, B, C, \ldots$. For conciseness of notation, we may omit the logical *and*-connector, writing AB instead of $A \wedge B$, and overlining formulas will indicate negation, i.e. $\overline{A}$ means $\neg A$. Let Ω denote the set of *possible worlds* over $\mathcal{L}$; Ω will be taken here simply as the set of all propositional interpretations over $\mathcal{L}$. $\omega \models A$ means that the propositional formula $A \in \mathcal{L}$ holds in the possible world $\omega \in \Omega$; then ω is called a *model* of A, and the set of all models of A is denoted by $\mathit{Mod}(A)$. For propositions $A, B \in \mathcal{L}$, $A \models B$ holds iff $\mathit{Mod}(A) \subseteq \mathit{Mod}(B)$, as usual. By slight abuse of notation, we will use ω both for the model and the corresponding conjunction of all positive or negated atoms. This will allow us to use ω both as an interpretation and a proposition, which will ease notation a lot. Since $\omega \models A$ means the same for both readings of ω, no confusion will arise.

For subsets Θ of Σ, let $\mathcal{L}(\Theta)$ or short $\mathcal{L}_\Theta$ denote the propositional language defined by Θ, with associated set of interpretations $\Omega(\Theta)$ or short Ω_Θ. Note that while each sentence of $\mathcal{L}(\Theta)$ can also be considered as a sentence of $\mathcal{L}$, the interpretations $\omega^\Theta \in \Omega(\Theta)$ are not elements of $\Omega(\Sigma)$ if $\Theta \neq \Sigma$. But each interpretation $\omega \in \Omega$ can be written uniquely in the form $\omega = \omega^\Theta \omega^{\overline{\Theta}}$ with concatenated $\omega^\Theta \in \Omega(\Theta)$ and $\omega^{\overline{\Theta}} \in \Omega(\overline{\Theta})$, where $\overline{\Theta} = \Sigma \backslash \Theta$ is the complement of Θ in Σ. Note that the syntactical reading of interpretations as conjunctions makes perfect sense here: According to this reading, ω is a conjunction of ω^Θ and $\omega^{\overline{\Theta}}$ (with omitted $\wedge$ symbol). ω^Θ is called the *reduct* of ω to Θ (Delgrande 2017). If $\Omega' \subseteq \Omega$ is a subset of models, then $\Omega'|_\Theta = \{\omega^\Theta | \omega \in \Omega'\} \subseteq \Omega(\Theta)$ restricts Ω' to a subset of $\Omega(\Theta)$. In the following, we will often consider the case that Σ_1, Σ_2 are disjoint subsignatures of Σ, then we write ω^i instead of ω^{Σ_i} for the reducts to ease notation.

By making use of a conditional operator $|$, we introduce the language $(\mathcal{L}|\mathcal{L})$ of *conditionals* over $\mathcal{L}$:

$$(\mathcal{L}|\mathcal{L}) = \{(B|A) \mid A, B \in \mathcal{L}\}.$$

Conditionals $(B|A)$ are meant to express plausible, defeasible rules "If A then plausibly (usually, possibly, probably, typically etc.) B". A conditional $(F|E)$ is called *self-fulfilling* if $E \models F$, i.e., there is no world that can falsify it. A popular semantic framework that is often used for interpreting conditionals is provided by ordinal conditional functions. *Ordinal conditional functions (OCFs)*, (also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ with $\kappa^{-1}(0) \neq \emptyset$, were introduced (in a more general form) first by (Spohn 1988). They express degrees of plausibility of propositional formulas A by specifying degrees of disbeliefs of their negations $\overline{A}$. More formally, we have $\kappa(A) := \min\{\kappa(\omega) \mid \omega \models A\}$, so that $\kappa(A \vee B) = \min\{\kappa(A), \kappa(B)\}$. A proposition A is believed if $\kappa(\overline{A}) > 0$ (which implies particularly $\kappa(A) = 0$). The *uniform OCF* κ_u is defined by $\kappa_u(\omega) = 0$ for all $\omega \in \Omega$.

Degrees of plausibility can also be assigned to conditionals by setting $\kappa(B|A) = \kappa(AB) - \kappa(A)$. A conditional $(B|A)$ is *accepted* in the epistemic state represented by κ, written as $\kappa \models (B|A)$, iff $\kappa(AB) < \kappa(A\overline{B})$, i.e. iff AB is more plausible than $A\overline{B}$. Conditional belief bases Δ (over $\mathcal{L}$) consist of finitely many conditionals from $(\mathcal{L} \mid \mathcal{L})$. Consistency of such a conditional belief base Δ can be defined in terms of OCFs (Pearl 1990): Δ is consistent iff there is an OCF κ such that $\kappa \models \Delta$. Using this definition, we focus on (strongly) consistent belief bases in the sense of (Pearl 1990; Goldszmidt and Pearl 1996) in order to elaborate our approach without having to deal with distracting technical particularities. The nonmonotonic inference relation $\mid\!\sim_\kappa$ induced by an OCF κ is given by (Spohn 1988)

$$A \mid\!\sim_\kappa B \text{ iff } A \equiv \bot \text{ or } \kappa(AB) < \kappa(A\overline{B}). \quad (1)$$

The *marginal of* κ *on* $\Theta \subseteq \Sigma$, denoted by $\kappa|_\Theta$, is defined by $\kappa|_\Theta(\omega^\Theta) = \kappa(\omega^\Theta)$ for any $\omega^\Theta \in \Omega(\Theta)$. Note that this marginalization is a special case of the general forgetful functor $\mathit{Mod}(\sigma)$ from Σ-models to Θ-models (Beierle and Kern-Isberner 2012) where σ is the inclusion from Θ to Σ.

To formalize inductive inference from conditional belief bases, (Kern-Isberner, Beierle, and Brewka 2020) introduced the notion of inductive inference operators. An *inductive inference operator* (on $\mathcal{L}$) is a mapping $\mathbf{C}$ that assigns to each conditional belief base $\Delta \subseteq (\mathcal{L} \mid \mathcal{L})$ an inference relation $\mid\!\sim_\Delta$ on $\mathcal{L}$, i.e.,

$$\mathbf{C} : \Delta \mapsto \mid\!\sim_\Delta,$$

such that the following properties hold:

Direct Inference (DI) if $(B|A) \in \Delta$ then $A \mid\!\sim_\Delta B$, and

Trivial Vacuity (TV) $A \mid\!\sim_\emptyset B$ implies $A \models B$.

An *inductive inference operator for OCFs* $\mathbf{C}^{ocf}$ maps each belief base to an OCF over $\Omega(\Sigma)$; the inferene relation $\mid\!\sim_\Delta$ asssigned to a belief base Δ is then the inference relation $\mid\!\sim_\Delta = \mid\!\sim_{\kappa_\Delta}$ induced by κ_Δ according to (1).

3 Conditional Syntax Splitting

Syntax splittings describe that a belief base contains completely independent information about different parts of the signature. Let us first recall the notion of syntax splitting as introduced in (Kern-Isberner, Beierle, and Brewka 2020). A conditional belief base Δ *splits* into subbases Δ_1, Δ_2 if there are disjoint subsignatures $\Sigma_1, \Sigma_2 \subseteq \Sigma$ such that $\Delta = \Delta_1 \cup \Delta_2$, $\Delta_i \subset (\mathcal{L}_i|\mathcal{L}_i), \mathcal{L}_i = \mathcal{L}(\Sigma_i)$ for $i = 1, 2$, $\Sigma_1 \cap \Sigma_2 = \emptyset$, and $\Sigma_1 \cup \Sigma_2 = \Sigma$. This is denoted as

$$\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2.$$

Syntax splittings were generalized in (Heyninck et al. 2023) to *conditional* syntax splittings, which allow sub-bases to share the atoms in a given subsignature Σ_3.

Definition 1 ((Heyninck et al. 2023))**.** *We say a conditional belief base Δ can be* split into subbases Δ_1,Δ_2 conditional on a subsignature Σ_3*, if there are $\Sigma_1, \Sigma_2 \subseteq \Sigma$ such that $\Delta_i = \Delta \cap (\mathcal{L}(\Sigma_i \cup \Sigma_3) \mid \mathcal{L}(\Sigma_i \cup \Sigma_3))$ for $i = 1, 2$, the signatures Σ_1, Σ_2 and Σ_3 are pairwise disjoint, and $\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$. This is denoted as*

$$\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3. \tag{2}$$

However, conditional syntax splittings in general do not ensure complete independence of Δ_1 and Δ_2 (Heyninck et al. 2023). To fix this, *safe* conditional syntax splittings were introduced.

Definition 2 ((Heyninck et al. 2023))**.** *A conditional belief base $\Delta = \Delta^1 \bigcup_{\Sigma_1,\Sigma_2} \Delta^2 \mid \Sigma_3$ can be* safely split into subbases Δ_1, Δ_2 conditional on a subsignature Σ_3*, writing:*

$$\Delta = \Delta_1 \overset{\mathsf{s}}{\bigcup_{\Sigma_1,\Sigma_2}} \Delta_2 \mid \Sigma_3 \tag{3}$$

if the following safety property *holds:*

$$\textit{for every } \omega^i\omega^3 \in \Omega(\Sigma_i \cup \Sigma_3), \textit{ there is an } \omega^{i'} \in \Omega(\Sigma^{i'})$$
$$\textit{s.t. } \omega^i\omega^{i'}\omega^3 \not\models \bigvee_{(F|E)\in\Delta_{i'}} E \wedge \neg F \textit{ for } i, i' \in \{1, 2\}, i \neq i'. \tag{4}$$

Safe conditional syntax splittings guarantee (conditional) independence of conditionals in Δ_1 and Δ_2. In essence, the safety property ensures that any complete conjunction over Σ_3 may not require the falsification of a conditional in Δ_1 or Δ_2. For a more detailed explanation on why this is necessary, see (Heyninck et al. 2023).

Note that unlike syntax splitting, conditional syntax splitting does not require the subbases Δ_1 and Δ_2 to be disjoint. For the remainder of this paper, we will use the notation introduced in the following straightforward proposition.

Proposition 3. *If $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ then*

$$\Delta = \Delta_{1\setminus 3} \,\dot\cup\, \Delta_{2\setminus 3} \,\dot\cup\, \Delta_3 \tag{5}$$
$$\textit{and } \Delta_{1\setminus 3}, \Delta_{2\setminus 3}, \Delta_3 \textit{ pairwise disjoint with} \tag{6}$$
$$\Delta_3 = \Delta_1 \cap \Delta_2 \tag{7}$$
$$\Delta_{1\setminus 3} = \Delta_1 \setminus \Delta_3 \tag{8}$$
$$\Delta_{2\setminus 3} = \Delta_2 \setminus \Delta_3. \tag{9}$$

Note that $\Delta_3 \subseteq (\mathcal{L}(\Sigma_3)|\mathcal{L}(\Sigma_3))$, more precisely $\Delta_3 = \Delta \cap (\mathcal{L}(\Sigma_3)|\mathcal{L}(\Sigma_3))$, and $\Delta_{i\setminus 3} \subseteq (\mathcal{L}(\Sigma_i \cup \Sigma_3)|\mathcal{L}(\Sigma_i \cup \Sigma_3))$ for $i \in \{1, 2\}$. Furthermore, for $\omega^i \in \Omega(\Sigma_i)$ and $A \in \mathcal{L}_i$ we have that

$$\omega^1\omega^2\omega^3 \models A \quad \text{iff} \quad \omega^i\omega^3 \models A. \tag{10}$$

We illustrate the notion of safe conditional syntax splitting with an example.

Example 4 (Δ^b)**.** *Consider $\Sigma = \{b, p, f, w\}$ representing (b)irds, (p)enguins, (f)lying entities, and (w)inged entities. Let $\Delta^b = \{(f|b), (\overline{f}|p), (b|p), (w|b)\}$ be a belief base describing the well known penguin triangle together with the expression that birds usually have wings. Then*

$$\Delta^b = \{(f|b), (\overline{f}|p), (b|p)\} \overset{\mathsf{s}}{\bigcup_{\{p,f\},\{w\}}} \{(w|b)\} \mid \{b\}$$

is a conditional syntax splitting with $\Sigma_1 = \{p, f\}, \Sigma_2 = \{w\}$, and $\Sigma_3 = \{b\}$. According to Proposition 3 we have $\Delta^b_3 = \emptyset$, $\Delta^b_1 = \Delta^b_{1\setminus 3} = \{(f|b), (\overline{f}|p), (b|p)\}$ and $\Delta^b_2 = \Delta^b_{2\setminus 3} = \{(w|b)\}$.

We can extend any $\omega^1 \in \Omega(\Sigma_1 \cup \Sigma_3)$ by any $\omega' \in \Omega(\Sigma_2)$ with $\omega' \models w$ without falsifying a conditional in Δ^b_2. Similarly we can extend any $\omega^2 \in \Omega(\Sigma_2 \cup \Sigma_3)$ by any $\omega'' \in \Omega(\Sigma_1)$ with $\omega'' \models \overline{p}f$ without falsifying a conditional in Δ^b_1. Thus, the splitting is safe.

Syntax splittings coincide with conditional syntax splittings conditional on $\Sigma_3 = \emptyset$.

Proposition 5. *Let Δ be a consistent belief base. We have*

$$\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2 \quad \textit{iff} \quad \Delta = \Delta_1 \overset{\mathsf{s}}{\bigcup_{\Sigma_1,\Sigma_2}} \Delta_2 \mid \emptyset.$$

Using the notion of conditional syntax splittings, the postulates conditional independence **(CInd)** and conditional relevance **(CRel)** for inference from belief bases with conditional syntax splitting have been introduced. They are inspired by the postulates (Rel) and (Ind) introduced in (Kern-Isberner, Beierle, and Brewka 2020). They describe that inference over Δ_1 and Δ_2 should be independent if we have full information, i.e., a full conjunction, on the “conditional pivot” Σ_3.

(CRel) (Heyninck et al. 2023) An inductive inference operator $C : \Delta \mapsto \mid\!\sim_\Delta$ satisfies **(CRel)** if for $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$, for $i \in \{1,2\}$, $A, B \in \mathcal{L}_{\Sigma_i}$, and a complete conjunction $E \in \mathcal{L}_{\Sigma_3}$ we have that

$$AE \mid\!\sim_\Delta B \quad \text{iff} \quad AE \mid\!\sim_{\Delta_i} B.$$

Thus, an inductive inference operator satisfies conditional relevance, if, for any safe conditional syntax splitting, inference in the language of $\Sigma_i \cup \Sigma_3$ is dependent only on the conditionals in Δ_i, i.e., only those conditionals in that same language.

(CInd) (Heyninck et al. 2023) An inductive inference operator $C : \Delta \mapsto \mid\!\sim_\Delta$ satisfies **(CInd)** if for $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$, for $i, j \in \{1,2\}, i \neq j$, for any $A, B \in \mathcal{L}_{\Sigma_i}$, $D \in \mathcal{L}_{\Sigma_j}$, and a complete conjunction $E \in \mathcal{L}_{\Sigma_3}$ such that $DE \not\mid\!\sim_\Delta \bot$ we have

$$AE \mid\!\sim_\Delta B \quad \text{iff} \quad ADE \mid\!\sim_\Delta B.$$

The requirement that $DE \not\mid\!\sim_\Delta \bot$ was added here. Otherwise, (CInd) would require that $A \mid\!\sim_\Delta \bot$ for every formula $A \in \mathcal{L}_{\Sigma_1} \cup \mathcal{L}_{\Sigma_2}$. Conditional independence requires that, given complete knowledge of Σ_3, inferences in the language of $\Sigma_i \cup \Sigma_3$ are independent of any formula over the language of Σ_j.

The postulate (CSynSplit) is the combination of (CRel) and (CInd):

(CSynSplit) (Heyninck et al. 2023) An inductive inference operator satisfies **(CSynSplit)** if it satisfies (CRel) and (CInd).

Conditional syntax splitting is closely related to the notion of conditional κ-independence for OCFs.

Definition 6 ((Heyninck et al. 2023),(Spohn 2012))**.** *Let $\Sigma_1, \Sigma_2, \Sigma_3 \subseteq \Sigma$ where Σ_1, Σ_2 and Σ_3 are pairwise disjoint and let κ be an OCF. Σ_1, Σ_2 are* conditionally κ-independent given Σ_3*, in symbols $\Sigma_1 \perp\!\!\!\perp_\kappa \Sigma_2 | \Sigma_3$, if for all $\omega^1 \in \Omega(\Sigma_1), \omega^2 \in \Omega(\Sigma_2)$, and $\omega^3 \in \Omega(\Sigma_3)$, it holds that $\kappa(\omega^1|\omega^2\omega^3) = \kappa(\omega^1|\omega^3)$.*

The following lemma provides another useful characterization of conditional κ-independence.

Lemma 7. *Let $\Sigma_1, \Sigma_2, \Sigma_3 \subseteq \Sigma$ where Σ_1, Σ_2 and Σ_3 are pairwise disjoint and let κ be an OCF. Σ_1, Σ_2 are conditionally κ-independent given Σ_3 iff for all $A \in \mathcal{L}(\Sigma_1), B \in \mathcal{L}(\Sigma_2)$ and complete conjunction $C \in \mathcal{L}(\Sigma_3)$ it holds that*

$$\kappa(ABC) = \kappa(AC) + \kappa(BC) - \kappa(C).$$

Proof. Note that the equation $\kappa(\omega^1|\omega^2\omega^3) = \kappa(\omega^1|\omega^3)$ from Definition 6 is equivalent to

$$\kappa(\omega^1\omega^2\omega^3) = \kappa(\omega^1\omega^3) + \kappa(\omega^2\omega^3) - \kappa(\omega^3) \tag{11}$$

by applying the definition of ranks of conditionals (cf. Section 2). Note that while κ is built over Σ the ω^i are treated here as conjunctions over their respective signatures. We show both directions of the "iff" separately.

Direction $\Rightarrow$: Let Σ_1, Σ_2 be conditionally κ-independent given Σ_3. Then (11) holds for all $\omega^1 \in \Omega(\Sigma_1), \omega^2 \in \Omega(\Sigma_2)$, $\omega^3 \in \Omega(\Sigma_3)$. Now let $\omega^1\omega^3$ be the world with minimal rank in the models of AC. Assume the same for $\omega^2\omega^3$ and BC. Note that since C is a complete conjunction over Σ_3, ω^3 must be a world with minimal rank in the models of C. Thus we can rewrite (11) to

$$\kappa(\omega^1\omega^2\omega^3) = \kappa(AC) + \kappa(BC) - \kappa(C) \tag{12}$$

Clearly $\omega^1\omega^2\omega^3 \models ABC$. We now show that $\omega^1\omega^2\omega^3$ is also has minimal rank with this property. Towards a contradiction assume $\omega^1\omega^2\omega^3$ did not have minimal rank with this property. Then there is some ω' with $\omega' \models ABC$ and $\kappa(\omega') < \kappa(\omega^1\omega^2\omega^3)$. Since Σ_1, Σ_2 and Σ_3 are disjoint, ω' can be split into $\omega'^1 \in \Omega(\Sigma_1)$, $\omega'^2 \in \Omega(\Sigma_2)$, $\omega'^3 \in \Omega(\Sigma_3)$. Thus it must hold, that

$$\kappa(\omega'^1\omega'^2\omega'^3) = \kappa(\omega'^1\omega'^3) + \kappa(\omega'^2\omega'^3) - \kappa(\omega'^3) \tag{13}$$

Since $\kappa(\omega') < \kappa(\omega^1\omega^2\omega^3)$ it must hold that $\kappa(\omega'^1\omega'^3) < \kappa(\omega^1\omega^3)$ or $\kappa(\omega'^2\omega'^3) < \kappa(\omega^2\omega^3)$ or $\kappa(\omega'^3) > \kappa(\omega^3)$. The first inequality can not hold, as $\omega'^1\omega'^3 \models AC$ but as per our assumption $\omega^1\omega^3$ is minimal with this property. Analogously the second inequality cannot hold. The third inequality does not hold either as C is a full conjunction and therefore $\omega^3 = \omega'^3$. Thus $\kappa(\omega') < \kappa(\omega^1\omega^2\omega^3)$ can not hold and $\omega^1\omega^2\omega^3$ must have minimal rank with $\omega^1\omega^2\omega^3 \models ABC$. Then we can rewrite (12) to

$$\kappa(ABC) = \kappa(AC) + \kappa(BC) - \kappa(C) \tag{14}$$

completing the proof for this direction.

Direction $\Leftarrow$: For the other direction assume (14) holds for all $A \in \mathcal{L}(\Sigma_1), B \in \mathcal{L}(\Sigma_2)$ and complete conjunction $C \in \mathcal{L}(\Sigma_3)$. Let $\omega^1 \in \Omega(\Sigma_1), \omega^2 \in \Omega(\Sigma_2)$, and $\omega^3 \in \Omega(\Sigma_3)$. As we have stated previously all worlds can be represented by a full conjunction of all literals of their signature. Let A be such a conjunction for ω^1, B for ω^2 and C for ω^3. Then (14) is equivalent to (11) completing the proof. □

As for probabilities, conditional independence for OCFs expresses that information on Σ_2 is redundant for Σ_1 if full information on Σ_3 is available and used. We can now characterize (CInd) and (CRel) for inference operators for OCFs as follows:

Proposition 8 ((Heyninck et al. 2023))**.** *An inductive inference operator for OCFs $\mathbf{C}^{ocf} : \Delta \mapsto \kappa_\Delta$ satisfies (CInd) iff for any $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ we have $\Sigma_1 \perp\!\!\!\perp_{\kappa_\Delta} \Sigma_2 | \Sigma_3$.*

Proposition 9 ((Heyninck et al. 2023))**.** *An inductive inference operator for OCFs $\mathbf{C}^{ocf} : \Delta \mapsto \kappa_\Delta$ satisfies (CRel) iff for any $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$, we have $\kappa_{\Delta_i} = \kappa_\Delta |_{\Sigma_i \cup \Sigma_3}$ for $i \in \{1,2\}$.*

Proposition 8 lets us use conditional κ-independence to characterize (CInd), while Proposition 9 allows us to characterize (CRel) in terms of marginalization of OCFs, both of which are useful for showing that an inference operator for OCFs satisfies (CInd) or (CRel), respectively.

4 Conditional Semantic Splitting

Now we will define the new concept of conditional semantic splitting, generalizing the notion of semantic splitting (Beierle, Haldimann, and Kern-Isberner 2021) and show that c-representations (Kern-Isberner 2001; Kern-Isberner 2004)) satisfy conditional semantic splitting.

4.1 Model Combinations and Semantic Splittings

For the rest of this paper, we focus on OCF-based semantics and first introduce the notion of model combinations for ranking models.

Definition 10 (model combination)**.** *Let M_1, M_2 be sets of OCFs over Σ.* Model combinations *of M_1 and M_2, denoted by $M_1 \oplus M_2$ and by $M_1 \ominus M_2$, respectively, are given by*

$$M_1 \oplus M_2 = \{\kappa \,|\, \kappa(\omega) = \kappa_1(\omega) + \kappa_2(\omega), \kappa_1 \in M_1, \kappa_2 \in M_2\}$$
$$M_1 \ominus M_2 = \{\kappa \,|\, \kappa(\omega) = \kappa_1(\omega) - \kappa_2(\omega), \kappa_1 \in M_1, \kappa_2 \in M_2\}$$

Note that in general, $M_1 \oplus M_2$ or $M_1 \ominus M_2$ may contain functions that are not ranking functions because, e.g., no ω is mapped to 0. We consider different subclasses of ranking models for conditional belief bases in this paper, e.g., system Z ranking functions or c-representations. The following definition provides a joint formal concept for focusing on such subclasses.

Definition 11. *An (OCF based)* semantics *Sem for conditional belief bases is a function mapping a belief base Δ over Σ to a set of models $Mod_{\Sigma}^{Sem}(\Delta) \subseteq Mod_{\Sigma}(\Delta)$ where $Mod_{\Sigma}(\Delta) = \{\kappa \mid \kappa \models \Delta\}$.*

A conditional semantic splitting of Δ depends on the combination of models given by an OCF-based semantics *Sem* and generalizes the notion of semantic splitting (Beierle, Haldimann, and Kern-Isberner 2021). Semantic splittings and conditional semantic splittings apply the notion of splittings to the model level, yielding a desirable splitting property to evaluate OCF-based semantics.

Definition 12 (conditional semantic splitting)**.** $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ *is a* conditional semantic splitting *of Δ for a semantic Sem if*

$$Mod_{\Sigma}^{Sem}(\Delta) = Mod_{\Sigma}^{Sem}(\Delta_1) \oplus Mod_{\Sigma}^{Sem}(\Delta_2) \ominus Mod_{\Sigma}^{Sem}(\Delta_3).$$

This yields the base for the following postulate.

(CSemSplit) An OCF-based semantic *Sem* satisfies (**CSemSplit**) if every safe splitting $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ is also a conditional semantic splitting of Δ

Example 13. *System P is an axiom system stating desirable properties for nonmonotonic reasoning with conditionals (Adams 1975; Kraus, Lehmann, and Magidor 1990). It also characterizes a semantic that maps a belief base Δ to all its models, i.e., $Mod_{\Sigma}^{System\ P}(\Delta) = Mod_{\Sigma}(\Delta)$. This semantics does not satisfy (**CSemSplit**) which can be illustrated with $\Delta = \Delta_1 \cup \Delta_2$ and $\Delta_1 = \{(a|\top)\}$ and $\Delta_2 = \{(b|\top)\}$. Obviously, $\{\Delta_1, \Delta_2\}$ is a syntax splitting and thus a safe conditional syntax splitting of Δ (Proposition 5). Then*

$$\kappa_1 = \{ab \mapsto 1, a\bar{b} \mapsto 0, \bar{a}b \mapsto 1, \bar{a}\bar{b} \mapsto 1\} \text{ accepts } \Delta_1, \text{ and}$$
$$\kappa_2 = \{ab \mapsto 1, a\bar{b} \mapsto 1, \bar{a}b \mapsto 0, \bar{a}\bar{b} \mapsto 1\} \text{ accepts } \Delta_2,$$

but $\kappa_1 + \kappa_2 = \{ab \mapsto 2, a\bar{b} \mapsto 1, \bar{a}b \mapsto 1, \bar{a}\bar{b} \mapsto 2\}$ is not even a ranking function and would also not model Δ if it were normalized by reducing all ranks by 1.

Example 14. *System Z (Goldszmidt and Pearl 1996) is based on a notion of* tolerance *where a conditional $(B|A)$ is tolerated by a set of conditionals Δ if there is a world ω that verifies $(B|A)$ and falsifies no other conditional in Δ. The* ordered partition *$OP(\Delta) = (\Delta^0, \dots, \Delta^n)$ is defined by $\Delta^0 = \{\delta \in \Delta \mid \Delta$ tolerates $\delta\}$ and $OP(\Delta \setminus \Delta^0) = (\Delta^1 \dots \Delta^n)$. Let $Z_\Delta(\delta) = i$ iff $\delta \in \Delta^i$. The uniquely defined System Z ranking function κ_Δ^z is then defined via $\kappa_\Delta^z = \max\{Z_\Delta(\delta) \mid \omega$ falsifies $\delta, \delta \in \Delta\}$. Thus System Z yields a model semantics given by $Mod_{\Sigma}^{Sem}(\Delta) = \{\kappa_\Delta^z\}$.*

*This semantics also does not satisfy (**CSemSplit**). Consider Δ^b and the conditional syntax splitting in Example 4. We have $OP(\Delta^b) = \{\{(f|b), (w|b)\}, \{(b|p), (\overline{f}|p)\}\}$, $OP(\Delta_1^b) = \{\{(f|b)\}, \{(b|p), (\overline{f}|p)\}\}$ and $OP(\Delta_2^b) = \{\{(w|b)\}\}$. Then we get $\kappa_\Delta^z(\overline{p}b\overline{f}\overline{w}) = 1 \neq 2 = \kappa_{\Delta_1}^z(\overline{p}b\overline{f}) + \kappa_{\Delta_2}^z(b\overline{w}) - \kappa_{\Delta_3}^z(b)$. Thus (**CSemSplit**) is not satisfied.*

4.2 c-Representations Satisfy Conditional Semantic Splitting

Among the OCF models of Δ, c-representations are special ranking models obtained by assigning individual integer impacts to the conditionals in Δ and generating the world ranks as the sum of impacts of falsified conditionals.

Definition 15 (c-representation (Kern-Isberner 2001; Kern-Isberner 2004))**.** *A* c-representation *of $\Delta = \{(B_1|A_1), \dots, (B_n|A_n)\}$ is an OCF κ constructed from non-negative integer impacts $\eta_j \in \mathbb{N}_0$ assigned to each $(B_j|A_j)$ such that κ accepts Δ and is given by:*

$$\kappa(\omega) = \sum_{\substack{1 \leqslant j \leqslant n \\ \omega \models A_j \overline{B}_j}} \eta_j \tag{15}$$

c-Representations can conveniently be specified using a constraint satisfaction problem (for detailed explanations, see (Kern-Isberner 2001; Kern-Isberner 2004)):

Definition 16 ($CR(\Delta)$, (Kern-Isberner 2001; Beierle et al. 2018))**.** *Let $\Delta = \{(B_1|A_1), \dots, (B_n|A_n)\}$. The* constraint satisfaction problem for c-representations of Δ*, denoted by $CR(\Delta)$, is given by the conjunction of the constraints, for all $j \in \{1, \dots, n\}$:*

$$\eta_j \geqslant 0 \tag{16}$$
$$\eta_j > \min_{\omega \models A_j B_j} \sum_{\substack{k \neq j \\ \omega \models A_k \overline{B_k}}} \eta_k - \min_{\omega \models A_j \overline{B}_j} \sum_{\substack{k \neq j \\ \omega \models A_k \overline{B_k}}} \eta_k \tag{17}$$

Note that (16) expresses that falsification of conditionals should make worlds not more plausible, and (17) ensures that κ as specified by (15) accepts Δ. A solution of $CR(\Delta)$ is a vector $\vec{\eta} = (\eta_1, \dots, \eta_n)$ of natural numbers. $Sol(CR(\Delta))$ denotes the set of all solutions of $CR(\Delta)$. For $\vec{\eta} \in Sol(CR(\Delta))$ and κ as in Equation (15), κ is the *OCF induced by* $\vec{\eta}$ and is denoted by $\kappa_{\vec{\eta}}$. $CR(\Delta)$ is sound and complete (Kern-Isberner 2001; Beierle et al. 2018): For every $\vec{\eta} \in Sol(CR(\Delta))$, $\kappa_{\vec{\eta}}$ is a c-representation with $\kappa_{\vec{\eta}} \models \Delta$, and for every c-representation κ with $\kappa \models \Delta$, there is $\vec{\eta} \in Sol(CR(\Delta))$ such that $\kappa = \kappa_{\vec{\eta}}$. Thus c-representations yield an OCF-based model semantics

$$Mod_{\Sigma}^{c\text{-}rep}(\Delta) = \{\kappa_{\vec{\eta}} \mid \vec{\eta} \in Sol(CR(\Delta))\}.$$

ω	δ_1: $(f\|b)$	δ_2: $(\overline{f}\|p)$	δ_3: $(b\|p)$	δ_4: $(w\|b)$	impact on ω	$\kappa_{\vec{\eta}_1}(\omega)$	$\kappa_{\vec{\eta}_2}(\omega)$	$\kappa_{\vec{\eta}_3}(\omega)$
$b\,p\,f\,w$	v	f	v	v	η_2	2	4	5
$b\,p\,f\,\overline{w}$	v	f	v	f	$\eta_2+\eta_4$	3	7	12
$b\,p\,\overline{f}\,w$	f	v	v	v	η_1	1	3	4
$b\,p\,\overline{f}\,\overline{w}$	f	v	v	f	$\eta_1+\eta_4$	2	6	11
$b\,\overline{p}\,f\,w$	v	–	–	v	0	0	0	0
$b\,\overline{p}\,f\,\overline{w}$	v	–	–	f	η_4	1	3	7
$b\,\overline{p}\,\overline{f}\,w$	f	–	–	v	η_1	1	3	4
$b\,\overline{p}\,\overline{f}\,\overline{w}$	f	–	–	f	$\eta_1+\eta_4$	2	6	11
$\overline{b}\,p\,f\,w$	–	f	f	–	$\eta_2+\eta_3$	4	8	11
$\overline{b}\,p\,f\,\overline{w}$	–	f	f	–	$\eta_2+\eta_3$	4	8	11
$\overline{b}\,p\,\overline{f}\,w$	–	v	f	–	η_3	2	4	6
$\overline{b}\,p\,\overline{f}\,\overline{w}$	–	v	f	–	η_3	2	4	6
$\overline{b}\,\overline{p}\,f\,w$	–	–	–	–	0	0	0	0
$\overline{b}\,\overline{p}\,f\,\overline{w}$	–	–	–	–	0	0	0	0
$\overline{b}\,\overline{p}\,\overline{f}\,w$	–	–	–	–	0	0	0	0
$\overline{b}\,\overline{p}\,\overline{f}\,\overline{w}$	–	–	–	–	0	0	0	0
impacts:	η_1	η_2	η_3	η_4				
$\vec{\eta}_1$	1	2	2	1				
$\vec{\eta}_2$	3	4	4	3				
$\vec{\eta}_3$	4	5	6	7				

Table 1: Verification and falsification with induced impacts for Δ^b in Example 17.

For an impact vector $\vec{\eta}$, we will simply write $\vec{\eta}^1$ and $\vec{\eta}^2$ for the corresponding projections $\vec{\eta}|_{\Delta_1}$ and $\vec{\eta}|_{\Delta_2}$, and $(\vec{\eta}^1, \vec{\eta}^2)$ for their composition. Similarly we will write $(\vec{\eta}^1, \eta_j)$ for the composition of a vector with a singular natural number η_j.

Example 17 (Δ^b continued)**.** *$CR(\Delta^b)$ contains $\eta_i \geqslant 0$ for $i \in \{1,2,3,4\}$ and the following constraints:*

$$\eta_1 > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models bf}} \sum_{\substack{j\neq 1\\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models b\overline{f}}} \sum_{\substack{j\neq 1\\ \omega\models A_j\overline{B_j}}} \eta_j$$

$$\eta_2 > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models p\overline{f}}} \sum_{\substack{j\neq 2\\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models pf}} \sum_{\substack{j\neq 2\\ \omega\models A_j\overline{B_j}}} \eta_j$$

$$\eta_3 > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models pb}} \sum_{\substack{j\neq 3\\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models p\overline{b}}} \sum_{\substack{j\neq 3\\ \omega\models A_j\overline{B_j}}} \eta_j$$

$$\eta_4 > \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models bw}} \sum_{\substack{j\neq 4\\ \omega\models A_j\overline{B_j}}} \eta_j - \min_{\substack{\omega\in\Omega_\Sigma\\ \omega\models b\overline{w}}} \sum_{\substack{j\neq 4\\ \omega\models A_j\overline{B_j}}} \eta_j$$

Table 1 shows some solutions for Δ^b as well as their corresponding c-representations. For example $\vec{\eta}_1 = (1,2,2,1) \in Sol(CR(\Delta^b))$, $\vec{\eta}^1_1 = (1,2,2) \in Sol(CR(\Delta^b_{1\setminus 3}))$ and $\vec{\eta}^2_1 = (1) \in Sol(CR(\Delta^b_{2\setminus 3}))$.

A fundamental property of c-representations is that for any syntax splitting $\Delta = \Delta_1 \bigcup_{\Sigma_1,\Sigma_2} \Delta_2$ the composition of *any* impact vectors for the subbases yields an impact vector for Δ, and vice versa (Kern-Isberner, Beierle, and Brewka 2020). Before proving a generalization of this observation to conditional syntax splitting, we state two useful lemmas.

Lemma 18. *Let Δ be a conditional belief base, $\vec{\eta} \in Sol(CR(\Delta))$, and $(F|E)$ a conditional where $E \models F$. Then $(\vec{\eta}, \eta) \in Sol(CR(\Delta \cup \{(F|E)\}))$ where $\eta \in \mathbb{N}$, and in particular $\kappa_{\vec{\eta}}(\omega) = \kappa_{(\vec{\eta},\eta)}(\omega)$ for all $\omega \in \Omega$.*

Proof. Since $\omega \not\models E\overline{F}$ for all worlds ω, the impact η assigned to $(F|E)$ only has to satisfy $\eta \geqslant 0$, and it does not appear in the sum-expression (15) defining a c-representation. ☐

Lemma 19. *Let $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$. Then all conditionals in $\Delta_3 = \Delta_1 \cap \Delta_2$ are self-fulfilling.*

Proof. Let $i, i' \in \{1,2\}$ where $i \neq i'$ and $(B|A) \in \Delta_3, A, B \in \mathcal{L}(\Sigma_3)$. Towards a contradiction, assume there were some ω with $\omega \models A\overline{B}$. Then for $\omega^3 = \omega|_{\Sigma_3}$ it must also hold that $\omega^3 \models A\overline{B}$. Due to the safety property (4), ω^3 must have extensions $\omega^i \in \Omega(\Sigma_i)$ and $\omega^{i'} \in \Omega(\Sigma_{i'})$ such that no conditional in Δ_i respectively $\Delta_{i'}$ is falsified. Since $(B|A) \in \Delta_3$ and thus $(B|A) \in \Delta_1$ and $(B|A) \in \Delta_2$, we get $\omega^3 \not\models A\overline{B}$, contradicting our assumption. ☐

Note that in our example base Δ^b, Δ_3 is empty while Σ_3 is not. The crucial (conditional) link between Δ_1 and Δ_2 is given semantically by Σ_3.

The following proposition provides the key for showing that c-representations satisfy conditional semantic splitting.

Proposition 20. *For any $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$, where $\Delta_3 = \Delta_1 \cap \Delta_2$, we have $Sol(CR(\Delta)) = \{(\vec{\eta}^1, \vec{\eta}^2, \vec{\eta}^3) \mid \vec{\eta}^i \in Sol(CR(\Delta_{i\setminus 3})), i \in \{1,2\}; \vec{\eta}^3 \in \mathbb{N}^{|\Delta_3|}\}$, i.e.:*

$$Sol(CR(\Delta)) = Sol(CR(\Delta_{1\setminus 3})) \times Sol(CR(\Delta_{2\setminus 3})) \times \mathbb{N}^{|\Delta_3|}$$

Proof. We consider two cases. First we will consider the case that $\Delta_3 = \emptyset$. Then we have that $\Delta = \Delta_{1\setminus 3} \cup \Delta_{2\setminus 3}$.

Let $i, i' \in \{1,2\}$ and $i \neq i'$. Let $\Delta_{1\setminus 3} = \{(B_1|A_1), \ldots, (B_{n_1}|A_{n_2})\}$, $\Delta_{2\setminus 3} = \{(B_{n_1+1}|A_{n_1+1}), \ldots, (B_{n_1+n_2}|A_{n_1+n_2})\}$, $\Delta = \{(B_1|A_1), \ldots, (B_n|A_n)\}$, thus $n = n_1 + n_2$. Then, for $(B_j|A_j) \in \Delta$ we have $(B_j|A_j) \in \Delta_{i\setminus 3}$ iff $(B_j|A_j) \notin \Delta_{i'\setminus 3}$. We start with the following assumption:

(S1) $\vec{\eta}^1 \in Sol(CR(\Delta_{1\setminus 3}))$, $\vec{\eta}^2 \in Sol(CR(\Delta_{2\setminus 3}))$

Let us denote the constraint variables in $CR(\Delta_{1\setminus 3})$ with $\eta^1_1, \ldots, \eta^1_{n_1}$ and in $CR(\Delta_{2\setminus 3})$ with $\eta^2_{n_1+1}, \ldots, \eta^2_n$. Hence we can write the constraints (17) in $CR(\Delta_{i\setminus 3})$ as:

$$\eta^i_j > \underbrace{\min_{\substack{\omega\models\\ A_jB_j}} \sum_{\substack{k\neq j\\ \omega\models A_k\overline{B}_k\\ (B_k|A_k)\in\Delta_{i\setminus 3}}} \eta^i_k}_{V_{min}(j,i)} - \underbrace{\min_{\substack{\omega\models\\ A_j\overline{B}_j}} \sum_{\substack{k\neq j\\ \omega\models A_k\overline{B}_k\\ (B_k|A_k)\in\Delta_{i\setminus 3}}} \eta^i_k}_{F_{min}(j,i)} \quad (18)$$

Due to the safety property (4), $CR(\Delta_{1\setminus 3})$ does not mention any constraint variable from $CR(\Delta_{2\setminus 3})$ and vice versa, thus (S1) is equivalent to:

(S2) $(\vec{\eta}^1, \vec{\eta}^2) \in Sol(\Gamma 1), \quad \Gamma 1 = CR(\Delta_{1\setminus 3}) \cup CR(\Delta_{2\setminus 3})$

For $(B_j|A_j) \in \Delta_{i\setminus 3}$, let $V_{min}(j,i')$ and $F_{min}(j,i')$ be:

$$V_{min}(j,i') = \min_{\substack{\omega\models\\ A_jB_j}} \sum_{\substack{k\neq j\\ \omega\models A_k\overline{B}_k\\ (B_k|A_k)\in\Delta_{i'\setminus 3}}} \eta_k^{i'} \tag{19}$$

$$F_{min}(j,i') = \min_{\substack{\omega\models\\ A_j\overline{B}_j}} \sum_{\substack{k\neq j\\ \omega\models A_k\overline{B}_k\\ (B_k|A_k)\in\Delta_{i'\setminus 3}}} \eta_k^{i'} \tag{20}$$

Note that both $V_{min}(j,i')$ and $F_{min}(j,i')$ only involve impacts corresponding to falsified conditionals from the subbase the conditional $(B_j|A_j)$ does *not* belong to.

Due to the safety property (4), any world $\omega \in \Omega$ that minimizes the sum in $V_{min}(j,i')$, falsifies no conditionals in $\Delta_{i'\setminus 3}$. Analogously this holds for $F_{min}(j,i')$ and therefore we have $V_{min}(j,i') = F_{min}(j,i') = 0$.

Thus, adding $V_{min}(j,i') - F_{min}(j,i')$ to the right-hand side of (18) yields the following constraint having the same set of solutions as (18):

$$\begin{aligned}\eta_j^i > & \underbrace{\min_{\substack{\omega\models\\ A_jB_j}} \sum_{\substack{k\neq j\\ \omega\models A_k\overline{B}_k\\ (B_k|A_k)\in\Delta_{i\setminus 3}}} \eta_k^i}_{V_{min}(j,i)} + \underbrace{\min_{\substack{\omega\models\\ A_jB_j}} \sum_{\substack{k\neq j\\ \omega\models A_k\overline{B}_k\\ (B_k|A_k)\in\Delta_{i'\setminus 3}}} \eta_k^{i'}}_{V_{min}(j,i')} \\ & - \underbrace{\min_{\substack{\omega\models\\ A_j\overline{B}_j}} \sum_{\substack{k\neq j\\ \omega\models A_k\overline{B}_k\\ (B_k|A_k)\in\Delta_{i\setminus 3}}} \eta_k^i}_{F_{min}(j,i)} - \underbrace{\min_{\substack{\omega\models\\ A_j\overline{B}_j}} \sum_{\substack{k\neq j\\ \omega\models A_k\overline{B}_k\\ (B_k|A_k)\in\Delta_{i'\setminus 3}}} \eta_k^{i'}}_{F_{min}(j,i')}\end{aligned} \tag{21}$$

Because the minimizations in $V_{min}(j,i)$ and $V_{min}(j,i')$ (in $F_{min}(j,i)$ and $F_{min}(j,i')$, respectively) are independent from each other, the V_{min}-minimizations and the F_{min}-minimizations in (21) can be combined without changing the set of solutions. Together with Lemma 19 this yields the constraint:

$$\eta_j^i > \min_{\substack{\omega\models\\ A_jB_j}} \sum_{\substack{k\neq j\\ \omega\models A_k\overline{B}_k\\ (B_k|A_k)\in\Delta}} \eta_k^i - \min_{\substack{\omega\models\\ A_j\overline{B}_j}} \sum_{\substack{k\neq j\\ \omega\models A_k\overline{B}_k\\ (B_k|A_k)\in\Delta}} \eta_k^i \tag{22}$$

Because the constraints (18), (21), and (22) all have the same set of solutions, (S2) and thus also (S1) is equivalent to:

(S3) $(\vec{\eta}^1, \vec{\eta}^2) \in Sol(\Gamma 2)$, where $\Gamma 2$ is obtained from $\Gamma 1$ by replacing each constraint (18) by (22).

Using $\eta_1^1, \ldots, \eta_{n_1}^1, \ldots, \eta_{n_1+1}^2, \ldots, \eta_n^2$ as constraint variables for expressing $CR(\Delta)$, we observe that $\Gamma 2 = CR(\Delta)$.

Next we consider the case that $\Delta_3 \neq \emptyset$. Due to Lemmata 19 and 18 the impact η assigned to $(F|E)$ in $Sol(CR(\Delta_3))$ has no influence on $Sol(CR(\Delta_{1\setminus 3}))$ or on $Sol(CR(\Delta_{2\setminus 3}))$, and furthermore $Sol(CR(\Delta_3)) = \mathbb{N}^{|\Delta_3|}$ completing the proof. ☐

Proposition 20 shows that, just like for syntax splittings, for safe conditional syntax splittings, the impact vectors for the subbases can be calculated independently, yielding an impact vector for Δ through composition. We illustrate this with an example.

Example 21 (Δ^b continued)**.** *Recall the safe conditional syntax splitting from Example 4. According to Proposition 20, in order to obtain a solution for $CR(\Delta^b)$ it suffices to calculate a solution for $CR(\Delta_1^b)$ and $CR(\Delta_2^b)$ separately, where $CR(\Delta_1^b)$ consists of the first three constraints from Example 17 and $CR(\Delta_2^b)$ consists of the fourth one. Thus we have that, e.g., $\vec{\eta}_1 = (1,2,2,1) \in Sol(CR(\Delta^b))$ can be obtained by composing $\vec{\eta}_1^1 = (1,2,2) \in Sol(CR(\Delta_{1\setminus 3}^b))$ and $\vec{\eta}_1^2 = (1) \in Sol(CR(\Delta_{2\setminus 3}^b))$. I.e., for the projections of $\vec{\eta}_1$ we have $\vec{\eta}_1|\Delta_{1\setminus 3} = \vec{\eta}_1^1$, and $\vec{\eta}_1|\Delta_{2\setminus 3} = \vec{\eta}_1^2$. We can also compose $\vec{\eta}_2^1 = (3,4,4) \in Sol(CR(\Delta_{2\setminus 3}^b))$ and $\vec{\eta}_3^2 = (7) \in Sol(CR(\Delta_{2\setminus 3}^b))$ to obtain $\vec{\eta}_4 = (3,4,4,7) \in Sol(CR(\Delta^b))$.*

With Proposition 20 we can now show that c-representations satisfy conditional semantic splitting.

Proposition 22. *c-Representations satisfy (***CSemSplit***).*

Proof. Let κ be a c-representation for a belief base Δ and let $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$, where $\Delta_3 = \Delta_1 \cap \Delta_2$. We have to show that this is also a conditional semantic splitting, i.e. that

$$Mod_{\Sigma}^{Sem}(\Delta) = Mod_{\Sigma}^{c\text{-}rep}(\Delta_1) \oplus Mod_{\Sigma}^{c\text{-}rep}(\Delta_2) \ominus Mod_{\Sigma}^{c\text{-}rep}(\Delta_3) \tag{23}$$

holds. With Proposition 20 we have that every $\vec{\eta} \in Sol(CR(\Delta))$ can be split into $(\vec{\eta}^1, \vec{\eta}^2, \vec{\eta}^3)$ such that $\vec{\eta}^1 \in Sol(CR(\Delta_{1\setminus 3}))$, $\vec{\eta}^2 \in Sol(CR(\Delta_{2\setminus 3}))$, and $\vec{\eta}^3 \in Sol(CR(\Delta_3))$. Vice versa, for every $\vec{\eta}^1 \in Sol(CR(\Delta_{1\setminus 3}))$, $\vec{\eta}^2 \in Sol(CR(\Delta_{2\setminus 3}))$, and $\vec{\eta}^3 \in Sol(CR(\Delta_3))$ we have $(\vec{\eta}^1, \vec{\eta}^2, \vec{\eta}^3) \in Sol(CR(\Delta))$. Therefore we have that

$$\begin{aligned} Mod_{\Sigma}^{c\text{-}rep}(\Delta) = Mod_{\Sigma}^{c\text{-}rep}(\Delta_{1\setminus 3}) \oplus Mod_{\Sigma}^{c\text{-}rep}(\Delta_{2\setminus 3}) \\ \oplus Mod_{\Sigma}^{c\text{-}rep}(\Delta_3). \end{aligned} \tag{24}$$

With Lemmata 19 and 18 we have that $Mod_{\Sigma}^{c\text{-}rep}(\Delta_1) = Mod_{\Sigma}^{c\text{-}rep}(\Delta_{1\setminus 3})$, $Mod_{\Sigma}^{c\text{-}rep}(\Delta_2) = Mod_{\Sigma}^{c\text{-}rep}(\Delta_{2\setminus 3})$ and $Mod_{\Sigma}^{c\text{-}rep}(\Delta_3) = \{\kappa_u\}$, where κ_u is the uniform OCF mapping every world ω to 0 (cf. Section 2). Thus (24) is equivalent to (23) finishing the proof. ☐

5 Conditional Syntax Splitting and Inference w.r.t. Single c-Representations

We will now define model-based inductive inference operators assigning a c-representation κ to each Δ. Since every c-representation κ with $\kappa \models \Delta$ yields an inference relation expanding the beliefs in Δ, we employ a selection function for modelling the different possible choices of which c-representation should be selected.

Definition 23 (selection strategy σ, (Kern-Isberner, Beierle, and Brewka 2020))**.** *A selection strategy (for c-representations) is a function σ*

$$\sigma : \Delta \mapsto \vec{\eta}$$

assigning to each conditional belief base Δ an impact vector $\vec{\eta} \in Sol(CR(\Delta))$.

Definition 24 (inductive inference operator $\mathbf{C}^{c\text{-}rep}_{\sigma}$, (Kern-Isberner, Beierle, and Brewka 2020))**.** *An* inductive inference operator for c-representations with selection strategy σ *is a function*

$$\mathbf{C}^{c\text{-}rep}_{\sigma} : \Delta \mapsto \kappa_{\sigma(\Delta)}$$

where σ is a selection strategy for c -representations and, as before, $\mid\!\sim_{\kappa_{\sigma(\Delta)}}$ is obtained via Equation (1) from $\kappa_{\sigma(\Delta)}$.

Note that $\mathbf{C}^{c\text{-}rep}_{\sigma}$ is an inductive inference operator because each $\mid\!\sim_{\kappa_{\sigma(\Delta)}}$ satisfies both (Direct Inference) and (Trivial Vacuity). A recent example for a specific selection strategy are *minimal core c-representations* (Wilhelm, Kern-Isberner, and Beierle 2024).

In principle, for every Δ, a selection strategy may choose some impact vector independently from the choices for all other belief bases. The following property characterizes selection strategies that preserve the impacts chosen for subbases if Δ splits into these subbases.

(IP-CSP) A selection strategy σ is *impact preserving w.r.t. conditional belief base splitting* if, for $i \in \{1, 2\}$, we have $\sigma(\Delta_i) = \sigma(\Delta)|_{\Delta_i}$ for every safe conditional belief base splitting $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$.

We illustrate selection strategies with an example.

Example 25 (Δ^b continued)**.** *Recall Example 17. Let σ be a selection strategy satisfying (**IP-CSP**) with $\sigma(\Delta^b) = (1, 2, 2, 1)$. Then $\sigma(\Delta^b_{1\setminus 3}) = (1, 2, 2)$ and $\sigma(\Delta^b_{2\setminus 3}) = (1)$.*

In (Beierle and Kern-Isberner 2021) an algorithm is introduced for generating selection strategies satisfying an impact preserving postulate for (unconditional) syntax splittings, providing a basis for an algorithm for generating selection strategies satisfying (**IP-CSP**).

The following proposition relates c-representations to conditional syntax splitting via Proposition 8.

Proposition 26. *Let $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$, and κ a c-representation with $\kappa \models \Delta$. Then $\Sigma_1 \perp\!\!\!\perp_{\kappa} \Sigma_2 | \Sigma_3$.*

Proof. Let $\omega = \omega^1\omega^2\omega^3$ and let $\vec{\eta} \in Sol(\Delta)$ such that $\kappa = \kappa_{\vec{\eta}}$. Recall the definition of c-representations (15). We can rewrite (15) to

$$\kappa_{\vec{\eta}}(\omega) = \sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{1\setminus 3}}} \eta_j + \sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{2\setminus 3}}} \eta_j + \sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{3}}} \eta_j \tag{25}$$

By simply adding and subtracting the last sum of (25) we obtain the following equation.

$$\begin{aligned}\kappa_{\vec{\eta}}(\omega) = &\sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{1\setminus 3}}} \eta_j + \sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{2\setminus 3}}} \eta_j + \sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{3}}} \eta_j \\ &+ \sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{3}}} \eta_j - \sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{3}}} \eta_j\end{aligned} \tag{26}$$

Then we can combine the sums for $\Delta_{1\setminus 3}$ and $\Delta_{2\setminus 3}$ with the sum for Δ_3 to obtain sums for Δ_1 and Δ_2 respectively.

$$\kappa_{\vec{\eta}}(\omega) = \sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{1}}} \eta_j + \sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{2}}} \eta_j - \sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{3}}} \eta_j \tag{27}$$

Since Δ_1 is in $\mathcal{L}(\Sigma_1 \cup \Sigma_3)$, Δ_2 is in $\mathcal{L}(\Sigma_2 \cup \Sigma_3)$ and Δ_3 is in $\mathcal{L}(\Sigma_3)$ we can use (10) to simplify (27).

$$\kappa_{\vec{\eta}}(\omega) = \sum_{\substack{\omega^1\omega^3 \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{1}}} \eta_j + \sum_{\substack{\omega^2\omega^3 \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{2}}} \eta_j - \sum_{\substack{\omega^3 \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{3}}} \eta_j \tag{28}$$

Thus, we obtain with Proposition 20 and Lemmata 19 and 18

$$\kappa_{\vec{\eta}}(\omega^1\omega^2\omega^3) = \kappa_{\vec{\eta}^1}(\omega^1\omega^3) + \kappa_{\vec{\eta}^2}(\omega^2\omega^3) - \kappa_{\vec{\eta}^3}(\omega^3) \tag{29}$$

which, together with Proposition 20, is equivalent to

$$\kappa_{\vec{\eta}}(\omega^1\omega^2\omega^3) = \kappa_{\vec{\eta}}(\omega^1\omega^3) + \kappa_{\vec{\eta}}(\omega^2\omega^3) - \kappa_{\vec{\eta}}(\omega^3) \tag{30}$$

which is equivalent to $\kappa_{\vec{\eta}}(\omega^1|\omega^2\omega^3) = \kappa_{\vec{\eta}}(\omega^1|\omega^3)$, completing the proof. $\square$

Now we show that any inductive inference operator $\mathbf{C}^{c\text{-}rep}_{\sigma}$ based on an impact preserving selection strategy σ satisfies the property of conditional syntax splitting.

Proposition 27. *Let σ be a selection strategy that satisfies (**IP-CSP**). Then $\mathbf{C}^{c\text{-}rep}_{\sigma}$ satisfies (**CSynSplit**).*

Proof. Let $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$ and σ a selection strategy satisfying (**IP-CSP**). Let $i, i' \in \{1, 2\}$ with $i \neq i'$. Let η^i_j be the impact of $(B_j|A_j) \in \Delta_i$.

We show (**CInd**) first. Due to Proposition 26 we know that $\Sigma_1 \perp\!\!\!\perp_{\kappa_\Delta} \Sigma_2 | \Sigma_3$ holds. Thus, with Proposition 8, $\mathbf{C}^{c\text{-}rep}_{\sigma}$ satisfies (**CInd**). Note that it is not necessary for σ to satisfy (**IP-CSP**) in this step.

Next we show that $\mathbf{C}^{c\text{-}rep}_{\sigma}$ satisfies (**CRel**). Let $\omega^i\omega^3 \in \Omega(\Sigma_i \cup \Sigma_3)$. Note that here both κ_{Δ_i} and $\kappa|_{\Sigma_i\cup\Sigma_3}$ are defined on worlds in $\Omega(\Sigma_i \cup \Sigma_3)$. According to the marginalization of ranking functions (cf. Section 2) we have

$$\kappa|_{\Sigma_i\cup\Sigma_3}(\omega^i\omega^3) = \kappa(\omega^i\omega^3) = \min\{\sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta}} \eta_j \mid \omega \models \omega^i\omega^3\} \tag{31}$$

Due to the safety property (4), there is an extension $\omega^{i'}$ of $\omega^i\omega^3$ such that $\omega^i\omega^3\omega^{i'}$ falsifies no conditional in $\Delta_{i'}$. Therefore we can simplify (31) by only considering Δ_i as follows:

$$\kappa|_{\Sigma_i\cup\Sigma_3}(\omega^i\omega^3) = \sum_{\substack{\omega^i\omega^3 \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{i}}} \eta_j \tag{32}$$

Note that we no longer need to consider a minimum over worlds, since $\Delta_i \subseteq (\mathcal{L}(\Sigma_i \cup \Sigma_3)|\mathcal{L}(\Sigma_i \cup \Sigma_3))$ and $\omega^i\omega^3 \in \Omega(\Sigma_i \cup \Sigma_3)$ is a full conjunction, thus any minimal world that is a model of $\omega^i\omega^3$ falsifies the same conditionals in Δ_i

as $\omega^i\omega^3$. Because σ satisfies (**IP-CSP**) we have $\eta^i_j = \eta_j|_{\Delta_j}$ and thus (32) is equivalent to

$$\kappa|_{\Sigma_i \cup \Sigma_3}(\omega^i\omega^3) = \sum_{\substack{\omega^i\omega^3 \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_i}} \eta^i_j \tag{33}$$

which is the definition of $\kappa_{\Delta_i}(\omega^i\omega^3)$. This holds for all $\omega^i\omega^3$ and accordingly $\kappa_{\Delta_i} = \kappa|_{\Sigma_i\cup\Sigma_3}$ which, together with Proposition 9, implies (**CRel**). □

6 c-Inference Satisfies Conditional Syntax Splitting

c-Inference was introduced in (Beierle, Eichhorn, and Kern-Isberner 2016; Beierle et al. 2018) as the skeptical inference relation obtained by taking all c-representations of a belief base Δ into account.

Definition 28 (c-inference, $\mathrel{|\!\!\sim}^{c\text{-}sk}_{\boldsymbol{\Delta}}$, (Beierle, Eichhorn, and Kern-Isberner 2016))**.** *Let Δ be a belief base and let A, B be formulas. B is* a (skeptical) c-inference from A in the context of Δ*, denoted by $A \mathrel{|\!\!\sim}^{c\text{-}sk}_{\Delta} B$, iff $A \mathrel{|\!\!\sim}_{\kappa} B$ holds for all c-representations κ of Δ, yielding the inductive inference operator*

$$\mathbf{C}^{c\text{-}sk} : \Delta \mapsto \mathrel{|\!\!\sim}^{c\text{-}sk}_{\Delta}$$

Before proving that c-inference satisfies conditional syntax splitting, we show a proposition, stating the following observations. Consider a safe conditional syntax splitting of Δ into Δ_1 and Δ_2, and a c-representation $\kappa_{\vec{\eta}}$ determined by a solution vector $\vec{\eta} \in Sol(CR(\Delta))$ together with its projections $\kappa_{\vec{\eta}^1}$ and $\kappa_{\vec{\eta}^2}$ to Δ_1 and Δ_2, respectively. Then the rank of any formula F_i over the language $\mathcal{L}(\Sigma_i \cup \Sigma_3)$ of Δ_i under the projection $\kappa_{\vec{\eta}^i}$ coincides with the rank of the formula rank determined by $\kappa_{\vec{\eta}}$, while its rank under the other projection $\kappa_{\vec{\eta}^{i'}}$ evaluates to zero.

Proposition 29. *For any $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$, for all $\vec{\eta} \in Sol(CR(\Delta))$, $F_i \in \mathcal{L}(\Sigma_i \cup \Sigma_3)$, $i \in \{1, 2\}$, we have $\kappa_{\vec{\eta}^1}(F_2) = \kappa_{\vec{\eta}^2}(F_1) = 0$ and $\kappa_{\vec{\eta}}(F_i) = \kappa_{\vec{\eta}^i}(F_i)$.*

Proof. Let $i, i' \in \{1, 2\}, i \neq i'$. We show $\kappa_{\vec{\eta}^i}(F_{i'}) = \kappa_{\vec{\eta}^{i'}}(F_i) = 0$ first. Consider some world $\omega^i\omega^3 \in \Omega(\Sigma_i\cup\Sigma_3)$ with $\omega^i\omega^3 \models F_i$. Then due to the safety property (4) there is some $\omega^{i'} \in \Omega(\Sigma_{i'})$ such that $\omega^i\omega^3\omega^{i'}$ does not falsify any conditional in Δ_i. Then we have $\kappa_{\vec{\eta}^{i'}}(\omega^i\omega^3\omega^{i'}) = 0$ and thus $\kappa_{\vec{\eta}^{i'}}(F_i) = 0$.

Next we show $\kappa_{\vec{\eta}}(F_i) = \kappa_{\vec{\eta}^i}(F_i)$. We have

$$\kappa_{\vec{\eta}}(F_i) = \min\{ \sum_{\substack{\omega \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta}} \eta_j \mid \omega \models F_i\} \tag{34}$$

Let $\omega^i\omega^3 = \omega|_{\Sigma_1\cup\Sigma_3}$ then $\omega^i\omega^3 \models F_i$. Furthermore ω and $\omega^i\omega^3$ falsify the same conditionals in $\Delta_{i\setminus 3}$, since $\Delta_{i\setminus 3} \in (\mathcal{L}(\Sigma_i \cup \Sigma_3)|\mathcal{L}(\Sigma_i \cup \Sigma_3))$. Due to the safety property (4) there is some extension $\omega^{i'} \in \Omega(\Sigma_{i'})$ such that $\omega^i\omega^3\omega^{i'}$ does not falsify any conditional in $\Delta_{i'\setminus 3}$. Clearly $\omega^i\omega^3\omega^{i'}$ is a minimal world in the sense of (34) if $\omega^i\omega^3$ is a minimal world in the sense of (34). Since $\omega^i\omega^3\omega^{i'}$ does not falsify any conditional in $\Delta_{i'}$ we can omit $\Delta_{i'}$ from (34) in the following way:

$$\kappa_{\vec{\eta}}(F_i) = \min\{ \sum_{\substack{\omega^i\omega^3 \models A_j\overline{B}_j \\ (B_j|A_j)\in\Delta_{i\setminus 3}}} \eta^i_j \mid \omega^i\omega^3 \models F_i\} \tag{35}$$

Thus we have $\kappa_{\vec{\eta}}(F_i) = \kappa_{\vec{\eta}^i}(F_i)$. □

The next proposition shows that skeptical c-inference satisfies conditional syntax splitting. Note that since in general, the inference relation $\mathrel{|\!\!\sim}^{c\text{-}sk}_{\Delta}$ can not be represented by an OCF, no corresponding characterization of syntax splitting is applicable to it. Thus, the techniques used in the proofs of the propositions here are different from those used in previous proofs.

Proposition 30. $\mathbf{C}^{c\text{-}sk}$ *satisfies* **(CSynSplit)***).*

Proof. Let $\Delta = \Delta_1 \bigcup^{\mathsf{s}}_{\Sigma_1,\Sigma_2} \Delta_2 \mid \Sigma_3$. W.l.o.g. assume $A, B \in \mathcal{L}(\Sigma_1), C \in \mathcal{L}(\Sigma_2), \dot{B} \in \{B, \overline{B}\}$ and assume $D \in \mathcal{L}(\Sigma_3)$ is a complete conjunction with $CD \not\equiv \bot$.

To prove that $\mathbf{C}^{c\text{-}sk}$ satisfies **(CRel)** we need to show that $AD \mathrel{|\!\!\sim}^{c\text{-}sk}_{\Delta} B$ iff $AD \mathrel{|\!\!\sim}^{c\text{-}sk}_{\Delta_1} B$. By applying the definition of $\mathrel{|\!\!\sim}^{c\text{-}sk}_{\Delta}$ we obtain:

$$\forall \vec{\eta} \in Sol(CR(\Delta)) : \kappa_{\vec{\eta}}(ADB) < \kappa_{\vec{\eta}}(AD\overline{B})$$
$$\text{iff } \forall \vec{\eta}^1 \in Sol(CR(\Delta_{1\setminus 3})) : \kappa_{\vec{\eta}^1}(ADB) < \kappa_{\vec{\eta}^1}(AD\overline{B})$$

With Proposition 20 we have that every $\vec{\eta} \in Sol(CR(\Delta))$ can be split into $(\vec{\eta}^1, \vec{\eta}^2, \vec{\eta}^3) \in Sol(CR(\Delta))$, and vice versa every $\vec{\eta}^i \in Sol(CR(\Delta_{i\setminus 3}))$ has an extension $\vec{\eta}^{i'}$ such that $(\vec{\eta}^i, \vec{\eta}^{i'}, \vec{\eta}^3) \in Sol(CR(\Delta))$ for $i, i' \in \{1, 2\}, i \neq i'$. Therefore it suffices to show that

$$\kappa_{\vec{\eta}}(ADB) < \kappa_{\vec{\eta}}(AD\overline{B}) \text{ iff } \kappa_{\vec{\eta}^1}(ADB) < \kappa_{\vec{\eta}^1}(AD\overline{B}) \tag{36}$$

for all $\vec{\eta} = (\vec{\eta}^1, \vec{\eta}^2, \vec{\eta}^3) \in Sol(CR(\Delta))$. With Proposition 29 this follows directly because $\kappa_{\vec{\eta}}(AD\dot{B}) = \kappa_{\vec{\eta}^1}(AD\dot{B})$ since $A, B \in \mathcal{L}(\Sigma_1)$, $D \in \mathcal{L}(\Sigma_3)$ and $\Delta_1 \subseteq (\mathcal{L}(\Sigma_1 \cup \Sigma_3)|\mathcal{L}(\Sigma_1 \cup \Sigma_3))$.

Next we prove that $\mathbf{C}^{c\text{-}sk}$ satisfies **(CInd)**. We need to show $AD \mathrel{|\!\!\sim}^{c\text{-}sk}_{\Delta} B$ iff $ACD \mathrel{|\!\!\sim}^{c\text{-}sk}_{\Delta} B$. Again, due to to Proposition 20 it suffices to show that

$$\kappa_{\vec{\eta}}(ADB) < \kappa_{\vec{\eta}}(AD\overline{B}) \text{ iff } \kappa_{\vec{\eta}}(ACDB) < \kappa_{\vec{\eta}}(ACD\overline{B}) \tag{37}$$

for all $\vec{\eta} \in Sol(CR(\Delta))$. Since Proposition 26 states that Σ_1 and Σ_2 are conditionally $\kappa_{\vec{\eta}}$-independent given Σ_3 we have with Lemma 7 that $\kappa_{\vec{\eta}}(A\dot{B}CD) = \kappa_{\vec{\eta}}(A\dot{B}D) + \kappa_{\vec{\eta}}(CD) - \kappa_{\vec{\eta}}(D)$ and therefore (37) holds. □

Thus c-inference fully complies with (**CSynSplit**). Note that $\mathbf{C}^{c\text{-}sk}$ does not make use of selection strategies any more. We give an Example illustrating Propositions 29 and 30.

Example 31 (Δ^b continued)**.** *Recall Example 17. According to Proposition 29, for* $p \wedge f \in \mathcal{L}(\Sigma_1)$*, we get* $\kappa_{\vec{\eta}_1^1}(p \wedge \overline{f}) = 1$ *and* $\kappa_{\vec{\eta}_1^2}(p \wedge \overline{f}) = 0$ *from* $\kappa_{\vec{\eta}_1}(p \wedge \overline{f}) = 1$ *without having to compute* $\kappa_{\vec{\eta}_1^1}$ *or* $\kappa_{\vec{\eta}_1^2}$*. This also works in the other direction, where we do not have to compute* $\kappa_{\vec{\eta}_1}$ *if we have knowledge about* $\kappa_{\vec{\eta}_1^1}$*.*

Next consider again the safe conditional syntax splitting given in Example 4. Taking the constraints in Example 17 into account, it follows that $pb \mathrel{|\!\!\sim}^{c\text{-}sk}_{\Delta_1} \overline{f}$ *holds. With Proposition 30 we know that* $\mathbf{C}^{c\text{-}sk}$ *satisfies* **(CRel)** *and* **(CInd)**. *From* **(CRel)** *we conclude* $pb \mathrel{|\!\!\sim}^{c\text{-}sk}_{\Delta} \overline{f}$*. Furthermore, according to* **(CInd)***, we know that* $pbw \mathrel{|\!\!\sim}^{c\text{-}sk}_{\Delta} \overline{f}$ *and* $pb\overline{w} \mathrel{|\!\!\sim}^{c\text{-}sk}_{\Delta} \overline{f}$*.*

7 Conclusions and Future Work

For inductive reasoning from conditional belief bases, the concept of conditional syntax splitting has been introduced in the literature as a generalization of syntax splitting. It is applicable also to cases where the conditionals in the subbases share some atoms, and it leads to a formalization of the drowning effect which had been described previously only by means of examples.

In this paper, we extended the study of conditional splittings and introduced the concept of conditional semantic splitting for OCF-based semantics of conditional belief bases. We showed that c-representations, which exhibit notable properties desirable for nonmonotonic reasoning, satisfy the conditional semantic splitting postulate (CSemSplit). For inference based on single c-representations, we showed that the concept of selection strategies leads to inductive inference operators satisfying the conditional syntax splitting postulate (CSynSplit). Furthermore, we proved that c-inference which is obtained by taking all c-representations of a belief base into account also satisfies (CSynSplit) and thus fully complies with conditional syntax splitting.

Generally, the belief bases Δ considered in the propositions of this paper are assumed be be consistent in the sense that a ranking model for Δ exists that maps every world to a natural number. Our current and future work includes extending the study of splittings further to cover also belief bases satisfying only a weaker notion of consistency (cf. (Haldimann et al. 2023; Haldimann, Beierle, and Kern-Isberner 2024)) and to exploit the benefits of conditional splittings in implementations of inductive inference, e.g., in the reasoning platform InfOCF (Beierle, Eichhorn, and Kutsch 2017; Kutsch and Beierle 2021).

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