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Unified Foundations of Team Semantics via Semirings

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Abstract

Semiring semantics for first-order logic provides a way to trace how facts represented by a model are used to deduce satisfaction of a formula. Team semantics is a framework for studying logics of dependence and independence in diverse contexts such as databases, quantum mechanics, and statistics by extending first-order logic with atoms that describe dependencies between variables. Combining these two, we propose a unifying approach for analysing the concepts of dependence and independence via a novel semiring team semantics, which subsumes all the previously considered variants for first-order team semantics. In particular, we study the preservation of satisfaction of dependencies and formulae between different semirings. In addition we create links to reasoning tasks such as provenance, counting, and repairs.

1 Introduction

Team semantics offers a logical framework to study important concepts that arise in the presence of plurality of data such as dependence and independence. The birth of the area can be traced to the introduction of dependence logic in (Väänänen 2007). During the past decade, the expressivity and complexity theoretical aspects of logics in team semantics have been actively studied. Fascinating connections have been drawn to areas such as of database theory (Hannula, Kontinen, and Virtema 2020; Hannula and Kontinen 2016), verification (Gutsfeld et al. 2022), real valued computation (Hannula et al. 2020), and quantum foundations (Albert and Grädel 2022; Abramsky, Puljujärvi, and Väänänen 2021). The study has focused on logics in the first-order, propositional and modal team semantics, and more recently in the multiset (Durand et al. 2018a; Grädel and Wilke 2022) and probabilistic settings (Durand et al. 2018b). Prior to this work, these adaptations of team semantics have been studied in isolation from one another.

Data provenance provides means to describe the origins of data, allowing to give information about the witnesses to a query, or determining how a certain output is derived. Provenance semirings were introduced in (Green, Karvounarakis, and Tannen 2007) to devise a general framework that allows to uniformly treat extensions of positive relational algebra, where the tuples have annotations that reflect very diverse information. Some motivating examples of said relations come from incomplete and probabilistic databases, and bag semantics. This semiring framework captures a notion of data provenance called *how-provenance*, where the semiring operations essentially capture how each output is produced from the source. Following this framework, semiring semantics for full first-order logic (FO) were developed in (Grädel and Tannen 2017). The semiring semantics for FO refines, in particular, the classical Boolean semantics by allowing formulae to be evaluated as values from a semiring. This allows for example counting proof trees, or winning strategies in the model checking game for $\mathfrak{A}$ and ϕ.

In databases, dependencies are applied as integrity constraints (ICs) that specify sets of rules that the database needs to satisfy. Formal analysis of the rules is facilitated by viewing them as FO sentences that usually follow certain syntactic patterns. This approach is sometimes inadequate because query languages such as SQL operate with multisets (i.e., *bags*) of tuples instead of sets. Recently, (Chu et al. 2018) have formulated ICs, such as keys and foreign keys, over semirings to study SQL query equivalence. In probability theory, conditional independence has widespread applications; for instance, assumptions about conditional independence can simplify computations of joint probabilities of variables. It is known that dependency notions in database theory and probability theory are interlinked because many (but not all) such concepts can be rewritten in terms of information-theoretic measures such as conditional entropy and conditional mutual information (Lee 1987; Yeung 2008). However, we are not aware of previous works that use semirings to the same effect.

Similar to extending first-order logic with counting, we extend the semiring semantics of FO (Grädel and Tannen 2017) with the ability of comparing the semiring values of first-order formulae. Using this formalism we define concepts such as dependence and independence in a way that encompasses prior interpretations. The proposed formalism also provides a robust framework for studying the preservation of satisfaction and entailment for dependence statements when moving from one semiring to another. Such preservation results have previously been studied between database and probability theory (Geiger and Pearl 1993; Gyssens, Niepert, and Gucht 2014; Durand et al. 2018a; Kenig and Suciu 2022; Malvestuto 1986; Malvestuto 1992; Wong, Butz, and Wu 2000). Furthermore, we propose a unified approach to team semantics that involves annotating the elements of a team with elements from an arbitrary semiring. By doing so, the original team semantics and its quantitative variants can be recovered by choosing a suitable concrete semiring. The conversion to semiring team semantics enables provenance analysis and other reasoning tasks to be performed for the first time for expressive team-based logics.

2 Preliminaries

We fix a countably infinite set Var of *variables*. We use $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \dots$ to denote *first-order structures*, and write $A, B, C, \dots$ for their *domains*. An *assignment* (of $\mathfrak{A}$) is a function s that maps a finite set $D \subseteq$ Var of variables to some values (in A). We call D the *domain* of s, written $\mathrm{Dom}(s)$. For a variable x and a value a, we write $s[a/x]$ for the assignment with domain $\mathrm{Dom}(s) \cup \{x\}$ which maps x to a and otherwise agrees with s. For a tuple of variables $\vec{x}$ and an assignment s, we write $s(\vec{x})$ for the sequence obtained by mapping $\vec{x}$ pointwise by s. We also write $\vec{x}\vec{y}$ for the concatenation of two tuples $\vec{x}$ and $\vec{y}$.

A *team* X is a finite set of assignments s with a shared domain D. We call D the *domain* of X, written $\mathrm{Dom}(X)$. Given a first-order structure $\mathfrak{A}$, we say that X is a *team of* $\mathfrak{A}$, if A subsumes the ranges of each $s \in X$. Moving from single assignments to sets of assignments enables us to interpret dependency statements between variables:

A *conditional independence atom* is an expression of the form $\vec{y} \perp_{\vec{x}} \vec{z}$, where $\vec{x}, \vec{y}, \vec{z}$ are variable sequences (not necessarily of the same length). A team X *satisfies* $\vec{y} \perp_{\vec{x}} \vec{z}$, written $X \models \vec{y} \perp_{\vec{x}} \vec{z}$, if for all $s, s' \in X$ such that $s(\vec{x}) = s'(\vec{x})$ there exists $s'' \in X$ such that $s(\vec{x}\vec{y}) = s''(\vec{x}\vec{y})$ and $s'(\vec{z}) = s''(\vec{z})$. A *pure independence atom* is an expression of the form $\vec{x} \perp \vec{y}$, defined as $\vec{x} \perp_{\emptyset} \vec{y}$. A team X satisfies $\vec{x} \perp \vec{y}$, if for all $s, s' \in X$ there exists $s'' \in X$ such that $s(\vec{x}) = s''(\vec{x})$ and $s'(\vec{y}) = s''(\vec{y})$.

A *dependence atom* is an expression of the form $=\!(\vec{x}, \vec{y})$, where $\vec{x}$ and $\vec{y}$ are variable sequences. A team X *satisfies* $=\!(\vec{x}, \vec{y})$, if for all $s, s' \in X$, $s(\vec{x}) = s'(\vec{x})$ implies $s(\vec{y}) = s'(\vec{y})$.

An *inclusion atom* is an expression of the form $\vec{x} \subseteq \vec{y}$, where $\vec{x}$ and $\vec{y}$ are variables sequences of the same length. A team X *satisfies* $\vec{x} \subseteq \vec{y}$, if for all $s \in X$ there is $s' \in X$ such that $s(\vec{x}) = s'(\vec{y})$.

In the probabilistic team semantics setting, $\vec{y} \perp_{\vec{x}} \vec{z}$ is given the usual meaning of conditional independence in probability theory. Furthermore, the probabilistic interpretation of $\vec{x} \subseteq \vec{y}$ states that the marginal distributions of $\vec{x}$ and $\vec{y}$ are identical.

If α is an atom for which satisfaction by teams is defined, we extend this definition to first-order structures $\mathfrak{A}$ by saying that X *satisfies* α *under* $\mathfrak{A}$, written $\mathfrak{A} \models_X \alpha$, if X satisfies α.

3 Semiring Perspective on Dependencies

In this section, we generalise teams and dependencies using semirings.

3.1 Semirings

We start by briefly reviewing concepts related to semirings that are necessary for the present paper.

Definition 1 (Semiring)**.** A *semiring* is a tuple $K = (K, +, \cdot, 0, 1)$, where $+$ and $\cdot$ are binary operations on K, $(K, +, 0)$ is a commutative monoid with identity element 0, $(K, \cdot, 1)$ is a monoid with identity element 1, $\cdot$ left and right distributes over $+$, and $x \cdot 0 = 0 = 0 \cdot x$ for all $x \in K$. K is called *commutative* if $(K, \cdot, 1)$ is a commutative monoid. As usual, we often write ab instead of $a \cdot b$.

That is, semirings are rings which need not have additive inverses. We focus on the listed semirings that encapsulate the set, multiset, and distribution based team semantics:

- The *Boolean semiring* $\mathbb{B} = (\mathbb{B}, \vee, \wedge, 0, 1)$ models logical truth and is formed from the two-element Boolean algebra. It is the simplest example of a semiring that is not a ring.
- The *probability semiring* $\mathbb{R}_{\geq 0} = (\mathbb{R}_{\geq 0}, +, \cdot, 0, 1)$ consists of the non-negative reals with standard addition and multiplication.
- The *semiring of natural numbers* $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ consists of natural numbers with their usual operations.

Other examples include the semiring of multivariate polynomials $\mathbb{N}[X] = (\mathbb{N}[X], +, \cdot, 0, 1)$ which is the free commutative semirings generated by the indeterminates in X, the *tropical semiring* $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \min, +, \infty, 0)$ which consists of the reals expanded with infinity and has $\min$ and $+$ respectively plugged in for addition and multiplication, and the Lukasiewicz semiring $\mathbb{L} = ([0, 1], \max, \cdot, 0, 1)$, used in multivalued logic, which endows the unit interval with $\max$ addition and multiplication $a \cdot b := \max(0, a + b - 1)$.

Let $\leq$ be a partial order. A binary operator $*$ is said to be *monotone under* $\leq$ if $a \leq b$ and $a' \leq b'$ implies $a * a' \leq b * b'$. A *partially ordered semiring* is a tuple $K = (K, +, \cdot, 0, 1, \leq)$, where $(K, +, \cdot, 0, 1)$ is a semiring, $(K, \leq)$ is a partially ordered set, and $+, \cdot$ are monotone under $\leq$. Given a semiring $K = (K, +, \cdot, 0, 1)$, define a binary relation $\leq_K$ on K as $a \leq_K b$ if $\exists c : a + c = b$. This relation is a preorder; meaning it is reflexive and transitive. If $\leq_K$ is also antisymmetric, it is a partial order, called the *natural order* of K, and K is said to be *naturally ordered*. In this case, K endowed with its natural order is a partially ordered semiring.

If a semiring K satisfies $ab = 0$ for some $a, b \in K$ where $a \neq 0 \neq b$, we say that K has *divisors of* 0. On the other hand, a semiring K is considered $+$*-positive* if $a + b = 0$ implies that $a = b = 0$. If a semiring is both $+$-positive and has no divisors of 0, it is referred to as *positive*. For example, the modulo two integer semiring $\mathbb{Z}_2$ is not positive since it is not $+$-positive (even though it has no divisors of 0). Conversely, an example of a semiring with divisors of 0 is $\mathbb{Z}_4$. We can also examine the positivity of K by looking at its *characteristic mapping*, which is defined as the function $\xi_K : K \to \mathbb{B}$ such that

$$\xi_K(a) = \begin{cases} 1 & \text{if } a \neq 0, \\ 0 & \text{if } a = 0. \end{cases}$$

Proposition 2 (Proposition 6, (Grädel and Tannen 2017))**.** *A semiring K is positive if and only if its characteristic mapping is a homomorphism.*

In particular, note that the Boolean semiring $\mathbb{B}$, the probability semiring $\mathbb{R}_{\geq 0}$, and the semiring of natural numbers $\mathbb{N}$ are positive and naturally ordered.

$K = \mathbb{B}$		
x	y	$\mathbb{X}_1(s)$
a	a	1
a	b	1
b	a	0
b	b	0

$K = \mathbb{N}$		
x	y	$\mathbb{X}_2(s)$
a	a	2
a	b	0
b	a	0
b	b	5

$K = \mathbb{R}_{\geq 0}$		
x	y	$\mathbb{X}_3(s)$
a	a	1/4
a	b	3/4
b	a	0
b	b	0

Figure 1: K-teams $\mathbb{X}_i\colon \mathrm{As}(D, A) \to K$, $D = \{x, y\}$, $A = \{a, b\}$, representing a team, a multiteam, and a probabilistic team.

3.2 K-teams

Given a semiring K, the concept of a K-team is obtained by labeling each assignment of a team with an element from K. If a D is a set of variables and A a set, we define $\mathrm{As}(D, A)$ as the set all assignments $s : D \to A$.

Definition 3 (K-team)**.** A K*-team* is a function $\mathbb{X}\colon \mathrm{As}(D, A) \to K$, where $K = (K, +, \cdot, 0, 1)$ is a (commutative) semiring, D is a set of variables, and A is a set. The *support* of $\mathbb{X}$ is defined as $\mathrm{Sup}(\mathbb{X}) := \{s \mid \mathbb{X}(s) \neq 0\}$. Provided that we have an ordering on K, we say that $\mathbb{X}$ is a *subteam* of $\mathbb{Y}$ if $\mathbb{X}(s) \leq \mathbb{Y}(s)$ for every $s \in \mathrm{Sup}(\mathbb{X})$.

We can now reconceptualise the notion of a team as a K-team by associating each possible assignment with either a 1 or 0 label depending on whether or not it belongs to the team. When dealing with probabilistic teams, each assignment is labeled with a non-negative real number that can be interpreted as a probability by scaling the sum of these labels to one. For multiteams, each assignment is assigned a natural number. Figure 1 provides an illustration of these concepts.

3.3 Dependencies over K-teams: A Prologue

Our goal is to find a common language for expressing concepts such as dependence and independence in different K-teams. Referring back to the preliminaries section, the reader may observe that the fundamental dependency concepts in team semantics can be formalised using the language of first-order logic. This approach, however, becomes insufficient when dealing with multisets or probability distributions. For example, the concept of independence between two random variables involves counting, which is beyond the capabilities of first-order logic; this appears to be the case in the semiring context as well (Grädel et al. 2022). Therefore, we explore extensions of first-order logic to overcome such limitations. The following example hints at the direction we will take.

Example 4. We aim to find a common logical expression underlying both relational and probabilistic interpretations of conditional independence. To this end, fix a conditional independence atom $y \perp_x z$ over variables x, y, z.

In the relational context, viewing a team X with domain $\{x, y, z\}$ as a ternary relation $R = \{(s(x), s(y), s(z)) \mid s \in X\}$, we observe that X satisfies $y \perp_x z$ if and only if R satisfies the first-order sentence

$$\forall abcde(R(a, b, c) \land R(a, d, e) \to R(a, b, e)). \quad (1)$$

Moving to the probability context, two random variables y and z are conditionally independent given a random variable x if and only if for all values a, b, c,

$$P(y = b \mid x = a) \cdot P(z = c \mid x = a) = P(yz = bc \mid x = a). \quad (2)$$

Our strategy is to transform (2) into a "logical" sentence similar to (1). First, we remove conditional probabilities to obtain from (2) the equation

$$P(xy = ab) \cdot P(xz = ac) = P(xyz = abc) \cdot P(x = a).$$

Next, we model the probability distribution with a ternary function R mapping value triples (a, b, c) to the probabilities $P(xyz = abc)$, and rewrite a marginal probability $P(xy = ab)$ as the sum of probabilities $\sum_c R(a, b, c)$, arriving at

$$\sum_c R(a, b, c) \cdot \sum_b R(a, b, c) = R(a, b, c) \cdot \sum_{b,c} R(a, b, c).$$

By interpreting multiplication as conjunction and aggregate summation as existential quantification, and adding the universal quantification of triples, we arrive at the expression

$$\forall abc\big((\exists c R(a, b, c) \land \exists b R(a, b, c)) = (R(a, b, c) \land \exists bc R(a, b, c))\big). \quad (3)$$

This expression can be viewed as a "logical" sentence defining probabilistic independence. Note that it involves an equality statement between two formulae and is thus not a well-formed first-order sentence. However, if we replace the equality symbol = with the logical equivalence symbol $\leftrightarrow$, we obtain a first-order sentence which, after removing logical redundancies, transforms into

$$\forall abc\big((\exists c R(a, b, c) \land \exists b R(a, b, c)) \to (R(a, b, c))\big).$$

By renaming the existentially quantified variables, and dragging them in front of the quantifier-free part, we obtain precisely the first-order sentence (1) that we used to define relational conditional independence.

Based on the example, it appears that logical statements formulated in the manner of (3) can integrate diverse expressions of dependency concepts. To give more depth to this idea, we will dedicate the next section to the interpretation of first-order logic and its extension with equality statements between formulae within the semiring context. Following this, we will revisit the concept of dependencies over K-teams.

4 First-Order Logic with Formula Equality

We first review K-interpretations for first-order formulae from (Grädel and Tannen 2017). From now on, we consider only commutative semirings. This is necessary to properly interpret quantifiers within this context.

4.1 First-Order Interpretations

Fix a relational vocabulary $\tau = \{R, S, T, ...\}$. We denote by $\mathrm{ar}(R)$ the *arity* of a relational symbol R. A *relational atom* (resp. a *negated relational atom*) is an expression of the form $R(\vec{x})$ (resp. $\neg R(\vec{x})$) where $\vec{x}$ is a sequence of variables of length $\mathrm{ar}(R)$. An *equality atom* (resp. *negated equality atom*) is an expression of the form $x = y$ where x and y are variables. An atom or a negated atom is called a *literal*.

First-order formulae are the expressions formed by closing atoms by quantifiers $\exists, \forall$ and connectives $\wedge, \vee, \neg$ in the usual way. We use $\phi \rightarrow \psi$ as a shorthand for $\neg\phi \vee \psi$, and $\phi \leftrightarrow \psi$ as a shorthand for $(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. The *set of free variables* $\mathrm{Fr}(\theta)$ of an FO formula θ is defined in the usual way. We also write nnf for the standard negation normal form transformation of first-order formulae.

Let A be a set. An expression of the form $R(\vec{a})$ (resp. $\neg R(\vec{a})$), where $\vec{a} \in A^{\mathrm{ar}(R)}$, is called a *fact* (resp. *negated fact*) over A. The set of *literals* over A, denoted by Lit_A, is the set comprising all facts and negated facts over A.

Definition 5 ((Grädel and Tannen 2017))**.** Fix a semiring $K = (K, +, \cdot, 0, 1)$. A K-interpretation is a mapping $\pi \colon \mathsf{Lit}_A \rightarrow K$. Given variable assignments $s \colon \mathrm{Var} \rightarrow A$, it is extended to FO formulae as follows:

$$\llbracket R(\vec{x}) \rrbracket_{\pi,s} = \pi(R(s(\vec{x}))) \qquad \llbracket \phi \wedge \psi \rrbracket_{\pi,s} = \llbracket \phi \rrbracket_{\pi,s} \cdot \llbracket \psi \rrbracket_{\pi,s}$$
$$\llbracket \neg R(\vec{x}) \rrbracket_{\pi,s} = \pi(\neg R(s(\vec{x}))) \qquad \llbracket \phi \vee \psi \rrbracket_{\pi,s} = \llbracket \phi \rrbracket_{\pi,s} + \llbracket \psi \rrbracket_{\pi,s}$$
$$\llbracket \forall x \phi \rrbracket_{\pi,s} = \prod_{a \in A} \llbracket \phi \rrbracket_{\pi,s[a/x]} \qquad \llbracket \exists x \phi \rrbracket_{\pi,s} = \sum_{a \in A} \llbracket \phi \rrbracket_{\pi,s[a/x]}$$
$$\llbracket \neg\phi \rrbracket_{\pi,s} = \llbracket \mathrm{nnf}(\neg\phi) \rrbracket_{\pi,s} \qquad \llbracket x * y \rrbracket_{\pi,s} = \begin{cases} 1 & \text{if } s(x) * s(y) \\ 0 & \text{otherwise,} \end{cases}$$

where $* \in \{=, \neq\}$. If $\mathrm{Fr}(\phi)$ is empty, then ϕ is called a *sentence*. For sentences ϕ, we write $\llbracket \phi \rrbracket_\pi$ as a shorthand for $\llbracket \phi \rrbracket_{\pi,s_\varnothing}$, where $s_\varnothing$ is the empty assignment.

A K-interpretation π is called *model-defining* (Grädel and Tannen 2017) if for all facts $R(\vec{a})$ it holds that exactly one of $R(\vec{a})$ and $\neg R(\vec{a})$ is mapped to 0 by π, while the other is mapped to a value different from 0.

The compositional interpretation entails that semiring homomorphisms extend to formula interpretations. This property will be used in this paper to analyse mutual relationships between different interpretations of dependency concepts.

Proposition 6 ((Grädel and Tannen 2017))**.** *Let h be a semiring homomorphism from K_1 to K_2, and let $\pi_1 : \mathsf{Lit}_A \rightarrow K_1$ and $\pi_2 : \mathsf{Lit}_A \rightarrow K_2$ be interpretations such that $h \circ \pi_1 = \pi_2$. Then, $h(\llbracket \phi \rrbracket_{\pi_1}) = \llbracket \phi \rrbracket_{\pi_2}$ for every first-order sentence ϕ.*

Let $\mathfrak{A}$ be a standard first-order structure over τ, and let $\mathbb{B}$ be the Boolean semiring. The interpretation π that maps relational facts $R(\vec{a})$ (resp. negated relational facts $\neg R(\vec{a})$) to 1 (resp. 0) if $\vec{a} \in R^{\mathfrak{A}}$, and otherwise to 0 (resp. 1), is called the *canonical truth interpretation* of $\mathfrak{A}$, denoted $\pi_{\mathfrak{A}}$.

Proposition 7 ((Grädel and Tannen 2017))**.** *Let ϕ be a first-order sentence, and $\mathfrak{A}$ a structure. Then $\mathfrak{A} \models \phi$ iff $\llbracket \phi \rrbracket_{\pi_{\mathfrak{A}}} = 1$.*

4.2 Formula (In)equality

To express dependencies logically in a general semiring context, we introduce equality and inequality over FO formulae. Given $\phi, \psi \in \mathsf{FO}$, we extend the K-interpretation as follows:

$$\llbracket \phi * \psi \rrbracket_{\pi,s} = \begin{cases} 1 & \text{if } \llbracket \phi \rrbracket_{\pi,s} * \llbracket \psi \rrbracket_{\pi,s} \\ 0 & \text{otherwise,} \end{cases}$$

where $* \in \{=, \neq, \leq, \nleq\}$. For the (negated) formula inequality, we assume $(K, \leq)$ is a partially ordered semiring. We write $\bot?$ and $\not\bot?$ to denote the formula equalities of the form $\phi = \bot$ and $\phi \neq \bot$, respectively. For $C \subseteq \{\bot?, \not\bot?, =, \neq, \leq, \nleq\}$, we let $\mathsf{FO}(C)$ denote the extension of the logic of Grädel and Tannen with the formula equalities and inequalities in C occuring positively (i.e. in the scope of even number of negations) and without nesting. The *set of free variables* for a formula of the form $\phi * \psi$, $* \in \{=, \neq, \leq, \nleq\}$, is defined as $\mathrm{Fr}(\phi * \psi) = \mathrm{Fr}(\phi) \cup \mathrm{Fr}(\psi)$.

We extend nnf to formula (in)equalities by setting $\mathrm{nnf}(\neg(\phi = \psi)) := (\mathrm{nnf}(\phi) \neq \mathrm{nnf}(\psi))$ and $\mathrm{nnf}(\neg(\phi \leq \psi)) := (\mathrm{nnf}(\phi) \nleq \mathrm{nnf}(\psi))$. We can then extend the use of shorthands $\phi \rightarrow \psi$ and $\phi \leftrightarrow \psi$ for FO with formula (in)equalities.

Proposition 8. *Let K be a positive semiring, and let π be a model-defining K-interpretation. Let ϕ be a formula of $\mathsf{FO}(=, \neq, \leq, \nleq)$. Then, $\pi(\phi) = 0$ if and only if $\pi(\neg\phi) \neq 0$.*

Proof. The proof is by structural induction. If ϕ is an atom, the statement follows by the assumption that π is model-defining. If ϕ is of the form $\psi_0 \vee \psi_1$, then $\pi(\phi) = 0$ if and only if $\pi(\psi_0) = 0 = \pi(\psi_1)$ if and only if $\pi(\neg\psi_0) \neq 0 \neq \pi(\neg\psi_1)$ if and only if $\pi(\neg\phi) \neq 0$. The first and third "if and only if" follow by positivity of K, and the second by induction assumption. The remaining cases are analogous. □

Two sentences ϕ and ψ are *K-equivalent*, written $\phi \equiv_K \psi$, if $\llbracket \phi \rrbracket_\pi = \llbracket \psi \rrbracket_\pi$ for all model-defining K-interpretations π. The sentences ϕ and ψ are *equivalent*, written $\phi \equiv \psi$, if they are K-equivalent for all semirings K. Two logics $\mathfrak{L}$ and $\mathfrak{L}'$ are *equally expressive under K* (resp. *equally expressive*), denoted $\mathfrak{L} \equiv_K \mathfrak{L}'$ (resp. $\mathfrak{L} \equiv \mathfrak{L}'$), if all sentences from $\mathfrak{L}$ are K-equivalent (resp. equivalent) to some sentence from $\mathfrak{L}'$, and conversely all sentences from $\mathfrak{L}'$ are K-equivalent (resp. equivalent) to some sentence from $\mathfrak{L}$.

The following is a consequence of Proposition 8.

Proposition 9. *If ϕ and ψ are FO-formulae with formula (in)equalities, then $\phi \leq \psi \equiv_{\mathbb{B}} \phi \rightarrow \psi$ and $\phi = \psi \equiv_{\mathbb{B}} \phi \leftrightarrow \psi$.*

Corollary 10. $\mathsf{FO}(=, \neq, \leq, \nleq) \equiv_{\mathbb{B}} \mathsf{FO}$.

4.3 K-atoms

We are now ready to explore the idea of using logical statements as definitions of dependencies across various semirings. To do so, we will consider an atom α, like the dependence or independence atom, and define its interpretation over K-teams by referencing a definition of α stated in first-order logic with formula (in)equalities.

Consider a relation symbol R that does not belong to τ (and is of any arity). For a tuple $\vec{a} = (a_1, \ldots, a_{\mathrm{ar}(R)})$ and a tuple of indices $\vec{i} = (i_1, \ldots, i_k)$ from $1, \ldots, \mathrm{ar}(R)$, we set $\vec{a}_{\vec{i}} := (a_{i_1}, \ldots, a_{i_k})$. For tuples of indices $\vec{i}_1, \ldots, \vec{i}_n$ from $1, \ldots, \mathrm{ar}(R)$ and variable tuples $\vec{u}_1, \ldots, \vec{u}_n$ (such that the length of $\vec{u}_l$ is that of $\vec{i}_l$, for each $l \leq n$), we define a shorthand

$$\theta_{\vec{i}_1,\ldots,\vec{i}_n}(\vec{u}_1, \ldots, \vec{u}_n) := \exists \vec{x}(R(\vec{x}) \wedge \vec{x}_{\vec{i}_1} = \vec{u}_1 \wedge \cdots \wedge \vec{x}_{\vec{i}_n} = \vec{u}_n).$$

This shorthand formula expresses that there exists an R-fact such that its projections on sequences of positions $\vec{i}_1, \ldots, \vec{i}_n$ are $\vec{u}_1, \ldots, \vec{u}_n$. Considering dependence, independence, and inclusion atoms, we now define the following sentences:

$$\phi^{\vec{i}}_{\text{lit-}S} := \forall \vec{x}\big(R(\vec{x}) = \bot \vee \big(R(\vec{x}) \neq \bot \wedge S(\vec{x}_{\vec{i}})\big)$$

$$\phi_{\text{dep}}^{\vec{i},\vec{j}} \coloneqq \forall \vec{u}\vec{v}\vec{w}\Big(\big(\theta_{\vec{i},\vec{j}}(\vec{u},\vec{v}) \land \theta_{\vec{i},\vec{j}}(\vec{u},\vec{w})\big) = \bot \lor \vec{v} = \vec{w}\Big)$$

$$\phi_{\text{indep}}^{\vec{i},\vec{j},\vec{k}} \coloneqq \forall \vec{u}\vec{v}\vec{w}\Big(\big(\theta_{\vec{i},\vec{j}}(\vec{u},\vec{v}) \land \theta_{\vec{i},\vec{k}}(\vec{u},\vec{w})\big) = \big(\theta_{\vec{i}}(\vec{u}) \land \theta_{\vec{i},\vec{j},\vec{k}}(\vec{u},\vec{v},\vec{w})\big)\Big)$$

$$\phi_{\text{inc}}^{\vec{i},\vec{j}} \coloneqq \forall \vec{u}\Big(\theta_{\vec{i}}(\vec{u}) \leq \theta_{\vec{j}}(\vec{u})\Big).$$

In the superscript, we may replace each unary tuple (i) with i; e.g., write $\phi_{\text{lit-}S}^{i}$ instead of $\phi_{\text{lit-}S}^{(i)}$. The above formulae can often be simplified, as illustrated next.

Example 11. If R is ternary then $\phi_{\text{indep}}^{1,2,3}$ is of the form $\forall uvw\Big(\big(\theta_{1,2}(u,v) \land \theta_{1,3}(u,w)\big) = \big(\theta_1(u) \land \theta_{1,2,3}(u,v,w)\big)\Big)$, where $\theta_{1,2}(u,v)$, $\theta_{1,3}(u,w)$, $\theta_1(u,w)$, and $\theta_{1,3}(u,w)$ are respectively of the form

$$\exists x_1x_2x_3(R(x_1,x_2,x_3) \land x_1 = u \land x_2 = v),$$
$$\exists x_1x_2x_3(R(x_1,x_2,x_3) \land x_1 = u \land x_3 = w),$$
$$\exists x_1x_2x_3(R(x_1,x_2,x_3) \land x_1 = u),$$
$$\exists x_1x_2x_3(R(x_1,x_2,x_3) \land x_1 = u \land x_2 = v \land x_3 = w).$$

Clearly, these sentences are equivalent to the simpler forms $\exists x_3 R(u,v,x_3)$, $\exists x_2 R(u,x_2,w)$, $\exists x_2x_3 R(u,x_2,x_3)$, and $R(u,v,w)$, respectively. These equivalences can then be used to rewrite $\phi_{\text{indep}}^{1,2,3}$ more succinctly.

It can now be observed that ϕ_{dep}, ϕ_{indep} and ϕ_{inc} are $\mathbb{B}$-equivalent to the standard relational definitions of dependence, independence, and inclusion atoms. The following proposition is a consequence of Proposition 9. It can be proven by imitating the reasoning in Example 4.

Proposition 12. *The following equivalences hold:*

$$\phi_{\text{lit-}S}^{\vec{i}} \equiv_{\mathbb{B}} \forall \vec{x}\big(R(\vec{x}) \to S(\vec{x}_i)\big)$$
$$\phi_{\text{dep}}^{\vec{i},\vec{j}} \equiv_{\mathbb{B}} \forall \vec{u}\vec{v}\vec{w}(\theta_{\vec{i},\vec{j}}(\vec{u},\vec{v}) \land \theta_{\vec{i},\vec{j}}(\vec{u},\vec{w}) \to \vec{v} = \vec{w})$$
$$\phi_{\text{indep}}^{\vec{i},\vec{j},\vec{k}} \equiv_{\mathbb{B}} \forall \vec{u}\vec{v}\vec{w}(\theta_{\vec{i},\vec{j}}(\vec{u},\vec{v}) \land \theta_{\vec{i},\vec{k}}(\vec{u},\vec{w}) \to \theta_{\vec{i},\vec{j},\vec{k}}(\vec{u},\vec{v},\vec{w}))$$
$$\phi_{\text{inc}}^{\vec{i},\vec{j}} \equiv_{\mathbb{B}} \forall \vec{u}(\theta_{\vec{i}}(\vec{u}) \to \theta_{\vec{j}}(\vec{u}))$$

Having formalised key dependency concepts using logical statements, let us then move on to K-teams. Now, fix a total order $<$ on the variable set Var. Let $\mathbb{X}\colon \text{As}(D,A) \to K$ be a K-team with domain $V = \{x_1,\ldots,x_k\}$, where $x_1 < \cdots < x_k$. For each $s\colon A^V \to K$, where A^V is the set of functions from V to A, define a tuple $\vec{a}_s \coloneqq (s(x_1),\ldots,s(x_k))$. Let R be a relation symbol of arity k. Denote by $\pi_{\mathbb{X}} : \mathsf{Lit}_A \to K$ any K-interpretation such that $\pi_{\mathbb{X}}$ maps $R(\vec{a}_s)$ to $\mathbb{X}(s)$. For a tuple of variables $\vec{x} = (x_{i_1},\ldots,x_{i_n})$, write $\vec{i}_{\vec{x}}$ for the integer tuple $(i_1,\ldots,i_n)$. The K-interpretation of literals and dependencies is now defined as follows:

- Literals: $[\![T(\vec{x})]\!]_{\mathbb{X}} \coloneqq [\![\phi_{\text{lit-}T}^{\vec{i}_{\vec{x}}}]\!]_{\pi_{\mathbb{X}}}$.
- Dependence atoms: $[\![{=}(\vec{x},\vec{y})]\!]_{\mathbb{X}} \coloneqq [\![\phi_{\text{dep}}^{\vec{i}_{\vec{x}},\vec{i}_{\vec{y}}}]\!]_{\pi_{\mathbb{X}}}$.
- Independence atom: $[\![\vec{y} \perp_{\vec{x}} \vec{z}]\!]_{\mathbb{X}} \coloneqq [\![\phi_{\text{indep}}^{\vec{i}_{\vec{x}},\vec{i}_{\vec{y}},\vec{i}_{\vec{z}}}]\!]_{\pi_{\mathbb{X}}}$.
- Inclusion atom: $[\![\vec{x} \subseteq \vec{y}]\!]_{\mathbb{X}} \coloneqq [\![\phi_{\text{inc}}^{\vec{i}_{\vec{x}},\vec{i}_{\vec{y}}}]\!]_{\pi_{\mathbb{X}}}$.

A K-team $\mathbb{X}$ *satisfies* an atom α, written $\mathbb{X} \vDash \alpha$, if $[\![\alpha]\!]_{\mathbb{X}} \neq 0$.

For instance, independence atom for the probability semiring corresponds to the notion of conditional independence in probability theory, and for the Boolean semiring it corresponds to the notion of *embedded multivalued dependency* in database theory.

Example 13. Consider a pure independence atom $x \perp y$ for the three K-teams presented in Figure 1. The atom is interpreted in K-teams using the sentence $\phi_{\text{indep}}^{\varnothing,1,2}$. Similar to Example 11, we may rewrite this sentence in a simpler form:

$$\forall uv\Big(\big(\exists y R(u,y) \land \exists x R(x,v)\big) = \big(\exists xy R(x,y) \land R(u,v)\big)\Big) \quad (4)$$

Suppose $x < y$ according to the total order $<$ on variables. Considering the $\mathbb{B}$-team $\mathbb{X}_1$, the function $\pi_{\mathbb{X}_1}$ maps facts $R(a,a)$ and $R(a,b)$ to 1, and facts $R(b,a)$ and $R(b,b)$ to 0. Then, $\pi_{\mathbb{X}_1}$ interprets the formula equality in (4) as $(1 \land 1) = (1 \land 1)$ for $(u,v) \mapsto \{(a,a),(a,b)\}$, and as $(0 \land 1) = (1 \land 0)$ for $(u,v) \mapsto \{(b,a),(b,b)\}$. Hence $[\![x \perp y]\!]_{\mathbb{X}_1} = 1$, meaning that $\mathbb{X}_1 \vDash x \perp y$. An alternative way to obtain $\mathbb{X}_1 \vDash x \perp y$ is to use Propositions 7 and 12, noting that the model (over signature $\{R\}$) defined by $\mathbb{X}_1$ satisfies a first-order sentence that is $\mathbb{B}$-equivalent to $\phi_{\text{indep}}^{\varnothing,1,2}$. Using similar calculations we may further observe $\mathbb{X}_3$ satisfies $x \perp y$ while $\mathbb{X}_2$ does not. On the other hand, $\mathbb{X}_2$ is the only K-team of the three satisfying the dependence atom ${=}(x,y)$.

Recall that $\xi_K : K \to \mathbb{B}$ is the characteristic mapping that associates non-zero values of K with 1 and zero with 0. The following proposition shows that this mapping preserves the truth of all $\mathsf{FO}(=,\not\perp?,\leq)$-formulae.

Proposition 14. *Let π be a K-interpretation over a positive semiring K. Then for all $\mathsf{FO}(=,\not\perp?,\leq)$-definable ϕ, $[\![\phi]\!]_{\xi_K\circ\pi} = 0$ implies $[\![\phi]\!]_{\pi} = 0$.*

Proof. The proof proceeds by structural induction on the structure of ϕ. We prove simultaneously that the implication can be strengthened to if and only if when $\phi \in \mathsf{FO}$. The cases for first-order literals follow directly from the definition of ξ_K, and the case for $\neg\phi$ is trivial.

The cases for formula equalities and inequalities follow from the positiveness of K together with the induction hypothesis. Below $\phi,\psi \in \mathsf{FO}$, and xor is the exclusive or.

$$\begin{aligned}[\![\phi = \psi]\!]_{\xi_K\circ\pi} = 0 &\Leftrightarrow [\![\phi]\!]_{\xi_K\circ\pi} = 0 \text{ xor } [\![\psi]\!]_{\xi_K\circ\pi} = 0\\ &\Leftrightarrow [\![\phi]\!]_{\pi} = 0 \text{ xor } [\![\psi]\!]_{\pi} = 0\\ &\Rightarrow [\![\phi = \psi]\!]_{\pi} = 0\end{aligned}$$

$$\begin{aligned}[\![\phi \leq \psi]\!]_{\xi_K\circ\pi} = 0 &\Leftrightarrow [\![\phi]\!]_{\xi_K\circ\pi} \neq 0 \text{ and } [\![\psi]\!]_{\xi_K\circ\pi} = 0\\ &\Leftrightarrow [\![\phi]\!]_{\pi} \neq 0 \text{ and } [\![\psi]\!]_{\pi} = 0\\ &\Rightarrow [\![\phi \leq \psi]\!]_{\pi} = 0\end{aligned}$$

The case for $[\![\phi \neq \bot]\!]_{\xi_K\circ\pi} = 0$ is similar.

The cases for $\land$, $\lor$, $\exists$, and $\forall$ follow from the positiveness of K together with the induction hypothesis, we show $\lor$:

$$\begin{aligned}[\![\phi \lor \psi]\!]_{\xi_K\circ\pi} = 0 &\Leftrightarrow [\![\phi]\!]_{\xi_K\circ\pi} = 0 \text{ and } [\![\psi]\!]_{\xi_K\circ\pi} = 0\\ &\Rightarrow [\![\phi]\!]_{\pi} = 0 \text{ and } [\![\psi]\!]_{\pi} = 0\\ &\Leftrightarrow [\![\phi \lor \psi]\!]_{\pi} = 0\end{aligned}$$

The implication above follows from the induction hypothesis and can be strengthened to an equivalence if $\phi, \psi \in \mathsf{FO}$. $\square$

Let us now define the *possibilistic collapse* of a K-team $\mathbb{X}$ is as the $\mathbb{B}$-team $\xi_K \circ \mathbb{X}$. We sometimes identify the possibilistic collapse with the team it defines (i.e., the set of assignments it maps to 1). As a consequence of the previous proposition, any $\mathsf{FO}(=, \not\perp?, \leq)$-definable atom is preserved under the possibilistic collapse.

Corollary 15. *Let $\mathbb{X}$ be a K-team over a positive semiring K, and let α be an $\mathsf{FO}(=, \not\perp?, \leq)$-definable atom. If $\mathbb{X}$ satisfies α, then its possibilistic collapse satisfies α. The converse direction holds true if α is $\mathsf{FO}(\perp?, \neq, \nleq)$-definable.*

For instance, in Example 13 we observed that the $\mathbb{B}$-team $\mathbb{X}_1$ satisfies the independence atom $x \perp y$. Since $\mathbb{X}_1$ is the possibilistic collapse of $\mathbb{X}_3$, this follows already by the fact that $\mathbb{X}_3$ satisfies the same atom, which in turn is $\mathsf{FO}(=, \not\perp?, \leq)$-definable. We also noted that the $\mathbb{N}$-team $\mathbb{X}_2$ in Figure 1 satisfies the dependence atom $=(x, y)$. By Corollary 15 this follows also from the fact that the possibilistic collapse of $\mathbb{X}_2$ satisfies the same $\mathsf{FO}(\perp?, \neq, \nleq)$-definable atom.

At this point we note that an alternative way to interpret the dependence atom $=(x, y)$ in a data table with duplicates (i.e., in an $\mathbb{N}$-team), would be to stipulate that x uniquely determines y and has no duplicates in the projection of the table to x and y (this would be a natural extension of the semiring interpretation of keys by (Chu et al. 2018, Definition 4.1)). However, such a notion of dependence would fail to satisfy the reflexivity rule of functional dependencies, which entails that $=(x, x)$ always holds.

5 Team Semantics

Next we explore how all the team semantics variants (and more) can be unified under the rubric of semirings. We first present an adaptation of team semantics for K-teams, and then consider K-interpretations of complex formulae.

For the notion of team semantics, a few useful concepts are needed. The *projection* $X_{\upharpoonright V}$ of X on a variable set $V \subseteq \mathrm{Dom}(X)$ is defined as the set $\{s_{\upharpoonright V} \mid s \in X\}$, where $s_{\upharpoonright V}$ is the projection of an assignment s on V defined in the usual way. For a set S and a variable, we define $X[S/x]$ as the team $\{s[a/x] \mid s \in X, a \in S\}$. For a set S and a function $F : X \to \mathcal{P}(S) \setminus \{\emptyset\}$, we define $X[F/x]$ as the team $\{s[a/x] \mid s \in X, a \in F(s)\}$.

We present the standard team semantics for FO in negation normal form. Note that in Definition 5, we allowed an arbitrary symbolic use of negation in the K-interpretation setting for convenience. In team semantics setting, it is customary (and more convenient) to work with formulae that are directly in negation normal form.

Definition 16 (Team semantics)**.** Let X be a team of a first-order structure $\mathfrak{A}$ over vocabulary τ. For $\phi \in \mathsf{FO}[\tau]$, we define when X *satisfies* ϕ *under* $\mathfrak{A}$, written $\mathfrak{A} \models_X \phi$, as follows ($\models_s$ refers to the usual Tarski semantics of FO):

$\mathfrak{A} \models_X l$	$\Leftrightarrow$	$\mathfrak{A} \models_s l$ for all $s \in X$ (l is a literal),
$\mathfrak{A} \models_X (\psi \wedge \theta)$	$\Leftrightarrow$	$\mathfrak{A} \models_X \psi$ and $\mathfrak{A} \models_X \theta$,
$\mathfrak{A} \models_X (\psi \vee \theta)$	$\Leftrightarrow$	$\mathfrak{A} \models_Y \psi$ and $\mathfrak{A} \models_Z \theta$ for some $Y, Z \subseteq X$ such that $Y \cup Z = X$,
$\mathfrak{A} \models_X \forall x \psi$	$\Leftrightarrow$	$\mathfrak{A} \models_{X[A/x]} \psi$,
$\mathfrak{A} \models_X \exists x \psi$	$\Leftrightarrow$	$\mathfrak{A} \models_{X[F/x]} \psi$ for some function $F : X \to \mathcal{P}(A) \setminus \{\emptyset\}$.

5.1 K-team Semantics

By re-examining team semantics through the lens of semirings, we can arrive at the following truth definition.

Definition 17 (K-team semantics)**.** Let $\mathbb{X}$ be a K-team of a first-order structure $\mathfrak{A}$ over vocabulary τ. For $\phi \in \mathsf{FO}[\tau]$, we define when $\mathbb{X}$ *satisfies* ϕ *under* $\mathfrak{A}$, written $\mathfrak{A} \models_{\mathbb{X}} \phi$:

$\mathfrak{A} \models_{\mathbb{X}} l$	$\Leftrightarrow$	$\mathfrak{A} \models_s l$ for all $s \in \mathrm{Sup}(\mathbb{X})$ (l is a literal),
$\mathfrak{A} \models_{\mathbb{X}} (\psi \wedge \theta)$	$\Leftrightarrow$	$\mathfrak{A} \models_{\mathbb{X}} \psi$ and $\mathfrak{A} \models_{\mathbb{X}} \theta$,
$\mathfrak{A} \models_{\mathbb{X}} (\psi \vee \theta)$	$\Leftrightarrow$	$\mathfrak{A} \models_{\mathbb{Y}} \psi$ and $\mathfrak{A} \models_{\mathbb{Z}} \theta$ for some $\mathbb{Y}, \mathbb{Z}$ such that $\forall s : \mathbb{Y}(s) + \mathbb{Z}(s) = \mathbb{X}(s)$,
$\mathfrak{A} \models_{\mathbb{X}} \forall x \psi$	$\Leftrightarrow$	$\mathfrak{A} \models_{\mathbb{Y}} \psi$, where $\mathbb{Y}$ is such that $\forall s, a : \mathbb{X}(s) = \mathbb{Y}(s[a/x])$
$\mathfrak{A} \models_{\mathbb{X}} \exists x \psi$	$\Leftrightarrow$	$\mathfrak{A} \models_{\mathbb{Y}} \psi$ for some $\mathbb{Y}$ such that $\forall s : \mathbb{X}(s) = \sum_a \mathbb{Y}(s[a/x])$.

For the Boolean semiring, the above definition gives the standard team semantics presented in Definition 16. For the semiring of natural numbers, we obtain multiteam semantics (Grädel and Wilke 2022), and for the probability semiring we obtain probabilistic team semantics (Hannula et al. 2020).

The extension of first-order logic with dependence atoms is called *dependence logic*. Similarly *independence logic* and *inclusion logic* are the extensions of FO with conditional independence atoms and inclusion atoms, respectively. The interpretations of relational and dependency atoms are as defined in Section 4.3, except that the definitions are of the form $[\![-]\!]_{\mathfrak{A},\mathbb{X}} := [\![-]\!]_{\pi_{\mathfrak{A},\mathbb{X}}}$ (instead of $[\![-]\!]_{\mathbb{X}} := [\![-]\!]_{\pi_{\mathbb{X}}}$), where $\pi_{\mathfrak{A},\mathbb{X}}$ is a model defining interpretation for $\mathfrak{A}$ that encodes both $\mathfrak{A}$ and $\mathbb{X}$. We then stipulate $\mathfrak{A} \models_{\mathbb{X}} \alpha$, if $[\![\alpha]\!]_{\mathfrak{A},\mathbb{X}} \neq 0$, when α is an atom or a literal. Note that the case for literals given in the above definition coincides with the definition of Section 4.3.

It has been observed that seminal "No-Go" theorems in quantum mechanics, such at the Bell's theorem or the Kochen-Specker theorem, can be formalised as a logical entailment $\Sigma \models \phi$, where $\Sigma \cup \{\phi\}$ is a collection of dependence or independence logic formulae (Albert and Grädel 2022; Abramsky, Puljujärvi, and Väänänen 2021). In this particular context, it does not make any difference whether one considers relational or probabilistic team semantics. Also in general the two are connected: for independence logic, satisfaction in probabilistic team semantics implies satisfaction in relational team semantics, and the converse holds for dependence logic (Albert and Grädel 2022, Theorem 3.5) and (Durand et al. 2018a). Using K-teams, these results can now be stated in the following more general form.

Given a collection of atoms C, we write $\mathsf{FO}(C)$ for the extension of (negation normal form) FO with atoms in C. We say that a semiring K is $+$*-dense* if for all nonzero $a \in K$ there exist nonzero $b, c \in K$ such that $a = b + c$.

Theorem 18. *Let C and D be collections of $\mathsf{FO}(=, \not\perp?, \leq)$-definable and $\mathsf{FO}(\perp?, \neq, \nleq)$-definable atoms, resp. Assume $\phi \in \mathsf{FO}(C)$ and $\psi \in \mathsf{FO}(D)$. Let $\mathfrak{A}$ be a first-order structure, $\mathbb{X}$ a K-team over a positive semiring K, and X the possibilistic collapse of $\mathbb{X}$. Then, $\mathfrak{A} \models_{\mathbb{X}} \phi \Rightarrow \mathfrak{A} \models_X \phi$. Moreover, if K is $+$-dense, then $\mathfrak{A} \models_{\mathbb{X}} \psi \Leftarrow \mathfrak{A} \models_X \psi$.*

Proof. The proof proceeds by structural induction on the formulae; Corollary 15 is the atomic case. The case for conjunction is trivial, and the cases for ∨, ∃, and ∀ are similar to each other. We show the case of ∨. Consider the following:

$$\mathfrak{A} \models_{\mathbb{Y}} \theta_1 \text{ and } \mathfrak{A} \models_{\mathbb{Z}} \theta_2 \text{ for } \mathbb{Y},\mathbb{Z} \text{ s.t. } \forall s : \mathbb{Y}(s) + \mathbb{Z}(s) = \mathbb{X}(s)$$
$$\mathfrak{A} \models_{Y} \theta_1 \text{ and } \mathfrak{A} \models_{Z} \theta_2 \text{ for some } Y, Z \text{ such that } Y \cup Z = X$$

The first line is by definition equivalent to $\mathfrak{A} \models_{\mathbb{X}} \theta_1 \vee \theta_2$, and the second line to $\mathfrak{A} \models_X \theta_1 \vee \theta_2$. Assuming the first line, we obtain the second line with Y and Z being supports of $\mathbb{Y}$ and $\mathbb{Z}$ by +-positiveness. Assuming the second line, we obtain the first line with Y and Z being supports of $\mathbb{Y}$ and $\mathbb{Z}$ by +-denseness. □

5.2 Algebraic K-team Semantics

Next we reformulate K-team semantics via constrained polynomials $[\![\phi]\!]_{\mathfrak{A},\mathbb{X}}$ over K which can be used for provenance analysis and various counting tasks. Due to the use of identities, constrained polynomials are terms over the expansion of $(K, +, \cdot, 0, 1)$ by suitable χ-functions giving access to identity between terms. Our approach can be used to reduce satisfaction in K-team semantics to the existential first-order theory of K. Similar reductions have been utilised in the case of Boolean and probabilistic team semantics to analyse the complexity of model checking and satisfiability (Hannula et al. 2019; Hannula et al. 2020; Hannula and Virtema 2022; Durand, Kontinen, and Väänänen 2022)

Let $\mathfrak{A}$ be a finite model with universe A. Let V be a finite set of variables and $\vec{a} \in A^V$. Below $\mathbb{X}(\vec{a})$ denotes a variable over K. Observe that by fixing interpretations for $\mathbb{X}(\vec{a})$, for all $\vec{a} \in A^V$, a unique K-team $\mathbb{X}$ is determined. The constrained polynomial $[\![\phi]\!]_{\mathfrak{A},\mathbb{X}}$ contains also other variables $\mathbb{Y}(\vec{b})$ and $\mathbb{Z}(\vec{c})$ that represent new teams that arise along the evaluation of disjunctions and the quantifiers, where $\vec{b} \in A^{V_1}$ and $\vec{c} \in A^{V_2}$, for $V_1, V_2 \supseteq V$. So $[\![\phi]\!]_{\mathfrak{A},\mathbb{X}}$ defines a function from K^n to K (where n is the number of variables of $[\![\phi]\!]_{\mathfrak{A},\mathbb{X}}$), which yields a value in K once values for all the free variables have been fixed.

Definition 19. Let $\mathfrak{A}$ be a finite model and V a finite set of variables. We define *K-interpretation* $[\![\cdot]\!]_{\mathfrak{A},\mathbb{X}}$ as follows. Below χ is the characteristic function of equality (with respect to 0 and 1 from K), and a and $\vec{a}$ range over A and tuples from A, resp. Note that each tuple from $\vec{a} \in A^V$ gives rise to an assignment $s\colon V \to A$ such that $\vec{a} = (s(x_1), \dots, s(x_n)) = \vec{a}_s$.

$$[\![\phi \vee \psi]\!]_{\mathfrak{A},\mathbb{X}} = [\![\phi]\!]_{\mathfrak{A},\mathbb{Y}} \cdot [\![\psi]\!]_{\mathfrak{A},\mathbb{Z}} \cdot \prod_{\vec{a}} \chi[\mathbb{Y}(\vec{a}) + \mathbb{Z}(\vec{a}) = \mathbb{X}(\vec{a})]$$
$$[\![\phi \wedge \psi]\!]_{\mathfrak{A},\mathbb{X}} = [\![\phi]\!]_{\mathfrak{A},\mathbb{X}} \cdot [\![\psi]\!]_{\mathfrak{A},\mathbb{X}}$$
$$[\![\forall x \phi]\!]_{\mathfrak{A},\mathbb{X}} = [\![\phi]\!]_{\mathfrak{A},\mathbb{Y}} \cdot \prod_{a, s:\mathrm{Dom}(\mathbb{X}) \to A} \chi[\mathbb{X}(\vec{a}) = \mathbb{Y}(\vec{a}_{s[a/x]})]$$
$$[\![\exists x \phi]\!]_{\mathfrak{A},\mathbb{X}} = [\![\phi]\!]_{\mathfrak{A},\mathbb{Y}} \cdot \prod_{s:\mathrm{Dom}(\mathbb{X}) \to A} \chi[\mathbb{X}(\vec{a}) = \sum_{a} \mathbb{Y}(\vec{a}_{s[a/x]})]$$

For first-order literals and atoms, we utilise the interpretations defined in Section 4.3 with the modification discussed in the previous subsection. Recall that the K-interpretation of dependencies and relational atoms is defined in terms of $\pi_{\mathfrak{A},\mathbb{X}} : \mathsf{Lit}_A \to K$ mapping $R(\vec{a})$ to $\mathbb{X}(\vec{a})$, where R is a relation symbol (not in the vocabulary of $\mathfrak{A}$) representing the team $\mathbb{X}$. We further assume that, except for R, $\pi_{\mathfrak{A},\mathbb{X}}$ is identical to the canonical truth interpretation $\pi_{\mathfrak{A}}$ of $\mathfrak{A}$.

Let $T(\vec{x})$ be a first-order literal and $\mathfrak{A}$ a structure. Then, $[\![T(\vec{x})]\!]_{\mathfrak{A},\mathbb{X}}$ can be expanded into:

$$\prod_{s:\mathrm{Dom}(\mathbb{X}) \to A} \Big(\chi[\mathbb{X}(\vec{a}_s) = 0] + \chi[\mathbb{X}(\vec{a}_s) \neq 0] \cdot T\big(s(\vec{x})\big)\Big) \quad (5)$$

where $\mathbb{X}(\vec{a})$ and $T(\vec{a})$ are interpreted according to $\pi_{\mathfrak{A},\mathbb{X}}$. Now, the definitions of literals and dependencies given in Section 4.3 can be imported into the algebraic semantics by viewing strings of the form $\mathbb{X}(\vec{a})$ as variables ranging over K.

It is straightforward to show that K-team semantics of Definition 17 coincides in the following sense with algebraic K-team semantics.

Proposition 20. $\mathfrak{A} \models_{\mathbb{X}} \phi$ *iff* $\mathrm{Ran}([\![\phi]\!]_{\mathfrak{A},\mathbb{X}}) \neq \{0\}$*, where* $\mathrm{Ran}([\![\phi]\!]_{\mathfrak{A},\mathbb{X}})$ *is the range of the function defined by* $[\![\phi]\!]_{\mathfrak{A},\mathbb{X}}$ *when the interpretations of* $\mathbb{X}(\vec{a})$ *are fixed according to* $\mathbb{X}$.

Note that any constrained polynomial $[\![\phi]\!]_{\mathfrak{A},\mathbb{X}}$ can be defined by an FO-formula over K, and thus checking $\mathfrak{A} \models_{\mathbb{X}} \phi$ can be reduced (in polynomial time) to the existential first-order theory of K with additional constants for $\mathbb{X}(\vec{a})$. E.g., satisfaction of a literal $T(\vec{x})$ can be expressed by the following formula ψ over an expansion of $(K, +, \cdot, 0, 1)$ with additional constants from K:

$$\psi := \bigwedge_{\mathfrak{A} \not\models_s T(\vec{x})} \mathbb{X}(\vec{a}_s) = 0.$$

Now $[\![T(\vec{x})]\!]_{\mathfrak{A},\mathbb{X}} \neq 0$ iff $\big(K, +, \cdot, 0, 1, \mathbb{X}(\vec{a}_1), \dots, \mathbb{X}(\vec{a}_n)\big) \models \psi$. It is worth noting that formalising K-team semantics of sentences can be done in existential first-order theory of K without additional constants from K.

6 Conclusions and Future Work

We defined an extension of FO under semiring semantics with the ability of comparing semiring values of first-order formulae. We used this formalism to define concepts such as dependence and independence in a way that encompasses prior interpretations and indicated its advantages in studying the preservation of satisfaction and entailment for dependence statements between different semirings. Such preservation results have previously been studied between database and probability theory. We proposed a unifying approach inspired by semiring provenance for analysing the concepts of dependence and independence via a novel semiring team semantics, which subsumes all the previously considered variants for first-order team semantics. We discovered general explanations for the preservation of satisfaction results from team-semantics literature. We conclude by exploring some applications and directions for future work.

6.1 Axiomatisations and Logical Implication

The notions of dependence and independence are known to exhibit remarkable similarity in their behavior across various contexts in which they are defined. One example of this are the *Armstrong axioms* (Armstrong 1974), which describe the laws of inference for functional dependence in relational databases. In this context, if every two tuples in a database that agree on an attribute set X also agree on an attribute set Y, we say that Y *functionally depends* on X. The Armstrong axioms seem to capture something more fundamental and universal than just this concept. For instance, if we consider Shannon's information measures, we can say that a random variable Y depends functionally on another random variable X whenever the conditional entropy $H(Y \mid X)$ of Y given X equals 0. Similarly, in linear algebra, we may say that a subspace Y of a vector space V depends functionally on another subspace X of V if every vector of Y is a linear combination of vectors in X. In all these cases, and in many others, the Armstrong axioms are sound and complete (see, e.g., (Galliani and Väänänen 2022)).

When it comes to the notion of independence, there are similarities but also differences. As for the similarities, the axioms of *marginal independence* (here, pure independence) $X \perp_{\emptyset} Y$ formulated by (Geiger, Paz, and Pearl 1991) in the context of probability theory, are known to be sound and complete in the database context (Kontinen, Link, and Väänänen 2013). This correspondence between logical implication in probability theory and database theory extends to the so-called *saturated conditional independence* (in databases, *multivalued dependency*) $Y \perp_X Z$, where $X \cup Y \cup Z$ has to cover all variables of the joint distribution (in databases, all attributes of the relation schema) (Wong, Butz, and Wu 2000), as well as their extension with functional dependencies (Kenig and Suciu 2022). Logical implication for the general conditional independence (in databases, embedded multivalued dependencies) however is not the same for probability distributions and database relations (Studený 1992).

It is noteworthy that this connection between database theory and probability theory seems to hold as long as there exists a common foundation through information theory. Indeed, marginal independence, saturated conditional independence, and functional dependence can in both contexts be interpreted through information-theoretic measures (Lee 1987; Galliani and Väänänen 2022). On the other hand, there does not seem to exist any evident information-theoretic interpretation for the embedded multivalued dependency of database theory. The semiring approach proposed in this paper manages to unify dependency concepts from various contexts; in particular, conditional independence from probability theory and database theory. In doing so, it offers the potential to shed new light on the underlying reasons behind said similarities and differences.

To illustrate what this sort of semiring approach might reveal, we provide an example that shows how the axiomatic properties of independence may sometimes hinge on the underlying algebraic properties.

Example 21. An element a of a (commutative) semiring $K = (K, +, \cdot, 0, 1)$ is *cancellative* if for all $b, c \in K$, $ab = ac$ implies $b = c$. It can be shown that the axioms of pure independence are sound for K-teams if every element $a \in K \setminus \{0\}$ is cancellative. If this condition fails, the *mixing rule* (Geiger, Paz, and Pearl 1991) of pure independence is not necessarily sound. This rule states that $x \perp yz$ can be derived from $x \perp y$ and $xy \perp z$. For a counterexample, the ring $\mathbb{Z}_4$ of integers modulo 4 contains a non-cancellative element $2 \neq 0$. If we define a $\mathbb{Z}_4$-team $\mathbb{X}$ as in Figure 2, we observe that $\mathbb{X}$ satisfies $x \perp y$ and $xy \perp z$, but fails to satisfy $x \perp yz$.

$K = \mathbb{Z}_4$

x	y	z	$\mathbb{X}(s)$
a_0	b_0	c_0	1
a_0	b_1	c_0	1
a_1	b_0	c_1	1
a_1	b_1	c_1	1
a_2	b_2	c_0	1
a_2	b_3	c_0	1
a_3	b_2	c_1	1
a_3	b_3	c_1	1

Figure 2: Mixing fails

6.2 Provenance and Counting Proofs

The introduction of the algebraic semantics is partially motivated by the fact that $[\![\phi]\!]_{\mathfrak{A},\mathbb{X}}$ can be used for provenance analysis and counting tasks. In provenance information is extracted from *tokens* (or *annotations*). In the K-team setting, each assignment is annotated with a token. Tokens are used to trace the origin of the truth value of a given formula by interpreting an expression of some sort. Our goal is to understand how a formula ends up being true in a first-order structure with K-team semantics.

Let K be a commutative positive semiring. If a formula ϕ is true in a non-empty K-team, we would like to obtain a polynomial expression involving the semiring values given to each assignment of the K-team that explains the truth of ϕ. If the formula is false we would like such expression to return 0.

Note that we already obtained a polynomial expression in Section 5.2, where the annotations played a role in the definition of algebraic K-team semantics. However, the literals' truth values lacked annotations. For a concrete K-team $\mathbb{X}$, the number of different ways of satisfying a formula ϕ over $\mathfrak{A}$ and $\mathbb{X}$ corresponds to the cardinality of the support of $[\![\phi]\!]_{\mathfrak{A},\mathbb{X}}$ (i.e., the number of assignments with domain $\mathrm{Dom}([\![\phi]\!]_{\mathfrak{A},\mathbb{X}}) \setminus \mathrm{Dom}(\mathbb{X})$ such that the expression returns a nonzero value). Moreover, $[\![\phi]\!]_{\mathfrak{A},\mathbb{X}}$ can be devised for counting the number of K-teams $\mathbb{X}$ that satisfy ϕ over $\mathfrak{A}$ (cf. (Haak et al. 2019)).

To trace provenance, we define the following sentence for literals in a similar manner as in Section 4.3:

$$\phi^{\vec{i}}_{\text{prov-}T} \coloneqq \forall \vec{x}\big(R(\vec{x}) = \bot \vee \big(R(\vec{x}) \wedge T(\vec{x}_{\vec{i}})\big)\big)$$

Then, we define K-team provenance semantics using an analogous interpretation for first-order literals and atoms as the one defined in Section 4.3. If $T(\vec{x})$ is a first-order literal, then $[\![T(\vec{x})]\!]_{\mathbb{X}}$ can be expanded into:

$$\prod_{s:\mathrm{Dom}(\mathbb{X})\to A} \Big(\chi[\mathbb{X}(\vec{a}_s) = 0] + \mathbb{X}(\vec{a}_s) \cdot T\big(s(\vec{x})\big)\Big).$$

This is extended for general formulae as in Section 5.2.

6.3 Repairs

To transform a database to an accurate reflection of the domain it is intended to model, some properties and conditions are imposed on the possible instances to avoid inconsistency. A notion of consistency of the database is then related to a set of ICs, which express some of the semantic structure that the data intends (or needs) to represent. It is common for a database to become inconsistent due to several reasons. When a database does not satisfy its ICs, one possible approach is to perform minimal changes to obtain a “similar” database that satisfies the constraints. Such a database is called a *repair* (Arenas, Bertossi, and Chomicki 1999), and to define it properly one has to precisely determine the meaning of “minimal change”. Several definitions have been proposed and studied in the literature, usually given in terms of a distance or partial order between database instances. Which notion to use may depend on the application.

We use K-team semantics to determine whether a given K-team $\mathbb{X}$ satisfies a set of ICs. Assuming we have a way to measure distances between K-teams, if the ICs are not satisfied we could ask for a *repair* of $\mathbb{X}$ that does. That is, a K-team $\mathbb{Y}$ such that the ICs are satisfied in $\mathbb{Y}$, and $\mathbb{X}$ and $\mathbb{Y}$ minimally differ in terms of the desired distance. In what follows, we restrict to ordered semirings and stipulate the existence of additive inverses (i.e., ordered rings).

Since K-team semantics allows to define dependencies in K-teams, one could ask for a notion of a repair that takes into account either dependence or independence. One possibility is to define a quantitative notion of non-independence to a K-team by assigning a value in the semiring using the weights of the assignments, indicating how far away we are from having independence. When looking at the independence atom defined in Section 4.3, we interpret that we have independence between $\vec{x}$ and $\vec{y}$ in a K-team $\mathbb{X}$ if the equality $\sum_s \mathbb{X}(s) \cdot \sum_{s(\vec{x}\vec{y})=\vec{a}\vec{b}} \mathbb{X}(s) = \sum_{s(\vec{x})=\vec{a}} \mathbb{X}(s) \cdot \sum_{s(\vec{y})=\vec{b}} \mathbb{X}(s)$ holds for every pair $\vec{a}, \vec{b}$. If instead $[\![x \perp y]\!]_{\mathbb{X}} = 0$, then at least one of these terms is false. Hence, for every pair $\vec{a}, \vec{b}$ for which the equality does not hold, we measure how far away they are from being equal. More precisely, we consider:

$$[\![\vec{x} \not\perp \vec{y}]\!]_{\mathbb{X}} = \sum_{\vec{a},\vec{b}} \Big|\sum_s \mathbb{X}(s) \cdot \sum_{s(\vec{x}\vec{y})=\vec{a}\vec{b}} \mathbb{X}(s) - \sum_{s(\vec{x})=\vec{a}} \mathbb{X}(s) \cdot \sum_{s(\vec{y})=\vec{b}} \mathbb{X}(s)\Big|$$

where the module $|a - b|$ is defined as $a - b$ if $a - b > 0$, and $b - a$ otherwise, for any $a, b \in K$.

We now present a natural way to define distance between K-teams using the values in the semiring, and then introduce some notions of K-team repairs.

Let $\mathbb{X}, \mathbb{Y}$ be two K-teams. We define the *symmetric difference of* $\mathbb{X}$ *and* $\mathbb{Y}$, denoted by $\mathbb{X} \triangle \mathbb{Y}$, as the K-team with weights $(\mathbb{X} \triangle \mathbb{Y})(s)$ defined as:

$$(\mathbb{X} \triangle \mathbb{Y})(s) = |\mathbb{X}(s) - \mathbb{Y}(s)|$$

Using this, we define a distance between $\mathbb{X}$ and $\mathbb{Y}$ as:

$$dist(\mathbb{X}, \mathbb{Y}) = \sum_s (\mathbb{X} \triangle \mathbb{Y})(s)$$

Notice that, if we already have some kind of norm or distance in K (or K^k), then we can consider instead said norm as a distance. Moreover, if we have a distance in K, then we do not need to ask for additive inverses in K.

Some notions of repairs that arise naturally in this context are the following: given a K-team $\mathbb{X}$ and a formula ϕ,

- A *symmetric difference repair* of $\mathbb{X}$ w.r.t. ϕ is a K-team $\mathbb{Y}$ that satisfies ϕ and is such that $dist(\mathbb{X}, \mathbb{Y}) \leq dist(\mathbb{X}', \mathbb{Y})$ for all K-teams $\mathbb{X}'$ satisfying ϕ. If K is the Boolean semiring, this notion becomes the cardinality-based repair known as the *C-repair* (Lopatenko and Bertossi 2007).
- A *subteam repair* (resp. *superteam repair*) of $\mathbb{X}$ w.r.t. ϕ is a K-team $\mathbb{Y}$ that is a subteam (resp. superteam) of $\mathbb{X}$ satisfying ϕ, and such that $dist(\mathbb{X}, \mathbb{Y}) \leq dist(\mathbb{X}', \mathbb{Y})$ for all subteams (resp. superteams) of $\mathbb{X}$ satisfying ϕ.
- Assuming ϕ is of the form $\vec{x} \perp \vec{y}$, we can also consider notions of repairs that minimise $[\![\vec{x} \not\perp \vec{y}]\!]_{\mathbb{X}\triangle\mathbb{Y}}$.

6.4 Complexity and K-machines

Similar to the way we generalised team semantics over semirings, there have been several approaches to do the same for computational complexity. A prominent related model of computation is the so-called *BSS-model* (Blum, Shub, and Smale 1989). A BSS-machine over a semiring K can be thought of as a Turing machine which has a tape of K-valued registers instead of just zeros and ones. The transition function then allows evaluating polynomial functions on a fixed interval of the tape in a single step. In their book (Blum et al. 1997), Blum, Cucker, Shub, and Smale predominantly use this model of computation to investigate questions in the realms of real and complex numbers and shed light on the differences between them, and the Turing model. Indeed, changing the underlying semiring often leads to profound complexity theoretic implications. Take the Hilbert’s 10th problem for example, the question whether a multivariate polynomial with integer coefficients has an integer solution is undecidable. However, asking for real solutions leads the problem to become decidable. Moreover, it is an open problem whether there exists a general decision procedure to check the existence of rational solutions.

It is a fascinating avenue for future work to investigate what general results can be proven for our formalisms in the context of BSS-complexity. In the Boolean setting most team-based logics are known to characterise NP (Durand, Kontinen, and Vollmer 2016) (and thus NP on BSS-machines with access to the Boolean semiring), while (Hannula et al. 2020) show a corresponding characterisation between probabilistic independence logic and NP on a variant of BSS-machines with access to the probabilistic semiring.

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