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Finest Syntax Splittings of Ranking Functions and Total Preorders on Worlds

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Abstract

The notion of syntax splitting was initially introduced by Parikh for belief sets, and one key observation is that every belief set has a unique finest syntax splitting, i.e., a syntax splitting that refines every other syntax splitting of that belief set. Later, the notion of syntax splitting was extended to ranking functions and total preorders on worlds (TPOs), which are two common models for belief states in the context of iterated belief revision. In this paper, we prove that ranking functions also have unique finest syntax splittings, i.e., every ranking function has a syntax splitting that refines all other syntax splittings of that ranking function. Using this, we can show that the syntax splittings of a ranking function κ are exactly the coarsenings of the finest splitting of κ. For TPOs we show that, in contrast to ranking functions, the coarsening of a syntax splitting of a TPO $\preceq$ is not necessarily a syntax splitting of $\preceq$. Despite that we can prove that every TPO has a unique finest syntax splitting that refines all other syntax splittings of that TPO.

1 Introduction

The notion of syntax splitting was first introduced in (Parikh 1999) for belief sets. For a given belief set K over a signature Σ, a syntax splitting of K is a partition S of Σ, such that the information in K can be expressed as conjunction of independent formulas that each use only atoms from one of the sub-signatures in S. A belief set having a syntax splitting means that the belief set is made up of independent information over different parts of the signature. Syntax splittings are useful properties of a belief set as they indicate that different parts of the belief set can be processed independently of each other. Syntax splittings has been found to be beneficial especially in the presence of belief change (Peppas et al. 2015; Kern-Isberner and Brewka 2017; Haldimann, Kern-Isberner, and Beierle 2020). One result in (Parikh 1999) is that every belief base has a unique finest syntax splitting, that refines all other syntax splittings of that belief base. As the coarsening of a syntax splitting is again a syntax splitting, the set of syntax splittings of a belief set K can be described as the set of all coarsenings of the finest splitting of K.

To handle not only propositional, but also conditional beliefs, the belief state of an agent (also called epistemic state) can be represented by more expressive frameworks like ranking functions (Spohn 1988) and total preorders on worlds (TPOs) (Katsuno and Mendelzon 1992; Darwiche and Pearl 1997). In (Kern-Isberner and Brewka 2017), the notion of syntax splitting was extended to ranking functions and TPOs, and corresponding syntax splitting postulates for the revision of ranking functions and TPOs were introduced.

In this paper we show that ranking functions also have finest syntax splittings. Analogously to belief sets, every ranking function κ has a unique finest syntax splitting that refines all other syntax splittings of κ. The coarsening of a syntax splitting of κ is again a syntax splitting. Thus, for ranking functions, we can describe the set of all syntax splittings of a ranking function κ as the set of all coarsenings of κ's finest syntax splitting. However, for TPOs on worlds, this does not work. There are examples where the coarsening of a syntax splitting of a TPO $\preceq$ is not a syntax splitting for $\preceq$. Nevertheless, we can show that every TPO $\preceq$ has a finest syntax splitting, that refines every other splitting of $\preceq$.

Among other things, the existence of finest splittings is beneficial for situations that exploit syntax splittings. For belief changes it is in some cases sufficient to focus on the finest splitting as many splitting properties carry over to coarser splittings automatically. In belief change or in nonmonotonic reasoning, focussing on finest splittings yields the maximal benefit of applying splitting techniques, e.g., (Parikh 1999; Kern-Isberner and Brewka 2017; Haldimann, Kern-Isberner, and Beierle 2020; Kern-Isberner, Beierle, and Brewka 2020).

In summary, the main contributions of this paper are showing that every ranking function has a unique finest syntax splitting, and that every TPO has a unique finest syntax splitting despite the observation that not every coarsening of a syntax splitting of a TPO is again a syntax splitting.

In Sec. 2 we present the background on ranking functions and TPOs, and in Sec. 3 we recall the definitions of syntax splittings. In Sec. 4 we show that ranking functions have finest syntax splittings, and in Sec. 5 we show that TPOs have finest syntax splittings before concluding in Sec. 6.

2 Logic, Ranking Functions, and TPOs

A *(propositional) signature* is a finite set Σ of identifiers. For a signature Σ, we denote the propositional language over Σ by $\mathcal{L}_\Sigma$. Usually, we denote elements of the signatures with lowercase letters $a, b, c, \ldots$ and formulas with uppercase letters $A, B, C, \ldots$. We may denote a conjunction $A \wedge B$ by AB and a negation $\neg A$ by $\overline{A}$ for brevity of notation. As usual, $\top$ denotes a tautology and $\perp$ an unsatisfiable formula. The set of interpretations over a signature Σ is denoted as Ω_Σ. Interpretations are also called *worlds*. An interpretation $\omega \in \Omega_\Sigma$ is a *model* of a formula $A \in \mathcal{L}_\Sigma$ if A holds in ω. This is denoted as $\omega \models A$. The set of models of a formula (over a signature Σ) is denoted as $Mod_\Sigma(A) = \{\omega \in \Omega_\Sigma \mid \omega \models A\}$. We will represent interpretations (or worlds) by complete conjunctions, e.g., the interpretation over $\Sigma_{abc} = \{a, b, c\}$ that maps a and c to *true* and b to *false* is represented by $a \wedge \neg b \wedge c$, or just $a\bar{b}c$. The complete conjunction assigned to a world is unique up to the order of the literals. Thus, every world $\omega \in \Omega_\Sigma$ is also a formula in $\mathcal{L}_\Sigma$.

A formula A *entails* a formula B, denoted by $A \models B$, if $Mod_\Sigma(A) \subseteq Mod_\Sigma(B)$. The *deductive closure* of a set of formulas F is $\mathrm{Cn}_\Sigma(F) = \{A \in \mathcal{L}_\Sigma \mid F \models A\}$; a set of formulas F is called *deductively closed* if $\mathrm{Cn}_\Sigma(F) = F$. A deductively closed set of formulas is also called a *belief set*.

A *ranking function*, also called *ordinal conditional function* (OCF), is a function $\kappa : \Omega_\Sigma \to \mathbb{N}_0$ such that $\kappa^{-1}(0) \neq \emptyset$; ranking functions were first introduced (in a more general form) by Spohn (Spohn 1988). The intuition of a ranking function is that the rank of a world is lower if the world is more plausible. Therefore, ranking functions can be seen as some kind of "implausibility measure". For a ranking function κ and a set X of worlds, $\min_{\omega \in X} \kappa(\omega)$ denotes the minimal rank $\kappa(\omega)$ among the worlds $\omega \in X$; for empty sets we define $\min_{\omega \in \emptyset} \kappa(\omega) = \infty$. Ranking functions are extended to formulas by $\kappa(A) = \min_{\omega \in Mod(A)} \kappa(\omega)$.

A *total preorder* (TPO) is a total, reflexive, and transitive binary relation. The meaning of a TPO $\preceq$ on Ω_Σ as model for an epistemic state is that if $\omega_1 \preceq \omega_2$ then ω_1 is at least as plausible as ω_2 for $\omega_1, \omega_2 \in \Omega_\Sigma$. The strict version of a TPO $\preceq$ is the relation $\prec$ defined by $\omega_1 \prec \omega_2$ iff $\omega_1 \preceq \omega_2$ and $\omega_2 \not\preceq \omega_1$. For a TPO $\preceq$ and a set X of worlds, $\min(X, \preceq)$ denotes the set of minimal worlds in X with respect to $\preceq$. TPOs on worlds are extended to formulas $A, B \in \mathcal{L}_\Sigma$ by defining $A \preceq B$ iff there is an $\omega_1 \in Mod_\Sigma(A)$ such that for every $\omega_2 \in Mod_\Sigma(B)$ it holds that $\omega_1 \preceq \omega_2$.

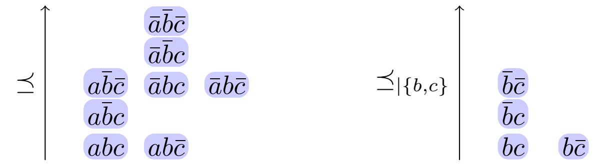
Marginalization realizes focusing on certain signature elements; among other things it is essential to syntax splitting for epistemic states.

Let κ be a ranking function over signature Σ. The marginalization of κ to a sub-signature $\Sigma' \subseteq \Sigma$ is the ranking function $\kappa_{|\Sigma'}$ defined by $\kappa_{|\Sigma'}(\omega) = \kappa(\omega)$ for $\omega \in \Omega_{\Sigma'}$ (Spohn 1988; Beierle and Kern-Isberner 2012). Note that ω is considered as a world over Σ' on the left hand side and as a formula over the larger signature Σ on the right hand side of the equation.

Let $\preceq$ be a TPO over signature Σ. The marginalization of $\preceq$ to a sub-signature $\Sigma' \subseteq \Sigma$ is the TPO $\preceq_{|\Sigma'}$ defined by $\omega_1 \preceq_{|\Sigma'} \omega_2$ iff $\omega_1 \preceq \omega_2$ for $\omega_1, \omega_2 \in \Omega_{\Sigma'}$ (Beierle and Kern-Isberner 2012; Kern-Isberner and Brewka 2017).

Example 1. *Consider the TPO $\preceq$ in Figure 1(a). The marginalization $\preceq_{|\{b,c\}}$ of $\preceq$ is shown in Figure 1(b).*

The marginalizations of OCFs and TPOs presented above are special cases of general forgetful functors $Mod(\varrho)$ from Σ-models to Σ'-models given in (Beierle and Kern-Isberner 2012) where $\Sigma' \subseteq \Sigma$ and $\varrho : \Sigma' \hookrightarrow \Sigma$ is the inclusion from Σ' to Σ. Informally, a forgetful functor forgets everything about the interpretation of the symbols in $\Sigma \setminus \Sigma'$ when mapping a Σ-model to a Σ'-model.



(a) TPO $\preceq$ with the syntax splitting $S = \{\{a\}, \{b, c\}\}$.

(b) Marginalization $\preceq_{|\{b,c\}}$ of $\preceq$ to $\{b, c\}$.

Figure 1: A TPO with syntax splitting and its marginalization.

The marginalization of a world $\omega \in \Omega_\Sigma$ to a sub-signature $\Sigma' \subseteq \Sigma$ is the valuation of the variables in Σ' as in ω and is denoted by $\omega_{|\Sigma'}$. E.g., the marginalization of $\omega = a\bar{b}c$ to $\Sigma' = \{b, c\}$ is $\omega_{|\Sigma'} = \bar{b}c$. For a world ω and $\Sigma_i \subseteq \Sigma$ the marginalization $\omega_{|\Sigma_i}$ may be abbreviated by ω^i and the marginalization $\omega_{|\Sigma \setminus \Sigma_i}$ by $\omega^{\neq i}$ if no confusion arises.

3 Syntax Splittings

An interesting feature of ranking functions and total preorders is the existence of syntax splittings. Syntax splittings were first introduced as a property of belief sets by Parikh.

Definition 1 (syntax splittings of belief sets (Parikh 1999))**.** *Let K be a belief set over Σ. A partitioning $\{\Sigma_1, \ldots, \Sigma_n\}$ of Σ is a* syntax splitting *for K, if there are formulas $A_1, \ldots, A_n$ with $K = \mathrm{Cn}_\Sigma(A_1, \ldots, A_n)$ and $A_i \in \mathcal{L}_{\Sigma_i}$ for $i = 1, \ldots, n$.*

Informally, the meaning of a belief set having a syntax splitting is that the belief set contains independent information over different parts of the signature. The partition of the signature in these parts is called a syntax splitting for the considered belief set. Syntax splittings are useful properties as they indicate that different parts of the belief state can be processed independently of each other. This can be used to formulate postulates for sensible reasoning and revision operators. Additionally, splitting belief states and processing their parts independently can make operations computationally more efficient. The notion of syntax splitting was extended to other representations of epistemic states such as TPOs and OCFs in (Kern-Isberner and Brewka 2017).

Definition 2 (syntax splitting for TPOs (Kern-Isberner and Brewka 2017))**.** *Let $\preceq$ be a TPO over a signature Σ. A partitioning $\{\Sigma_1, \ldots, \Sigma_n\}$ of Σ is a* syntax splitting *for $\preceq$ if, for $i = 1, \ldots, n$,*

$$\omega_1^{\neq i} = \omega_2^{\neq i} \quad \textit{implies} \quad \big(\omega_1 \preceq \omega_2 \textit{ iff } \omega_1^i \preceq_{|\Sigma_i} \omega_2^i\big).$$

This basically states that the order of two worlds that differ only in one partition of a syntax splitting does not depend on the actual variable assignment outside of this partition.

Example 2. *Consider the TPO $\preceq$ in Figure 1(a). It has the splitting $S = \{\{a\}, \{b, c\}\}$. The partitioning $T = \{\{a, b\}, \{c\}\}$ is not a splitting for $\preceq$ since $ab\bar{c} \preceq abc$ but $a\bar{b}\bar{c} \not\preceq a\bar{b}c$.*

Definition 3 (syntax splitting for ranking functions (Kern-Isberner and Brewka 2017))**.** *Let κ be a ranking function over*

Σ. A partitioning $\{\Sigma_1, \ldots, \Sigma_n\}$ of Σ is a syntax splitting *for κ if there are ranking functions $\kappa_i : \Sigma_i \to \mathbb{N}_0$ for $i = 1, \ldots, n$ such that $\kappa(\omega) = \kappa_1(\omega^1) + \cdots + \kappa_n(\omega^n)$. This is denoted as $\kappa = \kappa_1 \oplus \cdots \oplus \kappa_n$.*

4 Finest Syntax Splittings of OCFs

Partitions, and thus also syntax splittings, can be compared according to their fineness. Let $S = \{\Sigma_1, \ldots, \Sigma_n\}$ and $S' = \{\Sigma'_1, \ldots, \Sigma'_m\}$ be two partitions of the same set Σ. S' is called a *refinement* of S (or equivalently, S a *coarsening* of S') iff for any $\Sigma'_i \in S'$ there is a $\Sigma_j \in S$ such that $\Sigma'_i \subseteq \Sigma_j$. The relation X *is a refinement of* Y is a partial order on the set of partitions of Σ, i.e., it is reflexive, transitive, and antisymmetric. Among the syntax splittings of a belief set, we can always find a finest syntax splitting.

Proposition 1 ((Parikh 1999), Lemma 1)**.** *Let K be a belief set over a signature Σ. There is a syntax splitting $S^f = \{\Sigma^f_1, \ldots, \Sigma^f_n\}$ of K such that every other syntax splitting of K is a coarsening of S^f.*

Notably, the coarsening of a syntax splitting of a belief set K is again a syntax splitting for K. Using this, all syntax splittings of a belief base can be described as coarsenings of the belief base's finest splitting.

Proposition 2 ((Parikh 1999), Remark p. 4)**.** *Let K be a belief set over a signature Σ and $S = \{\Sigma_1, \ldots, \Sigma_n\}$ be a syntax splitting of K. Then every coarsening of S is a syntax splitting of K.*

The following proposition is a direct consequence of Propositions 1 and 2.

Proposition 3. *Let K be a belief set over Σ. There is a unique syntax splitting $S^f = \{\Sigma^f_1, \ldots, \Sigma^f_n\}$ of K such that the coarsenings of S^f are exactly the syntax splittings of K.*

Analogously to belief sets, we can show that ranking functions have finest syntax splittings.

Proposition 4. *Let κ be a ranking function over signature Σ. There is a syntax splitting $S^f = \{\Sigma^f_1, \ldots, \Sigma^f_n\}$ of κ such that every other syntax splitting of κ is a coarsening of S^f.*

Proof. Towards a contradiction assume that there is a ranking function κ over a signature Σ such that there is no syntax splitting S^f that refines all other syntax splittings of κ. Let $S = \{\Sigma_1, \ldots, \Sigma_n\}$ be a syntax splitting of κ such that no refinement of S (except S itself) is a syntax splitting of κ. We can always find such a splitting, because Σ has only finitely many partitions and $\{\Sigma\}$ is trivially a splitting of every ranking function. By assumption, S cannot be a refinement of every other syntax splitting of κ; therefore there is another syntax splitting $S' = \{\Sigma'_1, \ldots, \Sigma'_m\}$ of κ that is not a coarsening of S.

Let $IJ = \{(i,j) \mid \Sigma_i \in S,\ \Sigma'_j \in S' \text{ with } \Sigma_i \cap \Sigma'_j \neq \emptyset\}$. For any $(i,j) \in IJ$ let $\Sigma^*_{ij} = \Sigma_i \cap \Sigma'_j$. Now consider the set $S^* = \{\Sigma^*_{ij} \mid (i,j) \in IJ\}$. The set S^* is a partition of Σ, and it is a refinement of both S and S'.

Let ω_0 be a world with $\kappa(\omega_0) = 0$. For any $(i,j) \in IJ$ let $\omega_0^{\neq ij} \in \Omega_{\Sigma \setminus \Sigma^*_{ij}}$ be the world with the truth assignment of ω_0 over $\Sigma \setminus \Sigma^*_{ij}$, and define $\kappa^*_{ij}(\omega_x) = \kappa(\omega_0^{\neq ij}\omega_x)$ for worlds $\omega_x \in \Omega_{\Sigma_{ij}}$. For $(i,j) \in IJ$ the function κ^*_{ij} is a ranking function over Σ^*_{ij} as $\kappa^*_{ij}(\omega_0^{ij}) = \kappa(\omega_0) = 0$. We will show that S^* is a syntax splitting of κ with $\kappa = \bigoplus_{(i,j) \in IJ} \kappa_{ij}$, contradicting the assumption that S has no refinement.

Let ω be any world. We have

$$\kappa(\omega) = \kappa_1(\omega^1) + \cdots + \kappa_n(\omega^n) \tag{1}$$

as S is a syntax splitting of κ. Furthermore, we have

$$\kappa_i(\omega^i) = \kappa_i(\omega^i) + \underbrace{\sum_{k=1;\ k\neq i}^{n} \kappa_k(\omega_0^k)}_{=0} = \kappa(\omega^i \omega_0^{\neq i}). \tag{2}$$

Let $\omega_{i \nearrow \Sigma} = \omega^i \omega_0^{\neq i}$. Using the syntax splitting S' we obtain

$$\kappa(\omega_{i \nearrow \Sigma}) = \kappa_1(\omega^1_{i \nearrow \Sigma}) + \cdots + \kappa_m(\omega^m_{i \nearrow \Sigma}).$$

Let $J_i = \{j \in \{1, \ldots, m\} \mid \Sigma_i \cap \Sigma'_j \neq \emptyset\}$ be the set of indices j such that Σ_i and Σ'_j have at least one common element. Note that $J_i = \{j \mid (i,j) \in IJ\}$. For $j \in \{1, \ldots, m\}$ with $j \notin J_i$ we have $\kappa_j(\omega^j_{i \nearrow \Sigma}) = \kappa_j(\omega^j_0) = 0$. Therefore,

$$\kappa(\omega_{i \nearrow \Sigma}) = \sum_{j \in J_i} \kappa_j(\omega^j_{i \nearrow \Sigma}). \tag{3}$$

As in (2), we have

$$\kappa_j(\omega^j_{i \nearrow \Sigma}) = \kappa_j(\omega^j_{i \nearrow \Sigma}) + \sum_{l=1;\ l \neq j}^{n} \kappa_l(\omega^l_0) = \kappa(\omega^j_{i \nearrow \Sigma}\, \omega_0^{\neq j}). \tag{4}$$

Observe that the valuation of world $\omega^j_{i \nearrow \Sigma}\ \omega_0^{\neq j}$ over Σ coincides with the valuation of ω over the atoms in $\Sigma_i \cap \Sigma_j$; and that the valuation of $\omega^j_{i \nearrow \Sigma}\ \omega_0^{\neq j}$ coincides with the valuation of ω_0 over the atoms in $\Sigma \setminus (\Sigma_i \cap \Sigma_j)$. Therefore, $\omega^j_{i \nearrow \Sigma}\ \omega_0^{\neq j} = \omega_0^{\neq ij}\omega^{ij}$ which implies

$$\kappa(\omega^j_{i \nearrow \Sigma}\ \omega_0^{\neq j}) = \kappa(\omega_0^{\neq ij}\omega^{ij}) = \kappa^*_{ij}(\omega^{ij}). \tag{5}$$

In summary, we get $\kappa(\omega) \overset{(1)}{=} \kappa_1(\omega^1) + \cdots + \kappa_n(\omega^n) \overset{(2)}{=} \sum_{i=1}^n \kappa(\omega_{i \nearrow \Sigma}) \overset{(3)}{=} \sum_{i=1}^n \sum_{j \in J_i} \kappa_j(\omega^j_{i \nearrow \Sigma}) = \sum_{(i,j) \in IJ} \kappa_j(\omega^j_{i \nearrow \Sigma}) \overset{(4),(5)}{=} \sum_{(i,j) \in IJ} \kappa_{ij}(\omega^{ij})$ and therefore $\kappa = \bigoplus_{(i,j) \in IJ} \kappa_{ij}$. □

As the coarsening of a syntax splitting of a ranking function κ is again a syntax splitting for κ, we can obtain results similar to Propositions 2 and 3 for ranking functions: All syntax splittings of a ranking function are coarsenings of the ranking function's finest splitting.

Proposition 5. *Let κ be a ranking function over a signature Σ and $S = \{\Sigma_1, \ldots, \Sigma_n\}$ be a syntax splitting of κ. Then every coarsening of S is a syntax splitting of κ.*

Proof. Let $\kappa = \kappa_1 \oplus \cdots \oplus \kappa_n$ be a ranking function with syntax splitting $\{\Sigma_1, \ldots, \Sigma_n\}$. Let $i, j \in \{1, \ldots, n\}$. Then $\{\Sigma_k \mid k \in \{1, \ldots, n\} \setminus \{i,j\}\} \cup \{\Sigma_i \cup \Sigma_j\}$ is a syntax

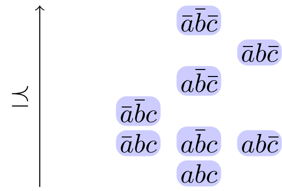


Figure 2: TPO $\preceq$ on $\Sigma = \{a, b, c\}$ with syntax splitting $\{\{a\}, \{b\}, \{c\}\}$. The partitions $\{\{a, b\}, \{c\}\}$, $\{\{a\}, \{b, c\}\}$, and $\{\{a, c\}, \{b\}\}$ are no syntax splittings of $\preceq$.

splitting of $\kappa = \bigoplus_{k=1;\ k\neq i;\ k\neq j}^{n} \kappa_k \quad \oplus \quad (\kappa_i \oplus \kappa_j)$. Therefore, combining two sub-signatures in a syntax splitting of a ranking function κ results in another syntax splitting for κ.

Any coarsening of S can be obtained by iteratively combining sub-signatures in S. Hence, if S is a syntax splitting of a ranking function κ then every coarsening of S is a syntax splitting of κ. $\square$

Proposition 6. *Let κ be a ranking function over Σ. There is a unique syntax splitting $S^f = \{\Sigma_1^f, \dots, \Sigma_n^f\}$ of κ such that the coarsenings of S^f are exactly the syntax splittings of κ.*

Proof. Follows directly from Propositions 4 and 5. $\square$

5 Finest Syntax Splittings of TPOs

One important difference between syntax splittings on TPOs and syntax splittings on ranking functions is that coarsenings of a syntax splitting of a TPO $\preceq$ are not necessarily syntax splittings of $\preceq$.

Example 3 ((Haldimann, Beierle, and Kern-Isberner 2021; Haldimann, Beierle, and Kern-Isberner 2023)). *The TPO $\preceq$ displayed in Figure 2 has the syntax splitting $S = \{\{a\}, \{b\}, \{c\}\}$. The coarsenings $\{\{a, b\}, \{c\}\}$, $\{\{a\}, \{b, c\}\}$, and $\{\{a, c\}, \{b\}\}$ of S are no syntax splittings of $\preceq$.*

Because coarsenings of syntax splittings of a TPO may not be syntax splittings, the characterization of all syntax splittings of a TPO as coarsenings of a finest syntax splitting is not possible for TPOs. Nevertheless, we can define the finest splitting of a TPO $\preceq$ to be the syntax splitting of $\preceq$ that refines every other splitting of $\preceq$, and we can still prove that every TPO has a finest syntax splitting.

Proposition 7. *Let $\preceq$ be a TPO on worlds over Σ. There is a syntax splitting $S^f = \{\Sigma_1^f, \dots, \Sigma_n^f\}$ of $\preceq$ such that every other syntax splitting of $\preceq$ is a coarsening of S^f.*

Proof. Towards a contradiction assume that there is a TPO $\preceq$ over a signature Σ such that there is no syntax splitting S^f that refines all other syntax splittings of κ. Let $S = \{\Sigma_1, \dots, \Sigma_n\}$ be a syntax splitting of $\preceq$ such that no refinement of S (except S itself) is a syntax splitting of $\preceq$. Note that S exists because Σ has only finitely many partitions and $\{\Sigma\}$ is trivially a splitting of every TPO. By assumption, S cannot be a refinement for every other syntax splitting of $\preceq$; thus, there is another syntax splitting $S' = \{\Sigma'_1, \dots, \Sigma'_m\}$ of $\preceq$ such that S is not a refinement of S'.

Now consider the set $S^* = \{\Sigma_i \cap \Sigma'_j \mid \Sigma_i \in S,\ \Sigma'_j \in S' \text{ with } \Sigma_i \cap \Sigma'_j \neq \emptyset\}$. The set S^* is a partition of Σ , and it is a refinement of both S and S'. We will show that S^* is a syntax splitting of $\preceq$ contradicting the assumption that S has no refining syntax splitting. To show that S^* is a splitting of $\preceq$, we show that for every $\Sigma_k^* \in S^*$ and any $\omega_1, \omega_2 \in \Omega_\Sigma$ it holds that $\omega_1^{\neq k} = \omega_2^{\neq k}$ implies $(\omega_1 \preceq \omega_2 \text{ iff } \omega_1^k \preceq_{|\Sigma_k^*} \omega_2^k)$.

Let $\Sigma_k^* \in S^*$ and $\omega_1, \omega_2 \in \Omega_\Sigma$ with $\omega_1^{\neq k} = \omega_2^{\neq k}$. By the construction of S^* there are $\Sigma_i \in S$ and $\Sigma'_j \in S'$ such that $\Sigma_k^* = \Sigma_i \cap \Sigma'_j$. Because $\omega_1^{\neq k} = \omega_2^{\neq k}$ we have that $\omega_1^{\neq i} = \omega_2^{\neq i}$ and $\omega_1^{\neq j} = \omega_2^{\neq j}$. We show that $\omega_1 \preceq \omega_2$ iff $\omega_1^k \preceq_{|\Sigma_k^*} \omega_2^k$ by distinguishing two cases.

Case 1: $\omega_1^k \preceq_{|\Sigma_k^*} \omega_2^k$. Let

$$\omega_1^{k\nearrow i} = \min(\{\omega_x^i \in \Omega_{\Sigma_i} \mid \omega^i_{x|\Sigma_k^*} = \omega_1^k\},\ \preceq_{|\Sigma_i})$$
$$\omega_2^{k\nearrow i} = \omega_2^k\ (\omega_1^{k\nearrow i}{}_{|\Sigma_i \setminus \Sigma_k^*}).$$

The worlds $\omega_1^{k\nearrow i}$ and $\omega_2^{k\nearrow i}$ coincide on $\Sigma_i \setminus \Sigma_k^*$. The world $\omega_1^{k\nearrow i}$ coincides with ω_1^k on Σ_k^*; the world $\omega_2^{k\nearrow i}$ coincides with ω_2^k on Σ_k. Because $\omega_1^{k\nearrow i}$ is chosen minimally and $\omega_1^k \preceq_{|\Sigma_k^*} \omega_2^k$ we have that $\omega_1^{k\nearrow i} \preceq_{|\Sigma_i} \omega_2^{k\nearrow i}$. Let

$$\omega_1^{k\nearrow i\nearrow\Sigma} = \omega^{\neq i}\ \omega_1^{k\nearrow i} \quad \text{and} \quad \omega_2^{k\nearrow i\nearrow\Sigma} = \omega^{\neq i}\ \omega_2^{k\nearrow i}.$$

The worlds $\omega_1^{k\nearrow i\nearrow\Sigma}$ and $\omega_2^{k\nearrow i\nearrow\Sigma}$ coincide on $\Sigma \setminus \Sigma_i$. We have $\omega_1^{k\nearrow i\nearrow\Sigma} \preceq \omega_2^{k\nearrow i\nearrow\Sigma}$ because $\omega_1^{k\nearrow i} \preceq_{|\Sigma_i} \omega_2^{k\nearrow i}$ and S is a syntax splitting of $\preceq$ containing Σ_i.

By construction the worlds $\omega_1^{k\nearrow i\nearrow\Sigma}$ and $\omega_2^{k\nearrow i\nearrow\Sigma}$ coincide on $\Sigma \setminus \Sigma'_j$. The world $\omega_1^{k\nearrow i\nearrow\Sigma}$ coincides with ω_1 on Σ'_j; the world $\omega_2^{k\nearrow i\nearrow\Sigma}$ coincides with ω_2 on Σ'_j. Using that S' is a syntax splitting of $\preceq$ containing Σ'_j we have $\omega_{1|\Sigma'_j} \preceq_{|\Sigma'_j} \omega_{2|\Sigma'_j}$. Because the worlds ω_1 and ω_2 coincide on $\Sigma \setminus \Sigma'_j$, we can use the syntax splitting again to show that $\omega_1 \preceq \omega_2$.

Case 2: $\omega_1^k \not\preceq_{|\Sigma_k} \omega_2^k$, i.e., $\omega_1^k \succ_{|\Sigma_k} \omega_2^k$. Let

$$\omega_2^{k\nearrow i} = \min(\{\omega_x^i \in \Omega_{\Sigma_i} \mid \omega^i_{x|\Sigma_k^*} = \omega_2^k\},\ \preceq_{|\Sigma_i})$$
$$\omega_1^{k\nearrow i} = \omega_1^k\ (\omega_2^{k\nearrow i}{}_{|\Sigma_i \setminus \Sigma_k^*}).$$

Analogously to Case 1 we can show that $\omega_1 \succ \omega_2$ which is equivalent to $\omega_1 \not\preceq \omega_2$. $\square$

6 Conclusions and Future Work

We showed that there is a finest syntax splitting for every ranking function. We also showed that there is a finest syntax splitting for every TPO on worlds, even though coarsenings of syntax splittings of TPOs may not be syntax splittings. Our current work includes using the results of this paper and algorithms for finding syntax splittings (Bräuer 2022) in nonmonotonic reasoning and belief change. Further on, extending our results on syntax splittings to infinite signatures and other belief representation settings, e.g., ranking functions with infinite ranks, are subject to future work.

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