Proceedings of the 20th International Conference on Principles of Knowledge Representation and Reasoning
Main Track

Iteration of Iterated Belief Revision

Nicolas Schwind[1], **Sébastien Konieczny**[2], **Ramón Pino Pérez**[2]
[1]National Institute of Advanced Industrial Science and Technology, Tokyo, Japan
[2]CRIL, CNRS – Université d'Artois, France
nicolas-schwind@aist.go.jp, konieczny@cril.fr, pinoperez@cril.fr

Abstract

The behavior of iterated belief revision operators with respect to iteration has been characterized by a set of four postulates proposed by Darwiche and Pearl. These postulates give constraints on a single iteration step, and this is not enough to forbid some pathological operators. In this paper, we propose a generalization of these postulates to solve this issue and we study its implications. One surprising consequence is that, for TPO-representable operators (i.e., for operators defined as transitions on total pre-orders on interpretations), there are very few operators that satisfy this generalization.

1 Introduction

Belief revision aims to correct our current beliefs when reliable new pieces of information arrive and contradict these beliefs. This capability to accommodate conflicting evidence in our beliefs is essential if we want to build truly autonomous robots. For instance, if we send a robot to a new planet, we will teach the robot what we know about this planet, including its physics. The robot will use this information to make decisions, elaborate plans, and so on. But if, upon arrival, the robot discovers evidence that contradicts some of these beliefs, it must be able to modify its beliefs accordingly.

The standard AGM (Alchourrón, Gärdenfors, Makinson) characterization of belief revision (Alchourrón, Gärdenfors, and Makinson 1985; Gärdenfors 1988; Katsuno and Mendelzon 1991; Fermé and Hansson 2011) has been shown to be the right one, as multiple representation theorems have demonstrated that all natural constructive ways to define belief revision operators satisfy all AGM postulates. Furthermore, this approach has been shown to have tight links with other logical reasoning frameworks, such as non-monotonic inference relations (Kraus, Lehmann, and Magidor 1990; Lehmann and Magidor 1992; Gärdenfors 1990) or possibilistic logic (Dubois, Lang, and Prade 1994; Dubois and Prade 1991).

Capturing the core of the belief revision problem is an impressive achievement, but it does not end the story. Nearly all postulates have been criticized, and interesting new operators can be defined when we remove or weaken some of these postulates.

However, even the basic belief revision problem has received criticism for the standard AGM characterization, as it does not provide any rationality constraint on the iteration of the revision process. Belief revision is inherently an iterative process, as each new piece of information received requires a revision of current beliefs. Many efforts were made during the nineties to address this iteration issue.

While alternative postulates and generalized operators have been proposed (Konieczny and Pino Pérez 2008; Konieczny, Medina Grespan, and Pino Pérez 2010), the approach by Darwiche and Pearl (1997) is widely regarded as providing a suitable framework for iterated belief revision. In this approach, the agent's epistemic state Ψ is represented by more complex objects than simple logical theories. From Ψ, the agent's current beliefs, $Bel(\Psi)$, can be extracted. Additionally, Ψ contains information on the relative plausibility of currently disbelieved information, which guides the iterated revision process. To ensure rational behavior during the iteration of belief revision, four additional postulates are considered. These postulates aim to guarantee the conservation of conditional (counterfactual) information throughout the iteration process.

The AGM postulates only consider one iteration of belief revision, while the DP postulates consider two iterations and provide conservation properties between any epistemic states in the n+1 and n+2 iterations. It is expected that linking these two consecutive steps will recursively give constraints on a complete sequence of any number of iterations. While this is partly true, these constraints may not be strong enough to completely forbid all inappropriate behaviors [1].

For instance, it has been observed in (Konieczny and Pino Pérez 2008) that the independence of syntax expressed for one iteration does not imply the independence of syntax for any number of iterations. Therefore, an interesting question is whether linking the n+1 and n+2 iterations, as done in the DP operators, is sufficient for characterizing any number of iterations.

In this paper, we investigate this issue and show that this is not the case. We go one iteration further by providing adaptations of the DP postulates when linking the n+2 and n+3 iterations. We demonstrate that this rules out very unintuitive operators and study the consequences of these postu-

[1]Note that an important improvement of the DP approach is the P postulate proposed in (Jin and Thielscher 2007; Booth and Meyer 2006).

lates for TPO-representable operators, i.e., for operators defined as transitions on total pre-orders on interpretations.

These new postulates allow us to forbid operators that are not homogeneous enough, i.e., whose behavior can change drastically from one iteration to the next. Thus, they correspond to operators that maintain a similar behavior iteration after iteration.

To illustrate why non-homogeneous operators are problematic, let us consider the following construction on TPO-representable operators. We will consider two well-known operators from the literature: Nayak's lexicographical revision (Nayak 1994) $\circ_L$, and Boutilier's natural revision (Boutilier 1996) $\circ_N$. These can be argued as being the extreme cases of operators allowed by DP postulates, where natural revision is the one that minimally changes the preorder at each revision, and lexicographic revision is the one that maximally changes it, resulting in a drastically different behavior. Now, let us build a new belief revision operator as follows: for each possible formula of the language (up to logical equivalence), select at random $\circ_L$ or $\circ_N$ (let's say $\circ_L$ with a probability of 0.5, and $\circ_N$ otherwise), and for any epistemic state (TPO) Ψ, use the operator that corresponds to its beliefs ($Bel(\Psi)$). We will denote this operator as $\circ_{L/N}$.

This operator satisfies all DP postulates, so it is acceptable according to the standard characterization. However, this operator exhibits a very peculiar behavior when used in a sequence of revisions. At each iteration, there is an equal chance that we use lexicographic or natural revision, resulting in a drastically different behavior. This is not related to the new piece of information, current beliefs, or the sequence of iterations. It is just determined by a random draw, which is not appropriate. These kinds of operators will not be allowed by our proposed postulates.

We will propose new postulates for iterated revision with homogeneous behavior and provide the corresponding characterization results. We will also examine some surprising consequences for TPO-representable operators.

2 Preliminaries

We consider a propositional language $\mathcal{L}_P$ built up from a finite (and of cardinality at least 2) set of propositional variables P and the usual connectives. The set of consistent formulae is denoted by $\mathcal{L}_P^*$. The symbol $\bot$ (resp. $\top$) is the Boolean constant always false (resp. true). An interpretation (or world) is a mapping from P to $\{0, 1\}$. The set of all worlds on $\mathcal{L}_P$ is denoted by Ω. $\models$ denotes logical entailment, $\equiv$ logical equivalence, and $[\varphi]$ denotes the set of models of the formula φ. Given a set of worlds $E \subseteq \Omega$, we denote by γ_S any formula such that $[\gamma_S] = S$. When S is a singleton set $S = \{\omega\}$, γ_S is abreviated as γ_ω.

In iterated belief change, it is standard to assume that the current set of beliefs of an agent is represented by an *epistemic state*. An epistemic state allows one to represent the current beliefs of the agent and some conditional information guiding the revision process. In all generality, an epistemic state can be any object Ψ from which the set of beliefs of the agent can be extracted through a mapping Bel, so that $Bel(\Psi)$ is a propositional formula from $\mathcal{L}_P$. Formally:

Definition 1 (Epistemic Space)**.** *An* epistemic space $\mathcal{E}$ *is a tuple* $\langle U, Bel\rangle$*, where* U *is a set and* Bel *is a mapping* $Bel : U \to \mathcal{L}_P^*$*.*

A simple example is the TPO-based epistemic space:

Example 1. *Let us define the* TPO*-based epistemic space*[2] *The* TPO-based epistemic space *is the epistemic space* $\mathcal{E}_{tpo} = \langle U_{tpo}, Bel\rangle$ *where:*

- U_{tpo} *is the set of all* TPO*s over the set of all worlds from* Ω*;*
- Bel *is the mapping associating each* TPO $\preceq$ *from* U_{tpo} *with a formula* $\psi \in \mathcal{L}_P^*$ *such that* $[\psi] = \min(\Omega, \preceq)$*.*

A belief revision operator $\circ$ on an epistemic space $\mathcal{E} = \langle U, Bel\rangle$ associates every epistemic state Ψ from U and every consistent formula μ with a new epistemic state from U, denoted by $\Psi \circ \mu$. Thus, $\circ$ is a mapping $\circ : U \times \mathcal{L}_P^* \to U$. In the rest of the paper, when we refer to a revision operator $\circ$ without specifying the epistemic space on which it is defined, we will assume that $\circ$ is defined on some arbitrary epistemic space. Furthermore, we will focus on TPO-representable revision operators, which can be viewed as transitions between TPOs, elements of the TPO-based epistemic space. For a precise definition and to see that this is just a subclass of DP iterated revision operators, please refer to (Schwind, Konieczny, and Pino Pérez 2022).

Let us recall the set of postulates which are expected for such operators to have a good iterative behavior (Darwiche and Pearl 1997):

Definition 2 (DP operator (Darwiche and Pearl 1997))**.** *A revision operator* $\circ$ *is a* DP operator *if the following properties are satisfied, for each epistemic state* Ψ *and all formulae* μ, μ', α*:*

(R*1) $Bel(\Psi \circ \mu) \models \mu$
(R*2) *If* $Bel(\Psi) \wedge \mu \not\models \bot$*, then* $Bel(\Psi \circ \mu) \equiv Bel(\Psi) \wedge \mu$
(R*3) *If* $\mu \not\models \bot$*, then* $Bel(\Psi \circ \mu) \not\models \bot$
(R*4) *If* $\mu \equiv \mu'$*, then* $Bel(\Psi \circ \mu) \equiv Bel(\Psi \circ \mu')$
(R*5) $Bel(\Psi \circ \mu) \wedge \mu' \models Bel(\Psi \circ (\mu \wedge \mu'))$
(R*6) *If* $Bel(\Psi \circ \mu) \wedge \mu' \not\models \bot$*,*
then $Bel(\Psi \circ (\mu \wedge \mu')) \models Bel(\Psi \circ \mu) \wedge \mu'$
(C1) *If* $\alpha \models \mu$*, then* $Bel((\Psi \circ \mu) \circ \alpha) \equiv Bel(\Psi \circ \alpha)$
(C2) *If* $\alpha \models \neg\mu$*, then* $Bel((\Psi \circ \mu) \circ \alpha) \equiv Bel(\Psi \circ \alpha)$
(C3) *If* $Bel(\Psi \circ \alpha) \models \mu$*, then* $Bel((\Psi \circ \mu) \circ \alpha) \models \mu$
(C4) *If* $Bel(\Psi \circ \alpha) \not\models \neg\mu$*, then* $Bel((\Psi \circ \mu) \circ \alpha) \not\models \neg\mu$

The postulates (R*1-R*6) are a direct adaptation of the standard KM postulates to epistemic states. The remaining four postulates, (C1-C4), add constraints w.r.t. iteration.

Darwiche and Pearl also provided a characterization of DP operators in terms of TPOs over worlds:

Definition 3 (DP assignment)**.** *Given an epistemic space* $\mathcal{E} = \langle U, Bel\rangle$*, a mapping* $\Psi \mapsto \preceq_\Psi$ *associating each epistemic state* $\Psi \in U$ *with a* TPO[3] *over worlds* $\preceq_\Psi$ *is a* DP assignment *if and only if for all worlds* $\omega, \omega' \in \Omega$*:*

[2] TPO stands for total preorder.

[3] For each TPO $\preceq$, $\simeq$ denotes the corresponding indifference relation, and $\prec$ the corresponding strict ordering.

(1) *If* $\omega \models Bel(\Psi)$ *and* $\omega' \models Bel(\Psi)$*, then* $\omega \simeq_\Psi \omega'$

(2) *If* $\omega \models Bel(\Psi)$ *and* $\omega' \not\models Bel(\Psi)$*, then* $\omega \prec_\Psi \omega'$

(CR1) *If* $\omega \models \mu$ *and* $\omega' \models \mu$*, then* $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi \circ \mu} \omega'$

(CR2) *If* $\omega \not\models \mu$ *and* $\omega' \not\models \mu$*, then* $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi \circ \mu} \omega'$

(CR3) *If* $\omega \models \mu$ *and* $\omega' \not\models \mu$*, then* $\omega \prec_\Psi \omega' \Rightarrow \omega \prec_{\Psi \circ \mu} \omega'$

(CR4) *If* $\omega \models \mu$ *and* $\omega' \not\models \mu$*, then* $\omega \preceq_\Psi \omega' \Rightarrow \omega \preceq_{\Psi \circ \mu} \omega'$

Theorem 1 ((Darwiche and Pearl 1997))**.** *An operator* $\circ$ *is a DP operator if and only if there exists a DP assignment* $\Psi \mapsto \preceq_\Psi$ *such that for each epistemic state* Ψ *and each formula* μ*,* $[Bel(\Psi \circ \mu)] = \min([\mu], \preceq_\Psi)$*.*

When a DP assignment exists for $\circ$, we will say it *corresponds to* $\circ$. As a matter of fact, when such assignment exists it is unique[4].

Conditions *(1)* and *(2)* above define the notion of *faithful* assignment (Darwiche and Pearl 1997; Katsuno and Mendelzon 1991). Conditions *(CR1-CR4)* correspond to the iteration postulates (C1-C4). They impose constraints on the TPO $\preceq_{\Psi \circ \mu}$: *(CR1)* and *(CR2)*, the "rigidity" conditions, say that the order between models of μ is preserved and the order between models of $\neg\mu$ is also preserved. *(CR3)* and *(CR4)* say that there is no worsening between the models of μ and the models of $\neg\mu$.

Let us now give three important instances of DP operators.

Example 2. *We consider the Boutilier's natural revision operator* $\circ_N$ *defined over the epistemic space* $\mathcal{E}_{tpo}$*. This operator associates each* TPO $\Psi \in U_{tpo}$ *and each formula* μ *with a* TPO $\Psi \circ_N \mu$ *that satisfies* $\min(\Psi \circ_N \mu) = \min([\mu], \preceq_\Psi)$ *and the following condition:*

(Nat) *If* $\omega, \omega' \notin \min([\mu], \Psi)$*,*
then $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi \circ_N \mu} \omega'$*,*

where $\preceq_\Psi$ *denotes* Ψ *and* $\preceq_{\Psi \circ_N \mu}$ *denotes* $\Psi \circ_N \mu$*.*

That is, Boutilier's natural revision operator on $\mathcal{E}_{tpo}$ consists in selecting the set of all models of μ that are minimal according to an input TPO, and defining this set as the first level of the revised TPO while leaving the rest of the TPO unchanged.

Example 3. *Another example is the Nayak's lexicographic operator* $\circ_L$ *defined also in the epistemic space* $\mathcal{E}_{tpo}$*. It is defined by* $\min(\Psi \circ_L \mu) = \min([\mu], \Psi)$*, conditions (CR1-CR4) and:*

(Lex) *If* $\omega \models \mu$ *and* $\omega' \not\models \mu$*, then* $\omega \prec_{\Psi \circ_L \mu} \omega'$

Lexicographic revision moves all models of μ below all models of $\neg\mu$, and keeps the relationships between worlds of μ (resp. of $\neg\mu$) unchanged.

Example 4. *Another example is the Booth and Meyers' restrained operator* $\circ_R$ *defined also in the epistemic space* $\mathcal{E}_{tpo}$*. It is defined by* $\min(\Psi \circ_R \mu) = \min([\mu], \Psi)$*, conditions (CR1-CR2) and the two following conditions:*

(PR) *If* $\omega \models \mu$ *and* $\omega' \not\models \mu$*,*
then $\omega \preceq_\Psi \omega' \implies \omega \prec_{\Psi \circ_R \mu} \omega'$

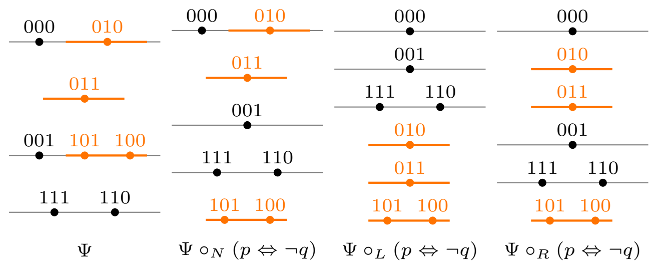

Figure 1: Natural $\circ_N$, lexicographic $\circ_L$ & restrained $\circ_R$ revision.

(DR) *If* $\omega \models \mu$*,* $\omega' \not\models \mu$ *and* $\omega \notin \min([\mu], \preceq_\Psi)$*, then* $\omega' \prec_\Psi \omega \implies \omega' \prec_{\Psi \circ_R \mu} \omega$

Note that (PR) is a strenghtening of both (CR3) and (CR4), so accordingly $\circ_R$ *is a DP operator.*

The restrained revision operator ensures that the models of μ that were as plausible as the models of its negation before the revision step become strictly more plausible after the revision, but it takes care not to make more changes than necessary (except that the minimal models of μ for Ψ are now the minimal models of $\Psi \circ_R \mu$, as required).

Let us now illustrate how the behaviors of these three revision operators differ from each other.

Example 5. *Let* $P = \{p, q, r\}$*. Figure 1 depicts a* TPO Ψ *over worlds*[5] *, and the revised* TPOs $\Psi \circ_N (p \Leftrightarrow \neg q)$*,* $\Psi \circ_L (p \Leftrightarrow \neg q)$ *and* $\Psi \circ_R (p \Leftrightarrow \neg q)$*. We have that* $Bel(\Psi) \equiv p \wedge q$*, and for the three operators,* $Bel(\Psi \circ_N (p \Leftrightarrow \neg q)) \equiv p \wedge \neg q$*, but the three associated* TPOs *are different. Therefore, in later iterations, their respective beliefs will differ.*

These are three well-known instances of DP revision operators, but there are many others (see, e.g., (Rott 2009)).

Note also that they are DP revision operators that are not TPO-representable, i.e., that require more complex structure, such as an OCF (Ordinal Conditional Function) (Spohn 1988), for being formally defined (Schwind, Konieczny, and Pino Pérez 2022).

3 Iteration of Iterated Postulates

So let us now propose our extended iteration postulates. The idea is to provide conditions between the $n+2$ and $n+3$ iterations to ensure that the change behavior remains the same throughout iterations and to characterize homogeneous operators. Let us first propose an initial version that gives rise to HDP (Homogeneous DP) operators.

Definition 4 (Homogenous DP revision operator)**.** *A DP operator* $\circ$ *is a* Homogeneous DP revision operator *(HDP revision operator for short) if the following properties are satisfied, for each epistemic state* Ψ *and all formulae* μ*,* α*,* α_1*,* α_2*,* β*:*

(CE1) *If* $\beta \models \mu$*, then* $Bel(\Psi \circ \mu \circ \alpha \circ \beta) \equiv Bel(\Psi \circ \alpha \circ \beta)$

(CE2) *If* $\beta \models \neg\mu$*, then* $Bel(\Psi \circ \mu \circ \alpha \circ \beta) \equiv Bel(\Psi \circ \alpha \circ \beta)$

[4]That is due to the fact that if an assignment satisfies $[Bel(\Psi \circ \mu)] = \min([\mu], \preceq_\Psi)$, then $\omega \preceq_\Psi \omega'$ iff $\omega \in [\Psi \circ \alpha_{\omega,\omega'}]$ (where $[\alpha_{\omega,\omega'}] = \{\omega, \omega'\}$), from which the unicity follows.

[5]A world ω is at the same or at a lower level than a world ω' iff $\omega \leq_\Psi \omega'$. So minimal (i.e., most plausible) worlds are at the lowest levels.

(CE3) *If* $Bel(\Psi \circ \alpha \circ \beta) \models \mu$*, then* $Bel(\Psi \circ \mu \circ \alpha \circ \beta) \models \mu$

(CE4) *If* $Bel(\Psi \circ \alpha \circ \beta) \not\models \neg\mu$*, then* $Bel(\Psi \circ \mu \circ \alpha \circ \beta) \not\models \neg\mu$

(CE5) *If* $\alpha_1 \wedge \beta \equiv \alpha_2 \wedge \beta$*,* $Bel(\Psi \circ \alpha_1) \models \neg\beta$ *and* $Bel(\Psi \circ \alpha_2) \models \neg\beta$*, then* $Bel(\Psi \circ \alpha_1 \circ \beta) \equiv Bel(\Psi \circ \alpha_2 \circ \beta)$

These postulates express rigidity and monotonicity conditions in a similar vein to the DP postulates (C1-C4). The difference is that while (C1-C4) express these conditions on conditional beliefs, (CE1-CE4) extends them to the level of *change* in conditional beliefs. Postulate (CE5) is dedicated to the homogeneity of change. It states that a formula β whose conjunction with two formulae α_1 and α_2 is identical will have exactly the same impact (i.e., will lead to the same beliefs) regardless of whether we revise by α_1 or α_2. Intuitively, this means that the plausibility of the formula $\alpha_1 \wedge \beta$ will change identically if we revise by α_1 or by α_2.

It is easy to see that (CE1-CE4) strengthen (C1-C4) (under a very light condition):

Proposition 1. *Let* $\circ$ *be an operator that satisfies* **(Tau)**.

(Tau) $\Psi \circ \top = \Psi$

For each $i \in \{1, \dots, 4\}$*, if an operator* $\circ$ *satisfies (CEi) then it satisfies (Ci).*

Proof. The proof is direct by replacing α with $\top$ in the statement of each postulate (CEi), from which we get the statement of (Ci). □

Let us see the corresponding conditions on the faithful assignment.

Definition 5 (Homogeneous DP assignment)**.** *A DP assignment* $\Psi \mapsto \preceq_\Psi$ *is a* Homogeneous DP assignment *(HDP assignment for short) if and only if for each epistemic state* Ψ*, all formulae* μ*,* α*,* α_1*,* α_2*, and all worlds* $\omega, \omega' \in \Omega$*:*

(CRE1) *If* $\omega \models \mu$ *and* $\omega' \models \mu$*, then* $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha} \omega'$

(CRE2) *If* $\omega \not\models \mu$ *and* $\omega' \not\models \mu$*, then* $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha} \omega'$

(CRE3) *If* $\omega \models \mu$ *and* $\omega' \not\models \mu$*, then* $\omega \prec_{\Psi\circ\alpha} \omega' \Rightarrow \omega \prec_{\Psi\circ\mu\circ\alpha} \omega'$

(CRE4) *If* $\omega \models \mu$ *and* $\omega' \not\models \mu$*, then* $\omega \preceq_{\Psi\circ\alpha} \omega' \Rightarrow \omega \preceq_{\Psi\circ\mu\circ\alpha} \omega'$

(CRE5) *If* $\omega \models \alpha_1 \wedge \alpha_2$*,* $\omega' \models \neg\alpha_1 \wedge \neg\alpha_2$ *and* $\omega \notin \min([\alpha_1], \preceq_\Psi) \cup \min([\alpha_2], \preceq_\Psi)$*, then* $\omega \preceq_{\Psi\circ\alpha_1} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha_2} \omega'$

(CRE1-CRE4) do not require a lot of explanation, since they can be compared to the standard *(CR1-CR4)*, and it can be seen that we go one step further. Instead of giving conditions between the current epistemic state and the one after one iteration, we give conditions between the epistemic states after one and two iterations.

However, it is more interesting to comment on *(CRE5)* since it provides another interpretation of the homogeneity of change. *(CRE5)* states that for non-minimal worlds of the new piece of information, each revision must lead to the same increase in plausibility. In other words, regardless of the formula used to revise the epistemic state, if ω is a model of this formula, its (potential) increase in plausibility (relative to non-models of the formula) will be the same.

Let us now state the correspondence between the new postulates and their semantical counterparts.

Proposition 2. *For each* $i \in \{1, \dots, 5\}$*, a DP revision operator* $\circ$ *satisfies (CEi) if and only if its corresponding DP assignment satisfies (CREi).*

Proof. Let $\circ$ be a DP revision operator, $\Psi \mapsto \preceq_\Psi$ be its corresponding DP assignment. Let Ψ be an epistemic state and μ, α, α_1, α_2 be formulae.

(Only if part) Let ω, ω' be two worlds. Assume that $\circ$ satisfies (CE1), and assume that $\omega, \omega' \models \mu$. Then $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega'$ iff $\omega \models Bel(\Psi \circ \mu \circ \alpha \circ \gamma_{\{\omega,\omega'\}})$ (by Theorem 1) iff $\omega \models Bel(\Psi \circ \alpha \circ \gamma_{\{\omega,\omega'\}})$ (by (CE1)) iff $\omega \preceq_{\Psi\circ\alpha} \omega'$ (by Theorem 1). So $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha} \omega'$, thus $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE1)*.

Assume that $\circ$ satisfies (CE2), and assume that $\omega, \omega' \not\models \mu$. Then $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega'$ iff $\omega \models Bel(\Psi \circ \mu \circ \alpha \circ \gamma_{\{\omega,\omega'\}})$ (by Theorem 1) iff $\omega \models Bel(\Psi \circ \alpha \circ \gamma_{\{\omega,\omega'\}})$ (by (CE2)) iff $\omega \preceq_{\Psi\circ\alpha} \omega'$ (by Theorem 1). So $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha} \omega'$, thus $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE2)*.

Assume that $\circ$ satisfies (CE3), and assume that $\omega \models \mu$, $\omega' \not\models \mu$, and $\omega \prec_{\Psi\circ\alpha} \omega'$. By Theorem 1, $[Bel(\Psi \circ \alpha \circ \gamma_{\{\omega,\omega'\}})] = \{\omega\}$. Then by (CE3) and since $Bel(\Psi \circ \alpha \circ \gamma_{\{\omega,\omega'\}}) \models \mu$, we get that $Bel(\Psi \circ \mu \circ \alpha \circ \gamma_{\{\omega,\omega'\}}) \models \mu$, which means that $Bel(\Psi \circ \mu \circ \alpha \circ \gamma_{\{\omega,\omega'\}}) = \{\omega\}$ (by (R*1) and (R*3)), from which we get that $\omega \prec_{\Psi\circ\mu\circ\alpha} \omega'$ by Theorem 1. We got that $\omega \prec_{\Psi\circ\alpha} \omega' \implies \omega \prec_{\Psi\circ\mu\circ\alpha} \omega'$, thus $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE3)*.

Assume that $\circ$ satisfies (CE4), and assume that $\omega \models \mu$, $\omega' \not\models \mu$, and $\omega \preceq_{\Psi\circ\alpha} \omega'$. By Theorem 1, $\omega \models Bel(\Psi \circ \alpha \circ \gamma_{\{\omega,\omega'\}})$, which means that $Bel(\Psi \circ \alpha \circ \gamma_{\{\omega,\omega'\}}) \not\models \neg\mu$. Then by (CE4) we get that $Bel(\Psi \circ \mu \circ \alpha \circ \gamma_{\{\omega,\omega'\}}) \not\models \neg\mu$, which means that $\omega \models Bel(\Psi \circ \mu \circ \alpha \circ \gamma_{\{\omega,\omega'\}})$ (by (R*1)), from which we get that $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega'$ by Theorem 1. We got that $\omega \preceq_{\Psi\circ\alpha} \omega' \implies \omega \preceq_{\Psi\circ\mu\circ\alpha} \omega'$, thus $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE4)*.

Assume that $\circ$ satisfies (CE5), and assume that $\omega \models \alpha_1 \wedge \alpha_2$, $\omega' \models \neg\alpha_1 \wedge \neg\alpha_2$ and $\omega \notin \min([\alpha_1], \preceq_\Psi) \cup \min([\alpha_2], \preceq_\Psi)$. We directly get that $\alpha_1 \wedge \gamma_{\{\omega,\omega'\}} \equiv \alpha_2 \wedge \gamma_{\{\omega,\omega'\}} (\equiv \gamma_\omega)$. And by Theorem 1 we know that $Bel(\Psi \circ \alpha_1) \models \neg\gamma_{\{\omega,\omega'\}}$ and $Bel(\Psi \circ \alpha_2) \models \neg\gamma_{\{\omega,\omega'\}}$. Then $\omega \preceq_{\Psi\circ\alpha_1} \omega'$ iff $\omega \models Bel(\Psi \circ \alpha_1 \circ \gamma_{\{\omega,\omega'\}})$ (by Theorem 1) iff $\omega \models Bel(\Psi \circ \alpha_2 \circ \gamma_{\{\omega,\omega'\}})$ (by (CE5)) iff $\omega \preceq_{\Psi\circ\alpha_2} \omega'$ (by Theorem 1). So $\omega \preceq_{\Psi\circ\alpha_1} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha_2} \omega'$, thus $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE5)*.

(If part) Let β be a formula. Note by Theorem 1 that $[Bel(\Psi \circ \alpha \circ \beta)] = \min([\beta], \preceq_{\Psi\circ\alpha})$ and $[Bel(\Psi \circ \mu \circ \alpha \circ \beta)] = \min([\beta], \preceq_{\Psi\circ\mu\circ\alpha})$.

Assume that $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE1)*, and that $\beta \models \mu$. By *(CRE1)*, we get that $\min([\beta], \preceq_{\Psi\circ\alpha}) = \min([\beta], \preceq_{\Psi\circ\mu\circ\alpha})$, so $Bel(\Psi \circ \alpha \circ \beta) \equiv Bel(\Psi \circ \alpha \circ \beta)$. Hence, $\circ$ satisfies (CE1).

Assume that $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE2)*, and that $\beta \models \neg\mu$. By *(CRE2)*, we get that $\min([\beta], \preceq_{\Psi\circ\alpha}) = \min([\beta], \preceq_{\Psi\circ\mu\circ\alpha})$, so $Bel(\Psi \circ \alpha \circ \beta) \equiv Bel(\Psi \circ \alpha \circ \beta)$. Hence, $\circ$ satisfies (CE2).

Assume that $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE3)*, and that $Bel(\Psi \circ \alpha \circ \beta) \models \mu$. This means that $\min([\beta], \preceq_{\Psi\circ\alpha}) \subseteq [\mu]$, so we can write that $\min([\beta], \preceq_{\Psi\circ\alpha}) = \min([\beta \wedge \mu], \preceq_{\Psi\circ\alpha})$. Now, assume toward a contradiction that $Bel(\Psi \circ \mu \circ \alpha \circ \beta) \not\models \mu$. This means that there exists a world $\omega' \models \neg\mu$ such that $\omega' \in \min([\beta], \preceq_{\Psi\circ\mu\circ\alpha})$, which precisely means that for every world $\omega \in [\beta]$, $\omega' \preceq_{\Psi\circ\mu\circ\alpha} \omega$. In particular, for every world $\omega \in [\beta \wedge \mu]$, $\omega' \preceq_{\Psi\circ\mu\circ\alpha} \omega$. From *(CRE3)* we get for every world $\omega \in [\beta \wedge \mu]$ that $\omega' \preceq_{\Psi\circ\alpha} \omega$, so if $\omega' \notin \min([\beta], \preceq_{\Psi\circ\alpha})$ then $\omega \notin \min([\beta], \preceq_{\Psi\circ\alpha})$. Yet $\omega' \models \neg\mu$ and we saw that $\min([\beta], \preceq_{\Psi\circ\alpha}) = \min([\beta \wedge \mu], \preceq_{\Psi\circ\alpha})$, so $\omega' \notin \min([\beta], \preceq_{\Psi\circ\alpha})$, and thus $\omega \notin \min([\beta], \preceq_{\Psi\circ\alpha})$, for every world $\omega \in [\beta \wedge \mu]$. We found that for every world $\omega \in [\beta \wedge \mu]$, $\omega \notin \min([\beta], \preceq_{\Psi\circ\alpha})$, which contradicts the fact that $\min([\beta], \preceq_{\Psi\circ\alpha}) = \min([\beta \wedge \mu], \preceq_{\Psi\circ\alpha})$. This shows that $Bel(\Psi \circ \mu \circ \alpha \circ \beta) \models \mu$, thus $\circ$ satisfies (CE3).

Assume that $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE4)*, and that $Bel(\Psi \circ \alpha \circ \beta) \not\models \neg\mu$. We need to show that $Bel(\Psi \circ \mu \circ \alpha \circ \beta) \not\models \neg\mu$. Then let ω be a world such that $\omega \in \min([\beta], \preceq_{\Psi\circ\alpha}) \cap [\mu]$. We have for every world $\omega' \models \beta$ that $\omega \preceq_{\Psi\circ\alpha} \omega'$. In particular, for every world $\omega' \models \beta \wedge \neg\mu$, $\omega \preceq_{\Psi\circ\alpha} \omega'$. By *(CRE4)*, for every world $\omega' \models \beta \wedge \neg\mu$, $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega'$. We fall into two cases:

Case 1: $\omega \in \min([\beta], \preceq_{\Psi\circ\mu\circ\alpha})$. In this case, $\omega \in [Bel(\Psi \circ \mu \circ \alpha \circ \beta)]$, so since $\omega \models \mu$ we directly get that $Bel(\Psi \circ \mu \circ \alpha \circ \beta) \not\models \neg\mu$.

Case 2: $\omega \notin \min([\beta], \preceq_{\Psi\circ\mu\circ\alpha})$. In that case, let ω'' be a world such that $\omega'' \in \min([\beta], \preceq_{\Psi\circ\mu\circ\alpha})$. We have that $\omega'' \prec_{\Psi\circ\mu\circ\alpha} \omega$. Yet we have seen for every world $\omega' \models \beta \wedge \neg\mu$ that $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega'$, which means that $\omega'' \models \beta \wedge \mu$. We got that $\omega'' \in \min([\beta], \preceq_{\Psi\circ\mu\circ\alpha}) \cap [\mu]$, thus $Bel(\Psi \circ \mu \circ \alpha \circ \beta) \not\models \neg\mu$.

In both cases, we got that $Bel(\Psi \circ \mu \circ \alpha \circ \beta) \not\models \neg\mu$, which shows that $\circ$ satisfies (CE4).

Assume that $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE5)*, and that $\alpha_1 \wedge \beta \equiv \alpha_2 \wedge \beta$, $Bel(\Psi \circ \alpha_1) \models \neg\beta$ and $Bel(\Psi \circ \alpha_2) \models \neg\beta$. Let ω, $\omega' \models \beta$. By Theorem 1 we know that $\omega \notin \min([\alpha_1], \preceq_\Psi) \cup \min([\alpha_2], \preceq_\Psi)$. Let us show that $\omega \preceq_{\Psi\circ\alpha_1} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha_2} \omega'$. Since $\alpha_1 \wedge \beta \equiv \alpha_2 \wedge \beta$, we fall into one of the following four cases: (i) $\omega, \omega' \models \alpha_1 \wedge \alpha_2$. Then $\omega \preceq_{\Psi\circ\alpha_1} \omega'$ iff $\omega \preceq_\Psi \omega'$ (by *(CR1)*) iff $\omega \preceq_{\Psi\circ\alpha_2} \omega'$ (by *(CR1)*). (ii) $\omega, \omega' \models \neg\alpha_1 \wedge \neg\alpha_2$. Then $\omega \preceq_{\Psi\circ\alpha_1} \omega'$ iff $\omega \preceq_\Psi \omega'$ (by *(CR2)*) iff $\omega \preceq_{\Psi\circ\alpha_2} \omega'$ (by *(CR2)*). (iii) $\omega \models \alpha_1 \wedge \alpha_2$ and $\omega' \models \neg\alpha_1 \wedge \neg\alpha_2$. Then $\omega \preceq_{\Psi\circ\alpha_1} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha_2} \omega'$ (by *(CRE5)*). (iv) $\omega \models \neg\alpha_1 \wedge \neg\alpha_2$ and $\omega' \models \alpha_1 \wedge \alpha_2$. This case is identical to case (iii) above, since ω, ω' play symmetrical roles.

In every case we get that $\omega \preceq_{\Psi\circ\alpha_1} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha_2} \omega'$. By Theorem 1, this means that $Bel(\Psi \circ \alpha_1) \equiv Bel(\Psi \circ \alpha_2)$. Hence, $\circ$ satisfies (CE5). $\square$

As a direct consequence, we get the following representation theorem for HDP operators:

Theorem 2. *A DP revision operator $\circ$ is an HDP revision operator if and only if there is a HDP assignment $\Psi \mapsto \preceq_\Psi$ such that for each epistemic state Ψ and each formula μ, $[Bel(\Psi \circ \mu)] = \min([\mu], \preceq_\Psi)$.*

Let us now focus on the consequences on TPO-representable revision operators.

First, concerning the three operators $\circ_L$, $\circ_N$, $\circ_R$, only $\circ_L$ satisfies the five postulates (CE1-CE5):

Proposition 3.

1. *$\circ_L$ satisfies (CE1-CE5)*
2. *$\circ_N$ satisfies (CE3-CE5), but not (CE1-CE2)*
3. *$\circ_R$ satisfies (CE1) and (CE3-CE5), but not (CE2).*

Proof Sketch. We only show that $\circ_N$ and $\circ_R$ do not satisfy (CE2) and that $\circ_N$ does not satisfy (CE1), but the rest of the proof is available at (Schwind, Konieczny, and Pérez 2023). It is enough from Proposition 2 to show that their corresponding assignments do not satisfy the semantic counterparts of (CE1) and (CE2). We do so by proving a counter-example in each case.

• Proof that $\Psi \mapsto \preceq^N_\Psi$ and $\Psi \mapsto \preceq^R_\Psi$ do not satisfy *(CRE2)*: Let $\star \in \{N, R\}$. Let ω_1, ω_2, ω_3 be three worlds, μ, α be two formulae such that $[\mu] = \{\omega_3\}$ and $[\alpha] = \{\omega_2, \omega_3\}$, and Ψ be any TPO where $\omega_1 \prec^\star_\Psi \omega_2 \prec^\star_\Psi \omega_3$. Note that $\omega_1, \omega_2 \not\models \mu$. On the one hand, we get that $\omega_2 \prec^\star_{\Psi\circ_\star\alpha} \omega_1$. On the other hand, we get that $\omega_3 \prec^\star_{\Psi\circ_\star\mu} \omega_1 \prec^\star_{\Psi\circ_\star\mu} \omega_2$, and thus $\omega_1 \prec^\star_{\Psi\circ_\star\mu\circ_\star\alpha} \omega_2$. So both assignments $\Psi \mapsto \preceq^N_\Psi$ and $\Psi \mapsto \preceq^R_\Psi$ do not satisfy *(CRE2)*.

• Proof that $\Psi \mapsto \preceq^N_\Psi$ does not satisfy *(CRE1)*: Let ω_1, ω_2, ω_3 be three worlds, μ, α be two formulae such that $[\mu] = \{\omega_1, \omega_2\}$ and $[\alpha] = \{\omega_2, \omega_3\}$, and Ψ be any TPO where $\omega_3 \prec^N_\Psi \omega_1 \simeq_\Psi \omega_2$. Note that $\omega_1, \omega_2 \models \mu$. On the one hand, we get that $\omega_1 \simeq^N_{\Psi\circ_N\alpha} \omega_2$. On the other hand, we get that $\omega_1 \simeq^N_{\Psi\circ_N\mu} \omega_2 \prec^N_{\Psi\circ_N\mu} \omega_3$, and thus $\omega_2 \prec^N_{\Psi\circ_N\mu\circ_N\alpha} \omega_1$. So the assignment $\Psi \mapsto \preceq^N_\Psi$ does not satisfy *(CRE1)*. $\square$

But let us try to identify exactly what the HDP operators exactly are. We will introduce a condition that will be useful in proving our results:

(Fun) If $(\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega \preceq_{\Psi_2} \omega')$,
then $(\omega \preceq_{\Psi_1\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi_2\circ\alpha} \omega')$

This condition, denoted by *(Fun)* (for "Functionality"), is a kind of relational property. It states that the only information that is relevant to define the new relation between two worlds is the current relation between these two worlds only. That is, we do not compare them to other worlds.

Proposition 4. *Let $\circ$ be a DP operator defined on the TPO-based epistemic space. If its DP assignment satisfies (CRE2), then it satisfies (Fun).*

Proof. Let $\Psi \mapsto \preceq_\Psi$ be a DP assignment, and assume that it satisfies *(CRE2)*. We need to show that it also satisfies *(Fun)*. So let Ψ_1, Ψ_2 be two TPOs, ω, ω' be two worlds, and α be a formula, and assume that $\omega \preceq_{\Psi_1} \omega'$ if and only if $\omega \preceq_{\Psi_2} \omega'$. We need to show that $\omega \preceq_{\Psi_1\circ\alpha} \omega'$ if and only if $\omega \preceq_{\Psi_2\circ\alpha} \omega'$.

Let $S \subseteq \Omega$ be the set of worlds defined such that $S \cap \{\omega, \omega'\} = \emptyset$ and $S \cup \{\omega, \omega'\} = \Omega$; and let us write

$S = \{\omega_1, \ldots, \omega_m\}$. Then let σ be the sequence of complete formulae $\sigma = (\gamma_{\omega_1}, \ldots, \gamma_{\omega_m})$. Now, for $i \in \{1, 2\}$, let $\Psi_i^1 = \Psi_i$ and for each $k \in \{1, \ldots, m-1\}$, let Ψ_i^{k+1} be defined as $\Psi_i^{k+1} = \Psi_i^k \circ \gamma_{\omega_k}$.

(Part 1) We first intend to prove that $\Psi_1^m = \Psi_2^m$. Let $i \in \{1, 2\}$, let us first prove that $\omega \preceq_{\Psi_i} \omega'$ if and only if $\omega \preceq_{\Psi_i^m} \omega'$. We do it by induction on the sequence $(\Psi_i^k)_{k \in \{1,\ldots,m\}}$. The base case where $k = 1$ is trivial: since $\Psi_i = \Psi_i^1$, we have that $\omega \preceq_{\Psi_i} \omega'$ if and only if $\omega \preceq_{\Psi_i^1} \omega'$. Now, let $k \in \{1, \ldots, m-1\}$, and assume that $\omega \preceq_{\Psi_i} \omega'$ if and only if $\omega \preceq_{\Psi_i^k} \omega'$. Since $\Psi_i^{k+1} = \Psi_i^k \circ \gamma_{\omega_k}$ by definition, and since $\omega, \omega' \not\models \gamma_{\omega_k}$, by *(CR2)*, we get that $\omega \preceq_{\Psi_i^k} \omega'$ if and only if $\omega \preceq_{\Psi_i^{k+1}} \omega'$. This shows by induction that, for each $i \in \{1, 2\}$,

$$\omega \preceq_{\Psi_i} \omega' \text{ if and only if } \omega \preceq_{\Psi_i^m} \omega' \tag{1}$$

Yet for each $i \in \{1, 2\}$, one can verify easily by construction of Ψ_i^m that for each $k \in \{1, \ldots, m-1\}$,

$$\omega_{k+1} \prec_{\Psi_i^m} \omega_k, \tag{2}$$

and that for each $k \in \{1, \ldots, m\}$,

$$\omega_k \prec_{\Psi_i^m} \omega \text{ and } \omega_k \prec_{\Psi_i^m} \omega' \tag{3}$$

Equations 1, 2 and 3 together with our initial assumption that $\omega \preceq_{\Psi_1} \omega'$, $\omega \preceq_{\Psi_2} \omega'$, fully characterize the TPOs Ψ_1^m and Ψ_2^m, and also show that Ψ_1^m and Ψ_2^m are the same TPO, which we denote by Ψ_*.

(Part 2) Now, we intend to prove that for each $i \in \{1, 2\}$, $\omega \preceq_{\Psi_i \circ \alpha} \omega'$ if and only if $\omega \preceq_{\Psi_* \circ \alpha} \omega'$. This is done as follows, similarly as in part 1 of this proof, by induction on the sequence $(\Psi_i^k)_{k \in \{1,\ldots,m\}}$, but this time taking advantage of *(CRE2)* instead of *(CR2)*. The base case where $k = 1$ is trivial: since $\Psi_i = \Psi_i^1$, we have that $\omega \preceq_{\Psi_i \circ \alpha} \omega'$ if and only if $\omega \preceq_{\Psi_i^1 \circ \alpha} \omega'$. Now, let $k \in \{1, \ldots, m-1\} \geq 1$, and assume that $\omega \preceq_{\Psi_i \circ \alpha} \omega'$ if and only if $\omega \preceq_{\Psi_i^k \circ \alpha} \omega'$. Since $\Psi_i^{k+1} = \Psi_i^k \circ \gamma_{\omega_k}$ by definition, and since $\omega, \omega' \not\models \gamma_{\omega_k}$, by *(CRE2)*, we get that $\omega \preceq_{\Psi_i^k \circ \alpha} \omega'$ if and only if $\omega \preceq_{\Psi_i^{k+1} \circ \alpha} \omega'$. Hence, $\omega \preceq_{\Psi_i^k \circ \alpha} \omega'$ if and only if $\omega \preceq_{\Psi_i^{k+1} \circ \alpha} \omega'$. This shows by induction that, for each $i \in \{1, 2\}$, $\omega \preceq_{\Psi_i \circ \alpha} \omega'$ if and only if $\omega \preceq_{\Psi_i^m \circ \alpha} \omega'$. But since $\Psi_1^m = \Psi_2^m = \Psi_*$, we can write that for each $i \in \{1, 2\}$,

$$\omega \preceq_{\Psi_i \circ \alpha} \omega' \text{ if and only if } \omega \preceq_{\Psi_* \circ \alpha} \omega' \tag{4}$$

Equation 4 shows that $\omega \preceq_{\Psi_1 \circ \alpha} \omega'$ if and only if $\omega \preceq_{\Psi_2 \circ \alpha} \omega'$, which shows that *(Fun)* is satisfied. □

Proposition 5. *If a DP assignment satisfies (Fun), then it satisfies (Lex).*

Proof. Let $\Psi \mapsto \preceq_\Psi$ be a DP assignment, and assume that it satisfies *(Fun)*. Assume toward a contradiction that it does not satisfy *(Lex)*. That is to say, there exists an epistemic state Ψ_1, two worlds ω, ω', and a formula α such that $\omega' \models \alpha$, $\omega \not\models \alpha$, and $\omega \preceq_{\Psi_1 \circ \alpha} \omega'$. By *(CR3)*, we know that $\omega \preceq_{\Psi_1} \omega'$. Let $S = \{\omega_* \models \alpha \mid \omega_* \prec_{\Psi_1} \omega'\}$, and let Ψ_1^S be an epistemic state such that $\omega_1 \preceq_{\Psi_1^S} \omega_2$ if and only if

$$\begin{cases} \omega_1 \preceq_{\Psi_1} \omega_2, & \text{if } \omega_1, \omega_2 \in S \text{ or } \omega_1, \omega_2 \notin S \quad (i) \\ \top, & \text{if } \omega_1 \in \Omega \setminus S \text{ and } \omega_2 \in S \quad (ii) \\ \bot, & \text{otherwise} \quad (iii) \end{cases}$$

That is, Ψ_1^S is any epistemic state built from Ψ_1, where in $\preceq_{\Psi_1^S}$ all worlds from S (i.e., all worlds strictly more plausible than ω' in $\preceq_{\Psi_1}$) are shifted on top of the TPO $\preceq_{\Psi_1^S}$ (ii), in a strict manner (iii), while keeping from $\preceq_{\Psi_1}$ all remaining relationships between worlds (i). Note by construction of Ψ_1^S that $\omega \preceq_{\Psi_1^S} \omega'$, since $\omega \preceq_{\Psi_1} \omega'$ and $\omega, \omega' \notin S$. Since the assignment satisfies *(Fun)*, and since $\omega \preceq_{\Psi_1} \omega'$, $\omega \preceq_{\Psi_1^S} \omega'$, and $\omega \preceq_{\Psi_1 \circ \alpha} \omega'$, we get that $\omega \preceq_{\Psi_1^S \circ \alpha} \omega'$. Yet by construction of Ψ_1^S we have that $\omega' \in \min([\alpha], \preceq_{\Psi_1^S})$, so by Theorem 1, we get that $\omega' \prec_{\Psi_1^S \circ \alpha} \omega$, which leads to a contradiction. □

As a direct consequence of Propositions 3.1, 4 and 5, we get that:

Theorem 3. $\circ_L$ *is the only* TPO*-representable HDP operator.*

So, there is only one HDP operator, Nayak's lexicographic revision. This provides an alternative characterization of this operator. However, this can also be interpreted as the fact that these conditions *(CE1-CE2)* are too strong. In fact, this is the case. This can be illustrated, for instance, by the property *(Fun)*, which is a consequence of *(CE2)* and requires homogeneity of the evolution of each possible world, whatever its plausibility.

But belief revision operators impose a very special treatment to minimal models (i.e., the most plausible ones) of the new piece of information, which have to become strictly more plausible than worlds of the negation of the new piece of information. So, requiring this similar behavior for any cases (even when these worlds are not the most plausible) leads to the strong change encoded by Nayak's lexicographic revision.

In fact, what we really want is not a homogeneous change in any situation but in all situations when the world is not among the most plausible ones. This additional condition will lead to another class of operators.

4 hDP Revision Operators

Let us focus on two weaker variations of (CE1) and (CE2):

Definition 6 (hDP revision operator)**.** *A DP operator* $\circ$ *is a* weak homogeneous DP operators *(hDP operator for short) if it satisfies (CE3), (CE4), (CE5), and the following properties, for each epistemic state* Ψ *and all formulae* μ, α, β:

(CE1w) *If* $\beta \models \mu$, $Bel(\Psi \circ \alpha) \models \neg\beta$ *and* $Bel(\Psi \circ \mu \circ \alpha) \models \neg\beta$, *then* $Bel(\Psi \circ \mu \circ \alpha \circ \beta)) \equiv Bel(\Psi \circ \alpha \circ \beta)$

(CE2w) *If* $\beta \models \neg\mu$, $Bel(\Psi \circ \alpha) \models \neg\beta$ *and* $Bel(\Psi \circ \mu \circ \alpha) \models \neg\beta$, *then* $Bel(\Psi \circ \mu \circ \alpha \circ \beta)) \equiv Bel(\Psi \circ \alpha \circ \beta)$

In both properties, we have added the conditions $Bel(\Psi \circ \alpha) \models \neg\beta$ and $Bel(\Psi \circ \mu \circ \alpha) \models \neg\beta$, which ensure that β is not believed after revising by α and by μ. Thus, we are talking about true counterfactual beliefs, and these two conditions are, like the DP conditions they come from, conditions for the preservation of counterfactual beliefs.

(CE1w) and (CE2w) correspond to the following semantical conditions:

Definition 7 (hDP assignment). *A DP assignment $\Psi \mapsto \preceq_\Psi$ is a* weak Homogeneous DP assignment *(hDP assignment for short) if and only if it satisfies conditions (CRE3), (CRE4) and (CRE5), and for each epistemic state Ψ, all formulae μ, α, and all worlds $\omega, \omega' \in \Omega$:*

(CRE1w) *If $\omega, \omega' \models \mu$ and $\omega, \omega' \notin \min([\alpha], \preceq_\Psi) \cup \min([\alpha], \preceq_{\Psi\circ\mu})$, then $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha} \omega'$*

(CRE2w) *If $\omega, \omega' \not\models \mu$ and $\omega, \omega' \notin \min([\alpha], \preceq_\Psi) \cup \min([\alpha], \preceq_{\Psi\circ\mu})$, then $\omega \preceq_{\Psi\circ\mu\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\alpha} \omega'$*

These additional conditions are maybe clearer here than in the postulates. They state that the worlds we are interested in when talking about preservation of their relationship are those that are not the most plausible under α for $\preceq_\Psi$ and $\preceq_{\Psi\circ\mu}$. These most plausible worlds will have a particular treatment imposed by the standard revision postulates, so we want to focus on the other ones and state that they will keep their relative relationship.

We can show that these conditions are true counterparts of the postulates (the proof is omitted for space reasons, but can be found at (Schwind, Konieczny, and Pérez 2023) and is similar to the part of the proof of Proposition 2 showing the correspondence between (CEi) and *(CREi)*, for each $i \in \{1, 2\}$):

Proposition 6. *For each $i \in \{1, 2\}$, a DP revision operator $\circ$ satisfies (CEiw) if and only if its corresponding DP assignment satisfies (CREiw).*

As a direct consequence, we get the following representation theorem for hDP revision operators:

Theorem 4. *A DP revision operator $\circ$ is a hDP operator if and only if there is a hDP assignment $\Psi \mapsto \preceq_\Psi$ such that for each epistemic state Ψ and each formula μ, $[Bel(\Psi \circ \mu)] = \min([\mu], \preceq_\Psi)$.*

Let us now focus on the consequences on TPO-representable revision operators. This time our three illustrative operators are member of this subclass of operators:

Proposition 7. *$\circ_L$, $\circ_N$ and $\circ_R$ are hDP operators.*

Proof. Using Propositions 3 and 6 it is enough to prove that $\Psi \mapsto \preceq^N_\Psi$ satisfies *(CRE1w)* and *(CRE2w)*, and that $\Psi \mapsto \preceq^R_\Psi$ satisfies *(CRE1w)*, where the assignments $\Psi \mapsto \preceq^N_\Psi$ and $\Psi \mapsto \preceq^R_\Psi$ denote the DP assignment corresponding to $\circ_N$ and $\circ_R$, respectively. Let Ψ be any epistemic state, μ, α be two formulae, and ω, ω' be two worlds.

• Proof that $\Psi \mapsto \preceq^N_\Psi$ satisfies *(CRE1w)* and *(CRE2w)*:

Assume that (a) $\omega, \omega' \notin \min([\alpha], \preceq^N_\Psi)$ and that (b) $\omega, \omega' \notin \min([\alpha], \preceq^N_{\Psi\circ_N\mu})$:

(CRE1w) Let $\omega, \omega' \models \mu$. We need to prove that $\omega \preceq^N_{\Psi\circ_N\alpha} \omega' \Leftrightarrow \omega \preceq^N_{\Psi\circ_N\mu\circ_N\alpha} \omega'$. The case when $\omega \models \alpha \Leftrightarrow \omega' \models \alpha$ is direct using *(CR1)*, *(CR2)* and *(CR4)*. Assume that $\omega \models \alpha$, $\omega' \not\models \alpha$. Then $\omega \preceq^N_{\Psi\circ_N\alpha} \omega'$ iff $\omega \preceq^N_{\Psi\circ_N} \omega'$ (by (a) and *(Nat)*) iff $\omega \preceq^N_{\Psi\circ_N\mu} \omega'$ (by *(CR1)*) iff $\omega \preceq^N_{\Psi\circ_N\mu\circ_N\alpha} \omega'$ (by (b) and *(Nat)*). The proof when $\omega \not\models \alpha$, $\omega' \models \alpha$ is identical since ω and ω' play symmetrical roles.

(CRE2w) Let $\omega, \omega' \not\models \mu$. The proof is identical to the one for *(CRE1w)* above, using *(CR2)* instead of *(CR1)*.

• Proof that $\Psi \mapsto \preceq^R_\Psi$ satisfies *(CRE2w)*:

Assume that $\omega, \omega' \not\models \mu$, that (a) $\omega, \omega' \notin \min([\alpha], \preceq^R_\Psi)$ and that (b) $\omega, \omega' \notin \min([\alpha], \preceq^R_{\Psi\circ_R\mu})$. The proof is identical to the one of Proposition 3 showing that $\Psi \mapsto \preceq^R_\Psi$ satisfies *(CRE1)* when $\omega \notin \min([\alpha], \preceq^R_\Psi)$, using *(CR2)* instead of *(CR1)*. And the proof when $\omega \not\models \alpha$, $\omega' \models \alpha$ is identical since ω and ω' play symmetrical roles. ☐

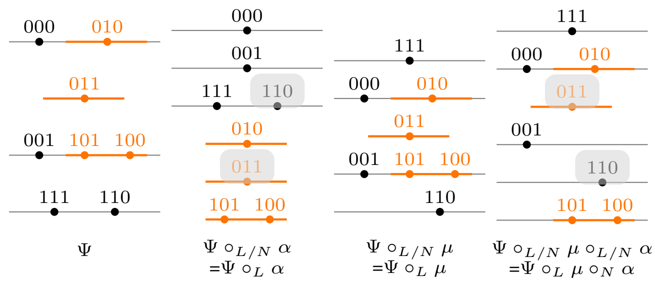

Figure 2: The $\circ_{L/N}$ operator is not a hDP operator

Let us now also show that the $\circ_{L/N}$ of the introduction is not a hDP operator.

Example 6. *Let us consider the operator $\circ_{L/N}$ of the introduction, that assigns randomly to each formula either the operator $\circ_L$ or $\circ_N$. And suppose that to the formula whose models are $\{111, 110\}$, it is $\circ_L$ that has been drawn, and to the formula whose models are $\{110\}$, it is $\circ_N$ that has been drawn. Now consider the same epistemic state Ψ as the one in Figure 2, and the formulae $\alpha = p \Leftrightarrow \neg q$ and $\mu = \neg p \vee \neg q \vee \neg r$.*

As the beliefs of Ψ are the models $\{111, 110\}$, we have to use $\circ_L$ to revise it. So we give in Figure 2 the results when we revise Ψ by α and by μ.

When we revise by μ, the beliefs of $\Psi \circ_{L/N} \mu$ are the model $\{110\}$. So this epistemic state has to be revised using $\circ_N$. And the result is the last order on the figure.

Now consider the two interpretations $\omega = 110$ and $\omega' = 011$. We have $\omega \models \mu$, $\omega' \models \mu$ and $\omega, \omega' \notin \min([\alpha], \preceq_\Psi) \cup \min([\alpha], \preceq_{\Psi\circ\mu})$. So by (CRE1w) we should have $\omega \preceq_{\Psi\circ_{L/N}\mu\circ_{L/N}\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi\circ_{L/N}\alpha} \omega'$. These two worlds are highlighted in Figure 2 and one can check that this is not the case.

Let us check if we can characterize more precisely this class. We introduce a weakening of *(Fun)*, where we restrict its scope to non-minimal worlds:

(FunW) If $\omega, \omega' \notin \min([\alpha], \preceq_{\Psi_1}) \cup \min([\alpha], \preceq_{\Psi_2})$
and $(\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega \preceq_{\Psi_2} \omega')$,
then $(\omega \preceq_{\Psi_1\circ\alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi_2\circ\alpha} \omega')$

And we can show that:

Proposition 8. *Let $\circ$ be a DP operator defined on the* TPO*-based epistemic space. If its DP assignment satisfies (CRE2w), then it satisfies (FunW).*

Proof. The proof is almost identical to the one of Proposition 4, by assuming the DP assignment $\Psi \mapsto \preceq_\Psi$ satisfies *(CRE2w)* instead of *(CRE2)* and by proving that it satisfies *(FunW)* instead of *(Fun)*. The only difference is that one initially assumes that the two TPOs Ψ_1, Ψ_2, the two worlds ω, ω' and the formula α are such that $\omega, \omega' \notin \min([\alpha], \preceq_{\Psi_1}) \cup \min([\alpha], \preceq_{\Psi_2})$. Doing so, one can easily see that all TPOs Ψ_1^k, Ψ_2^k ($k \in \{1, \ldots, m\}$) involved in the proof are such that $\omega, \omega' \notin \min([\alpha], \preceq_{\Psi_1^k}) \cup \min([\alpha], \preceq_{\Psi_2^k})$ and $\omega, \omega' \notin \min([\alpha], \preceq_{\Psi_1^k \circ \alpha}) \cup \min([\alpha], \preceq_{\Psi_2^k \circ \alpha})$, from which we can conclude that *(FunW)* is satisfied. □

Interestingly, we can generalize *(FunW)* to a version with a kind of additional anonymity condition (i.e. the worlds do not have to be the same for $\preceq_{\Psi_1}$ or $\preceq_{\Psi_2}$, but just to be in the same situations).

(FunWA) If ($\omega \models \alpha \Leftrightarrow \omega^2 \models \alpha$), ($\omega' \models \alpha \Leftrightarrow \omega^3 \models \alpha$), ($\omega, \omega' \notin \min([\alpha], \preceq_{\Psi_1})$), ($\omega^2, \omega^3 \notin \min([\alpha], \preceq_{\Psi_2})$), and ($\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2} \omega^3$),
then ($\omega \preceq_{\Psi_1 \circ \alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \alpha} \omega^3$).

Proposition 9. *A DP assignment satisfies (FunW) if and only if it satisfies (FunWA).*

Proof. The (if) part of the proof is direct by setting $\omega = \omega^2$ and $\omega' = \omega^3$. Let us show the (only if) part. Let $\Psi \mapsto \preceq_\Psi$ a DP assignment satisfying *(FunW)*, and let us show that *(FunWA)* is satisfied. So let Ψ_1, Ψ_2 be two epistemic states, α, β be two formulae and $\omega, \omega', \omega^2, \omega^3$ be four worlds, such that (i) ($\omega \models \alpha \Leftrightarrow \omega^2 \models \beta$), (ii) ($\omega' \models \alpha \Leftrightarrow \omega^3 \models \beta$), (iii) $\omega, \omega' \notin \min([\alpha], \preceq_{\Psi_1})$, and (iv) $\omega^2, \omega^3 \notin \min([\beta], \preceq_{\Psi_2})$. Assume that (v) $\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2} \omega^3$. We must prove that $\omega \preceq_{\Psi_1 \circ \alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \beta} \omega^3$. Let Ψ be any epistemic state such that (vi) ($\omega \simeq_\Psi \omega^2, \omega' \simeq_\Psi \omega^3$), (vii) $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi_1} \omega'$), and (viii) ω, ω', ω^2, $\omega^3 \notin \min([\alpha], \preceq_\Psi) \cup \min([\beta], \preceq_\Psi)$. First, from (vi) we can write that (ix) $\omega \preceq_\Psi \omega' \Leftrightarrow \omega^2 \preceq_\Psi \omega^3$. Second, by (i), (ii), (vi) and *(CR1-CR2)* we get that (x) $\omega \simeq_{\Psi \circ \alpha} \omega^2$ and $\omega' \simeq_{\Psi \circ \alpha} \omega^3$, Hence, by (ix) and (x) we get that (xi) ($\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi \circ \alpha} \omega'$) $\Leftrightarrow$ ($\omega^2 \preceq_\Psi \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi \circ \alpha} \omega^3$). By (ix) and (xi), we get that (xii) $\omega \preceq_{\Psi \circ \alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi \circ \alpha} \omega^3$. From (iii), (vii), (viii) and *(FunW)*, we get that (xiii) $\omega \preceq_{\Psi \circ \alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi_1 \circ \alpha} \omega'$, and so by (xii) and (xiii) we get that (xiv) $\omega \preceq_{\Psi_1 \circ \alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi \circ \alpha} \omega^3$. Third, by (v) and (vii) we get that (xv) $\omega \preceq_\Psi \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2} \omega^3$, by (ix) and (xv) we get that (xvi) $\omega^2 \preceq_\Psi \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi_2} \omega^3$, and then by (iv), (viii), (xvi) and *(FunW)*, we get that (xvii) $\omega^2 \preceq_{\Psi \circ \alpha} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \alpha} \omega^3$. Lastly, from (xiv) and (xvii), we get that $\omega \preceq_{\Psi_1 \circ \alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \alpha} \omega^3$. This shows that $\Psi \mapsto \preceq_\Psi$ satisfies *(FunWA)*. □

Let us give a generalisation of *(FunWA)*:

(FunWP) If ($\omega \models \alpha \Leftrightarrow \omega^2 \models \beta$), ($\omega' \models \alpha \Leftrightarrow \omega^3 \models \beta$), ($\omega, \omega' \notin \min([\alpha], \preceq_{\Psi_1})$, ($\omega^2, \omega^3 \notin \min([\beta], \preceq_{\Psi_2})$, and ($\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2} \omega^3$),
then ($\omega \preceq_{\Psi_1 \circ \alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \beta} \omega^3$).

Let us now show that in fact, *(FunWP)* does not bring more than *(FunWA)* for homogeneous operators:

Proposition 10. *A DP assignment satisfies (FunWA) and (CRE5) if and only if it satisfies (FunWP).*

Proof Sketch. The (if) part of the proof is direct by setting $\alpha = \beta$ to prove *(FunWA)*, and by setting $\omega = \omega^2$, $\omega' = \omega^3$ and $\Psi_1 = \Psi_2$ to prove *(CRE5)*. Let us show the (only if) part. Let $\Psi \mapsto \preceq_\Psi$ be a DP assignment satisfying *(FunWA)* and *(CRE5)*. Let Ψ_1, Ψ_2 be two epistemic states, α, β be two formulae, and ω, ω', ω^2, ω^3 be four worlds such that ($\omega \models \alpha \Leftrightarrow \omega^2 \models \beta$), ($\omega' \models \alpha \Leftrightarrow \omega^3 \models \beta$), ($\omega, \omega' \notin \min([\alpha], \preceq_{\Psi_1})$), ($\omega^2, \omega^3 \notin \min([\beta], \preceq_{\Psi_2})$), and ($\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2} \omega^3$). We must prove that ($\omega \preceq_{\Psi_1 \circ \alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \beta} \omega^3$). We only provide the proof in the case when all four worlds ω, ω', ω^2, ω^3 are pairwise distinct (the remainder of the proof can be found at (Schwind, Konieczny, and Pérez 2023)). Assume first that $\omega, \omega' \models \alpha$. Then, $\omega^2, \omega^3 \models \beta$. From *(CR1)* we get that $\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega \preceq_{\Psi_1 \circ \alpha} \omega'$ and $\omega^2 \preceq_{\Psi_2} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \beta} \omega^3$. Yet $\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2} \omega^3$, so $\omega \preceq_{\Psi_1 \circ \alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \beta} \omega^3$. The case when $\omega, \omega' \not\models \alpha$ is proved similarly by using *(CR2)* instead of *(CR1)*. So, assume now that $\omega \models \alpha$ and $\omega' \not\models \alpha$. Then, $\omega^2 \models \beta$ and $\omega^3 \not\models \beta$. Since all worlds ω, ω', ω^2, ω^3 are pairwise distinct, there exists a formula γ such that $[\gamma] = \{\omega, \omega^2\} \cup \min([\alpha], \preceq_{\Psi_1}) \cup \min([\beta], \preceq_{\Psi_2})$. Clearly, we have that $\omega \models \alpha \wedge \gamma$, $\omega' \models \neg\alpha \wedge \neg\gamma$, and $\omega \notin \min([\alpha], \preceq_{\Psi_1}) \cup \min([\gamma], \preceq_{\Psi_1})$. So by *(CRE5)*, we get that (i) $\omega \preceq_{\Psi_1 \circ \alpha} \omega' \Leftrightarrow \omega \preceq_{\Psi_1 \circ \gamma} \omega'$. Likewise, since $\omega^2 \models \beta \wedge \gamma$, $\omega^3 \models \neg\beta \wedge \neg\gamma$, and $\omega^2 \notin \min([\beta], \preceq_{\Psi_2}) \cup \min([\gamma], \preceq_{\Psi_2})$, by *(CRE5)* again we get that (ii) $\omega^2 \preceq_{\Psi_2 \circ \beta} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \gamma} \omega^3$. Lastly, since $\omega, \omega^2 \models \gamma$, $\omega', \omega^3 \not\models \gamma$, $\omega, \omega' \notin \min([\gamma], \preceq_{\Psi_1})$, $\omega^2, \omega^3 \notin \min([\gamma], \preceq_{\Psi_2})$ and $\omega \preceq_{\Psi_1} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2} \omega^3$, by *(FunWA)* we get that (iii) $\omega \preceq_{\Psi_1 \circ \gamma} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \gamma} \omega^3$. Hence, from (i-iii) we get that $\omega \preceq_{\Psi_1 \circ \alpha} \omega' \Leftrightarrow \omega^2 \preceq_{\Psi_2 \circ \beta} \omega^3$, which shows that $\Psi \mapsto \preceq_\Psi$ satisfies *(FunWP)* and concludes the proof. □

And interestingly this gives us two dichotomy results on the properties satisfied by the operators:

Lemma 1. *If a DP assignment satisfies (FunWP), then it satisfies (DR) or (Lex).*

Proof. Let $\Psi \mapsto \preceq_\Psi$ be a DP assignment and assume that it does not satisfy *(DR)*, i.e., there exists an epistemic state Ψ, a formula α and two worlds ω, ω' such that $\omega \models \alpha$, $\omega' \not\models \alpha$, $\omega \notin \min([\alpha], \preceq_\Psi)$, $\omega' \prec_\Psi \omega$ and $\omega \preceq_{\Psi \circ \alpha} \omega'$.

First, let us prove that $\omega \prec_{\Psi \circ \alpha} \omega'$. Toward a contradiction, assume that $\omega \simeq_{\Psi \circ \alpha} \omega'$. Let Ψ' be an epistemic state, ω^2, ω^3 be two worlds such that $\omega^2 \models \alpha$, $\omega^3 \not\models \alpha$, $\omega, \omega^2 \notin \min([\alpha], \Psi')$ and $\omega^3 \prec_{\Psi'} \omega' \prec_{\Psi'} \omega \prec_{\Psi'} \omega^2$. Since $\omega' \prec_\Psi \omega$, $\omega' \prec_{\Psi'} \omega$ and $\omega \simeq_{\Psi \circ \alpha} \omega'$, by *(FunW)* we get that (i) $\omega \simeq_{\Psi' \circ \alpha} \omega'$. Then since $\omega^3 \prec_{\Psi'} \omega^2$ and $\omega' \prec_{\Psi'} \omega$, by (i) and *(FunWA)* we get that (ii) $\omega^3 \simeq_{\Psi' \circ \alpha} \omega^2$. Yet $\omega \prec_{\Psi'} \omega^2$, so by *(CR1)* we get that (iii) $\omega \prec_{\Psi' \circ \alpha} \omega^2$. And $\omega^3 \prec_{\Psi'} \omega'$, so by *(CR2)* we get that (iv) $\omega^3 \prec_{\Psi' \circ \alpha} \omega'$. But (i-iv) contradict the transitivity of $\preceq_{\Psi' \circ \alpha}$. Hence, $\omega \prec_{\Psi \circ \alpha} \omega'$.

We intend now to prove that $\circ$ satisfies *(Lex)*, that is, let Ψ_2 be any epistemic state, β be any formula, ω^2, ω^3 be two worlds such that $\omega^2 \models \beta$ and $\omega^3 \not\models \beta$, and let us show that $\omega^2 \prec_{\Psi_2 \circ \beta} \omega^3$. The proof is direct when $\omega^2 \in \min([\beta], \preceq_{\Psi_2})$ by Theorem 1 or when $\omega^2 \prec_{\Psi_2} \omega^3$ by *(CR3)*, so assume that $\omega^2 \notin \min([\beta], \preceq_{\Psi_2})$ and $\omega^3 \preceq_{\Psi_2} \omega^2$.

Then we can verify that the preconditions for *(FunWP)* are satisfied, i.e., we have that $\omega \models \alpha$, $\omega' \not\models \alpha$, $\omega^2 \models \beta$, $\omega^3 \not\models \beta$, $\omega, \omega' \notin \min([\alpha], \preceq_\Psi)$, $\omega^2, \omega^3 \notin \min([\beta], \preceq_{\Psi_2})$, $\omega' \preceq_\Psi \omega$ and $\omega^3 \preceq_{\Psi_2} \omega^2$. Then by *(FunWP)* and since we have shown that $\omega \prec_{\Psi\circ\alpha} \omega'$, we get that $\omega^2 \prec_{\Psi_2\circ\beta} \omega^3$. This shows that $\circ$ satisfies *(Lex)* and concludes the proof. $\square$

Lemma 2. *If a DP assignment satisfies (FunWP), then it satisfies (PR) or (Nat).*

Proof. Let $\Psi \mapsto \preceq_\Psi$ be a DP assignment and assume that it does not satisfy *(PR)*, i.e., there exists an epistemic state Ψ, a formula α and two worlds ω, ω' such that $\omega \models \alpha$, $\omega' \not\models \alpha$, $\omega \preceq_\Psi \omega'$ and $\omega' \preceq_{\Psi\circ\alpha} \omega$. Let us first remark by Theorem 1 that $\omega \notin \min([\alpha], \preceq_\Psi)$. Then, let us first show that $\omega \simeq_\Psi \omega'$ and $\omega \simeq_{\Psi\circ\alpha} \omega'$. On the one hand, since $\omega \preceq_\Psi \omega'$, by *(CR4)* we get that $\omega \preceq_{\Psi\circ\alpha} \omega'$, and since $\omega' \preceq_{\Psi\circ\alpha} \omega$ we get that $\omega \simeq_{\Psi\circ\alpha} \omega'$. On the other hand, since $\omega' \preceq_{\Psi\circ\alpha} \omega$, by *(CR3)* we get $\omega' \preceq_\Psi \omega$, and since $\omega \preceq_\Psi \omega'$ we get $\omega \simeq_\Psi \omega'$.

Now, we want to prove that $\Psi \mapsto \preceq_\Psi$ satisfies *(Nat)*. So let Ψ_2 be an epistemic state, β be a formula and ω^2, ω^3 be two worlds such that $\omega^2, \omega^3 \notin \min([\beta], \preceq_{\Psi_2})$. We need to show that $\omega^2 \preceq_{\Psi_2} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi_2\circ\beta} \omega^3$. The case when $\omega^2 \models \beta \Leftrightarrow \omega^3 \models \beta$ is direct using *(CR1-CR2)*. So assume that $\omega^2 \models \beta$ and $\omega^3 \not\models \beta$. If $\omega^2 \prec_{\Psi_2} \omega^3$, by *(CR3)* we get that $\omega^2 \prec_{\Psi_2\circ\beta} \omega^3$. So assume that $\omega^3 \preceq_{\Psi_2} \omega^2$, and let us show that $\omega^3 \preceq_{\Psi_2\circ\beta} \omega^2$. Yet we can verify that the preconditions for *(FunWP)* are satisfied, i.e., we have that $\omega \models \alpha \Leftrightarrow \omega^2 \models \beta$, $\omega' \models \alpha \Leftrightarrow \omega^3 \models \beta$, $\omega, \omega' \notin \min([\alpha], \preceq_\Psi)$, $\omega^2, \omega^3 \notin \min([\beta], \preceq_{\Psi_2})$, $\omega' \preceq_\Psi \omega$ and $\omega^3 \preceq_{\Psi_2} \omega^2$. Then by *(FunWP)* and since $\omega' \preceq_{\Psi\circ\alpha} \omega$, we get that $\omega^3 \preceq_{\Psi_2\circ\beta} \omega^2$. The remaining case is when $\omega^2 \not\models \beta$ and $\omega^3 \models \beta$, which can be proved identically to the case when $\omega^2 \models \beta$ and $\omega^3 \not\models \beta$ by permuting ω^2 and ω^3, and since all preconditions for *(FunWP)* are also satisfied when permuting ω and ω'. We have shown that $\omega^2 \preceq_{\Psi_2} \omega^3 \Leftrightarrow \omega^2 \preceq_{\Psi_2\circ\beta} \omega^3$ in every case, which concludes the proof that $\Psi \mapsto \preceq_\Psi$ satisfies *(Nat)*. $\square$

As a consequence of the previous results, we get that:

Theorem 5. *$\circ_N$, $\circ_L$ and $\circ_R$ are the only* TPO*-representable hDP revision operators.*

So there are only three hDP operators: Nayak's lexicographic revision $\circ_L$, Boutilier's natural revision $\circ_N$, and Booth and Meyer's restrained revision $\circ_R$. They are the only choice if one wants to have an homogeneous behaviour for TPO-representable iterated belief revision operators.

Note that these three operators are called *elementary revision operators* in (Chandler and Booth 2023), and they are characterized (for TPO-representable operators) by an axiom of "Independence of Irrelevant Alternatives" ($\mathrm{IIA}^*_\preceq$), inspired by a well-known axiom from social choice. So it is interesting to obtain this class from these two different intuitions and constructions.

5 Conclusion

We have studied the consequences of pushing one iteration further than Darwiche and Pearl's postulates, i.e., instead of linking iterations n+1 and n+2, we linked iterations n+2 and n+3. We have shown that this leads to more restricted subclasses of operators, which rule out non-homogeneous operators.

We do not claim that non-homogeneous operators have to be completely forbidden since they may make sense in some applications where the state of mind of an agent changes during its life (depending on the sequence of iterations) and there are some non-homogeneities at some points.

However, for many "normal" cases, this homogeneity requirement seems more than natural. Thus, we wanted to start the study of these homogeneous operators in this work. The results were quite surprising since very few operators satisfy these requirements: there is only one homogeneous DP (HDP) revision operator, namely Nayak's lexicographic revision.

If we relax these (obviously strong) conditions of HPD revision, we obtain hDP revision operators that could be considered the most interesting class of this work. Even here, we have only three operators in this class, namely Nayak's lexicographic revision, Boutilier's natural revision, and Booth and Meyer's restrained revision. If we add the "separation" postulate **(P)** (whose semantical counterpart is denoted by *(PR)*), this rules out Boutilier's natural revision, and we have a characterization of the two remaining operators. This class is restricted but very meaningful since these three operators are the only TPO-representable operators that satisfy a property of locality (that we call functionality - *(FunW)*), which states that the relative plausibility of two (non-minimal) worlds is only a function of these two worlds.

Thus, the hDP postulates can be seen as very strong since they are satisfied by only three TPO-representable revision operators. However, we want to stress that this is caused by the combination of these postulates and the other constraints on revision operators, especially the special behavior that belief revision operators impose on minimal worlds of the new piece of information. If hDP postulates are applied to improvement operators, it is expected that the class of corresponding operators will be larger. However, the extent of this increase in the class size remains unknown and will be a subject for future investigation.

One possible relaxation could be to study operators that do not satisfy (CE2/CE2w). In fact this postulate has been criticized in several works (Spohn 1988; Konieczny and Pino Pérez 2000; Rodrigues 2005; Rodrigues, Gabbay, and Russo 2010; Schwind and Konieczny 2020) since in some cases it forces a stronger behaviour than expected, in particular by linking together all the information coming in one step of revision. So studying the class of homogeneous operators that do not satisfy (CE2w) seems interesting.

Another future work is related to these notions of homogeneity, locality, and functionality. In this work, we impose homogeneity using a (particular) functionality condition, but one can figure out other notions of functionality and homogeneity. Having a more general view of these two notions seems to be an interesting issue.

Acknowledgements

This work has benefited from the support of the AI Chair BE4musIA of the French National Research Agency (ANR-20-CHIA-0028). Nicolas Schwind has benefited from the support of the JSPS Kakenhi Grant Number JP20K11947.

References

Alchourrón, C. E.; Gärdenfors, P.; and Makinson, D. 1985. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic* 50:510–530.

Booth, R., and Meyer, T. A. 2006. Admissible and restrained revision. *Journal of Artificial Intelligence Research* 26:127–151.

Boutilier, C. 1996. Iterated revision and minimal change of conditional beliefs. *Journal of Philosophical Logic* 25(3):263–305.

Chandler, J., and Booth, R. 2023. Elementary belief revision operators. *Journal of Philosophical Logic* 52(1):267–311.

Darwiche, A., and Pearl, J. 1997. On the logic of iterated belief revision. *Artificial Intelligence* 89(1-2):1–29.

Dubois, D., and Prade, H. 1991. Epistemic entrenchment and possibilistic logic. *Artificial Intelligence* 50:223–239.

Dubois, D.; Lang, J.; and Prade, H. 1994. Possibilistic logic. In *Handbook of Logic in Artificial Intelligence and Logic Programming. Volume 3 : Nonmonotonic Reasoning and Uncertain Reasoning*. Oxford Science Publications.

Fermé, E., and Hansson, S. O. 2011. AGM 25 years: Twenty-five years of research in belief change. *Journal of Philosophical Logic* 40(2):295–331.

Gärdenfors, P. 1988. *Knowledge in flux*. MIT Press.

Gärdenfors, P. 1990. Belief revision and nonmonotonic logic: two sides of the same coin ? In *Proceedings of the ninth European Conference on Artificial Intelligence (ECAI'90)*, 768–773.

Jin, Y., and Thielscher, M. 2007. Iterated belief revision, revised. *Artificial Intelligence* 171(1):1–18.

Katsuno, H., and Mendelzon, A. O. 1991. Propositional knowledge base revision and minimal change. *Artificial Intelligence* 52:263–294.

Konieczny, S., and Pino Pérez, R. 2000. A framework for iterated revision. *Journal of Applied Non-Classical Logics* 10(3-4):339–367.

Konieczny, S., and Pino Pérez, R. 2008. Improvement operators. In *Proceedings of the 11th International Conference on Principles of Knowledge Representation and Reasoning (KR'08)*, 177–187.

Konieczny, S.; Medina Grespan, M.; and Pino Pérez, R. 2010. Taxonomy of improvement operators and the problem of minimal change. In *Proceedings of the 12th International Conference on Principles of Knowledge Representation and Reasoning (KR'10)*, 161–170.

Kraus, S.; Lehmann, D.; and Magidor, M. 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence* 44:167–207.

Lehmann, D., and Magidor, M. 1992. What does a conditional knowledge base entail? *Artificial Intelligence* 55:1–60.

Nayak, A. 1994. Iterated belief change based on epistemic entrenchment. *Erkenntnis* 41:353–390.

Rodrigues, O.; Gabbay, D.; and Russo, A. 2010. Belief revision. In *Handbook of Philosophical Logic: Volume 16*. Springer Netherlands. 1–114.

Rodrigues, O. 2005. Iterated revision and automatic similarity generation. In Artëmov, S. N.; Barringer, H.; d'Avila Garcez, A. S.; Lamb, L. C.; and Woods, J., eds., *We Will Show Them! Essays in Honour of Dov Gabbay, Volume Two*, 591–614. College Publications.

Rott, H. 2009. Shifting priorities: Simple representations for twenty-seven iterated theory change operators. In *Towards Mathematical Philosophy*, volume 28 of *Trends in logic*. Springer. 269–296.

Schwind, N., and Konieczny, S. 2020. Non-prioritized iterated revision: Improvement via incremental belief merging. In *Proceedings of the 17th International Conference on Principles of Knowledge Representation and Reasoning (KR'20)*, 738–747.

Schwind, N.; Konieczny, S.; and Pérez, R. P. 2023. Iteration of iterated belief revision (remaining proofs). https://nicolas-schwind.github.io/SKP-KR23-remaining-proofs.pdf.

Schwind, N.; Konieczny, S.; and Pino Pérez, R. 2022. On the representation of Darwiche and Pearl's epistemic states for iterated belief revision. In *Proceedings of the 19th International Conference on Principles of Knowledge Representation and Reasoning (KR'22)*, 320–330.

Spohn, W. 1988. *Ordinal Conditional Functions: A Dynamic Theory of Epistemic States*. Springer Netherlands. 105–134.