# Simple Contrapositive Assumption-Based Argumentation with Partially-Ordered Preferences

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**Abstract**

We show that assumption-based argumentation frameworks, based on contrapositive logics and partially-ordered preference functions, provide a solid platform for argumentation-based representation and reasoning. Two useful properties of the preference functions are identified (selectivity and max-lower-boundedness), and extended forms of attacks relations are supported (∃- and ∀-attacks), which assure several desirable properties and a variety of reasoning modes.

## 1 Introduction

Argumentation is a useful approach for modeling defeasible reasoning with many fruitful applications (see, e.g., (Thimm and García 2010; Kok et al. 2012; Young, Modgil, and Rodrigues 2016)). Assumption-based argumentation (ABA, (Bondarenko et al. 1997)) is a central approach in argumentation-based reasoning (Toni 2014; Čyras et al. 2018). In (Heyninck and Arieli 2020; Arieli and Heyninck 2021) it is shown that simple contrapositive ABA frameworks, a class of ABA frameworks (ABFs, for short) induced by logics that preserve the rule of contraposition and whose contrary operator is represented by a negation operator, are particularly suitable for reasoning in the presence of conflicting arguments and counterarguments.

So far, simple contrapositive ABFs were assumed to be either non-prioritized (Heyninck and Arieli 2020), or based on linear preference orders among the assumptions (Arieli and Heyninck 2021). However, in many settings, assuming a total order greatly limits the realistic modelling capabilities of a formal system, e.g., when agents do not know the actual preferences of each assumption or since different sources of information have different preferences over the assumptions. This is illustrated in the following example:

**Example 1.** Suppose that one wants to compare reviews of hotels in a certain city, not only by their final scores, but by taking into account several considerations, such as location, price, quality of service, etc. In this case, tuples of values are compared (for example, one hotel may be preferred over the other if more than half of its components are superior in the respective tuples), and hence the comparison is not strictly linear. We shall return to this in Examples 4, 5, and 9 below.

The present work takes simple contrapositive ABFs one step forwards and shows that the incorporation of partial orders for making preferences among arguments considerably extends the expressive power of such frameworks while preserving much of their properties shown in earlier works. Thus, for instance, we introduce several criteria for comparing sets of arguments, the elements of which are not necessarily mutually comparable with respect to the preference relations, and consider a new property of the preference setting ('selecting' setting, which requires that the aggregated value assigned to a set of values is one of these values), under which the set of the stable or preferred extensions of the ABF coincide with the preferred maximally consistent subsets of the set of assumptions. Together with another property ('max-lower-boundedness', which requires that the aggregated value assigned to a set of values is bounded by these values), further rationality postulates are guaranteed in this setting.

It is important to note that partially-ordered preference relations in ABFs have already been considered in the literature, most notably in ABA$^+$ systems (Čyras and Toni 2016; Čyras 2017). However, the latter is adequate only for the weakest link principle for comparing arguments (taking into consideration the least preferred assumptions of an argument), while we do not confine ourselves to a particular preference setting. Moreover, as the deducibility relation is closed under contraposition, we are able to assure some rationality postulates (like tolerance, see Section 5.2), which are not necessarily satisfied in other prioritized ABFs (such as ABA$^+$, see a discussion in (Čyras 2017)). Finally, the incorporation of partial orders allows us to consider new forms of attacks relations (∃-attacks and ∀-attacks), which are not supported by strict preferences (Arieli and Heyninck 2021). This enables some new types of reasoning which were not available previously.

## 2 Preliminaries

This section contains some background material. Further details can be found in (Heyninck and Arieli 2020; Arieli and Heyninck 2021).

In what follows we shall denote by $\mathcal{L}$ an arbitrary propositional language. Atomic formulas in $\mathcal{L}$ are denoted by $p, q, r$, compound formulas are denoted by $\psi, \phi, \sigma$, and sets of formulas in $\mathcal{L}$ are denoted by $\Gamma$, $\Delta$, $\Theta$ (possibly primed or indexed). The powerset of $\mathcal{L}$ is denoted by $\wp(\mathcal{L})$.

A *logic* is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language containing at least a negation $\neg$, conjunction $\wedge$, disjunction $\vee$, implication $\supset$, and the falsity constant $\mathsf{F}$, with their usual definitions, and where $\vdash$ is a consequence relation for $\mathcal{L}$, that is, a binary relation between sets of formulas and formulas in $\mathcal{L}$, satisfying the following conditions: (i) if $\psi \in \Gamma$ then $\Gamma \vdash \psi$ (Reflexivity), (ii) if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash \psi$ (Monotonicity), and (iii) if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$ then $\Gamma, \Gamma' \vdash \phi$ (Transitivity). We also assume, as usual, that the logic is closed under substitutions and it is nontrivial (i.e., there are $\Gamma \neq \emptyset$ and $\psi$ such that $\Gamma \nvdash \psi$).

We say that $\psi$ is $\vdash$-*tautological* if $\vdash \psi$, and that $\Gamma$ is $\vdash$-*consistent* if $\Gamma \nvdash \mathsf{F}$. When $\Gamma$ is finite we denote by $\bigwedge\Gamma$ (respectively, by $\bigvee\Gamma$), the conjunction (respectively, the disjunction) of all the formulas in $\Gamma$.

The following family of assumption-based argumentation frameworks (Bondarenko et al. 1997) is shown in (Heyninck and Arieli 2020) to be a useful setting for argumentative reasoning.

**Definition 1.** An *assumption-based framework* (ABF) is a tuple $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, where:
• $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is a (propositional) logic,
• $\Gamma$ (the *strict assumptions*) and $Ab$ (the *candidate* or *defeasible assumptions*) are distinct sets of $\mathcal{L}$-formulas, where the former is assumed to be $\vdash$-consistent and the latter is assumed to be nonempty,
• $\sim : Ab \to \wp(\mathcal{L})$ is a *contrariness operator*, assigning a finite set of $\mathcal{L}$-formulas to every defeasible assumption in $Ab$, such that for every consistent and non-tautological formula $\psi \in Ab \setminus \{\mathsf{F}\}$ it holds that $\psi \nvdash \bigwedge\!\sim\!\psi$ and $\bigwedge\!\sim\!\psi \nvdash \psi$.

A *simple contrapositive* ABF is an assumption-based framework $\langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ such that:
• $\mathfrak{L}$ is *explosive*: for every $\mathcal{L}$-formula $\psi$, the set $\{\psi, \neg\psi\}$ is $\vdash$-inconsistent (thus $\psi, \neg\psi \vdash \varphi$ for every $\varphi$),
• $\mathfrak{L}$ is *contrapositive*: it holds that $\vdash \neg\mathsf{F}$, and for every nonempty $\Gamma$ and $\psi$, we have that $\Gamma \vdash \neg\psi$ iff either $\psi = \mathsf{F}$ or for every $\varphi \in \Gamma$ it holds that $\Gamma \setminus \{\varphi\}, \psi \vdash \neg\varphi$,[1]
• for every formula $\psi$ it holds that $\sim\!\psi = \{\neg\psi\}$.

In (Arieli and Heyninck 2021), simple contrapositive ABFs are augmented with preferences among the defeasible assumptions. Intuitively, smaller values indicate higher preferences.

**Definition 2.** A *linearly ordered prioritized assumption-based framework* (a linear pABF, for short) is a pair $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$, where $\mathsf{ABF}$ is a simple contrapositive ABF and $\mathcal{P} = \langle g, f \rangle$ is a linear priority (or preference) setting, where:
• $g : Ab \to \mathbb{N}$ is a total function on $Ab$, called *linear allocation function*. We denote $g(\Delta) = \{g(\delta) \mid \delta \in \Delta\}$.
• $f$ is a *numeric aggregation function*, i.e.: a total function that maps multisets of non-negative natural numbers into a non-negative real number, such that $\forall x \in \mathbb{N}\ f(\{x\}) = x$. We also assume that an aggregation function is $\subseteq$-*coherent* in its values, namely, it is either non-decreasing with respect to the subset relation ($f(X') \leq f(X)$ whenever $X' \subseteq X$) or non-increasing with respect to the subset relation ($f(X') \geq f(X)$ whenever $X' \subseteq X$).

Intuitively, $g(\phi)$ represents the strength of the assumption $\phi$, where lower numbers indicate *higher* strengths. Aggregation functions then give a method to assign a single strength value to a set of assumptions on the basis of the strengths of the composite members.

Attacks in pABFs are defined by preferred counter defeasible information:

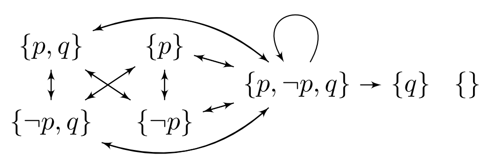
**Definition 3.** Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ be a linear pABF with $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, $\mathcal{P} = \langle g, f \rangle$, $\Delta, \Theta \subseteq Ab$, and $\psi \in Ab$.
• $\Delta$ *attacks* $\psi$ (w.r.t. $\Gamma$) iff $\Gamma, \Delta \vdash \neg\psi$. We say that $\Delta$ attacks $\Theta$ if $\Delta$ attacks some $\psi \in \Theta$.
• If $\Delta$ attacks $\psi$, the $\mathcal{P}$-*attacking value* of $\Delta$ on $\psi$ is:

$$\mathsf{val}_{f,g}(\Delta, \psi) = \min\{f(g(\Delta')) \mid \Delta' \text{ is a } \subseteq\text{-minimal subset of } \Delta \text{ that attacks } \psi\}.$$

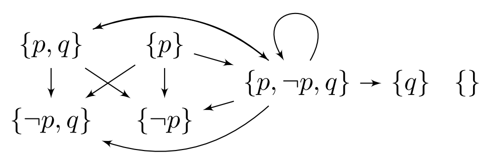
• $\Delta$ *linearly p-attacks* $\psi$ iff $\Delta$ attacks $\psi$ and $\mathsf{val}_{f,g}(\Delta, \psi) \leq f(g(\psi))$. We say that $\Delta$ linearly p-attacks $\Theta$ if $\Delta$ linearly p-attacks some $\psi \in \Theta$.

Thus, a set of assumptions $\Delta$ linearly p-attacks an assumption $\psi$ if $\Delta$ implies the contrary of $\psi$ and the aggregated value $\mathsf{val}_{f,g}(\Delta, \psi)$ is at least as strong as the value of the attacked assumption $f(g(\psi))$.

**Example 2.** Let $\mathfrak{L} = \mathsf{CL}$ (classical logic), $\Gamma = \emptyset$, $Ab = \{p, \neg p, q\}$, and $\sim\!\psi = \{\neg\psi\}$ for every $\psi \in Ab$. An attack diagram for this ABF is shown in Figure 1a.[2] Note that since in classical logic inconsistent sets of premises imply *any* conclusion, the classically inconsistent set $\{p, \neg p, q\}$ attacks all the other sets in the diagram. (For instance, $\{p, \neg p, q\}$ attacks $\{q\}$, since $p, \neg p, q \vdash \neg q$). [3]




(a) All the attacks (no preferences)




(b) Only the linear p-attacks

Figure 1: Diagrams for Example 2

Consider now the pABF that is obtained from this ABF, together with the allocation function $g(p) = 1$, $g(\neg p) = 2$,

[1] Classical logic, intuitionistic logic, and standard modal logics are all explosive and contrapositive.

[2] By Note 1 below, we include in the diagram only *closed sets* (i.e., only subsets $\Delta \subseteq Ab$ such that $\Delta = Ab \cap Cn_{\vdash}(\Gamma \cup \Delta)$) (see Definition 4). Thus, the set $\{p, \neg p\}$ is omitted from the diagram.

[3] Notice furthermore that the emptyset does *not* attack $\{p, \neg p\}$, as $\emptyset \nvdash p$ or $\emptyset \nvdash \neg p$: the attacks used in assumption-based argumentation are *pointed* in the sense that the contrary of a single assumption needs to be derived for an attack to take place.

$g(q) = 3$, and the aggregation $f = \max$. The diagram of the linear p-attack of the pABF is shown in Figure 1b.

The last definition gives rise to the following adaptation to pABFs of the usual Dung-style semantics (Dung 1995) for abstract argumentation frameworks.

**Definition 4.** Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ with $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, and let $\Delta \subseteq Ab$. We denote $Cn_\vdash(\Theta) = \{\psi \mid \Theta \vdash \psi\}$. Then:
• $\Delta$ is *closed* iff $\Delta = Ab \cap Cn_\vdash(\Gamma \cup \Delta)$.
• $\Delta$ is *conflict-free* iff there is no $\Delta' \subseteq \Delta$ that linearly p-attacks some $\psi \in \Delta$.
• $\Delta$ is *naive* iff it is closed and maximally conflict-free.
• $\Delta$ *defends* $\Delta' \subseteq Ab$ iff for every closed set $\Theta$ that linearly p-attacks $\Delta'$ there is $\Delta'' \subseteq \Delta$ that linearly p-attacks $\Theta$.
• $\Delta$ is *admissible* iff it is closed, conflict-free, and defends every $\Delta' \subseteq \Delta$.
• $\Delta$ is *complete* iff it is admissible and contains every $\Delta' \subseteq Ab$ that it defends.
• $\Delta$ is *well-founded* iff $\Delta = \bigcap\{\Theta \subseteq Ab \mid \Theta \text{ is complete}\}$.
• $\Delta$ is *grounded* iff it is $\subseteq$-minimally complete.
• $\Delta$ is *preferred* iff it is $\subseteq$-maximally admissible.
• $\Delta$ is *stable* iff it is closed, conflict-free, and linearly p-attacks all $\psi \in Ab \setminus \Delta$.

**Note 1.** As shown in (Heyninck and Arieli 2020; Arieli and Heyninck 2021), for (linearly ordered) simple contrapositive ABFs the closure requirement in Definition 4 is redundant. We shall therefore disregard it in what follows (see also Section 4.2).

The sets of the complete, naive, grounded, well-founded, preferred, and stable extensions of pABF are respectively denoted by $\mathsf{Cmp}(\mathsf{pABF})$, $\mathsf{Naive}(\mathsf{pABF})$, $\mathsf{Grd}(\mathsf{pABF})$, $\mathsf{WF}(\mathsf{pABF})$, $\mathsf{Prf}(\mathsf{pABF})$ and $\mathsf{Stb}(\mathsf{pABF})$. We denote by $\mathsf{Sem}(\mathsf{pABF})$ any of these sets.

The entailment relations that are induced from a pABF (with respect to a certain semantics) are defined as follows:

**Definition 5.** For $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Cmp}, \mathsf{WF}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$, we denote:

• $\mathsf{pABF} \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for *every* $\Delta \in \mathsf{Sem}(\mathsf{pABF})$.

• $\mathsf{pABF} \mid\!\sim^{\cup}_{\mathsf{Sem}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for *some* $\Delta \in \mathsf{Sem}(\mathsf{pABF})$.

**Example 3.** Consider again the ABF in Example 2, where $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$ and $Ab = \{p, \neg p, q\}$ (see also Figure 1a). Here, $\mathsf{Naive}(\mathsf{ABF}) = \mathsf{Prf}(\mathsf{ABF}) = \mathsf{Stb}(\mathsf{ABF}) = \{\{p, q\}, \{\neg p, q\}\}$,[4] thus $\mathsf{ABF} \mid\!\sim^{*}_{\mathsf{Sem}} q$ for every $* \in \{\cup, \cap\}$ and every $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$. Also, $\mathsf{Grd}(\mathsf{ABF}) = \mathsf{WF}(\mathsf{ABF}) = \{\emptyset\}$, since there are no unattacked arguments. Thus, when all the assumptions have the same priority, we have that for $* \in \{\cup, \cap\}$ and $\mathsf{Sem} \in \{\mathsf{Grd}, \mathsf{WF}\}$ it holds that $\mathsf{ABF} \mid\!\sim^{*}_{\mathsf{Sem}} \psi$ only if $\psi$ is a classical tautology.

When preferences are incorporated as in Example 2, we have that $\mathsf{Cmp}(\mathsf{pABF}) = \mathsf{Grd}(\mathsf{pABF}) = \mathsf{WF}(\mathsf{pABF}) = \mathsf{Prf}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF}) = \{\{p, q\}\}$. It follows that $\mathsf{pABF} \mid\!\sim^{*}_{\mathsf{Sem}} p$ and $\mathsf{pABF} \mid\!\sim^{*}_{\mathsf{Sem}} q$ for every semantics $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{WF}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$ and every $* \in \{\cup, \cap\}$. Note that in case that the preference value of $q$ is smaller than those of $p$ and $\neg p$, the set $\{p, \neg p, q\}$ does not attack the sets $\{q\}$ and $\{p, q\}$, in which case the set $\{q\}$ also belongs to $\mathsf{Cmp}(\mathsf{pABF})$. In this case $\mathsf{Grd}(\mathsf{pABF}) = \mathsf{WF}(\mathsf{pABF}) = \{\{q\}\}$, while $\mathsf{Prf}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF}) = \{\{p, q\}\}$.

[4] $\{p\}$ is not even complete, as it defends $q$, which is not in $\{p\}$.

## 3 Non-Linear Preferences

We now generalize the setting in the previous section to preferences that do not necessarily have a strict (linear) order. This considerably extends the expressive power of the ABFs, as demonstrated next.

**Example 4.** The following scenario resembles the motivating illustration in the introduction (Example 1). A tourist considers two restaurants $\mathtt{r_1}, \mathtt{r_2}$ and a coffeehouse $\mathtt{c}$, where one restaurant at the most may be visited. This may be represented by an ABF (based, e.g., on $\mathsf{CL}$) with a strict assumption $\neg(\mathtt{r_1} \wedge \mathtt{r_2})$ and the set $\{\mathtt{r_1}, \mathtt{r_2}, \mathtt{c}\}$ of defeasible assumptions.

In a linear comparison, only one numerical value can be attributed to each dining place, while in a comparison according to a partial order ratio one can refer to a vector of values taking into considerations several aspects, e.g., $\langle q, p, s \rangle$, representing food quality, price, and service. Suppose, for instance, that a website offers evaluations (on a descending scale of 1 to 5, i.e., 1 is the highest value) of these places along these three criteria. Suppose further that $\mathtt{r_1}$ is evaluated by $\langle 2, 3, 3 \rangle$, the scores of $\mathtt{r_2}$ are $\langle 4, 2, 2 \rangle$, and the scores of $\mathtt{c}$ are $\langle 3, 3, 3 \rangle$. One way to compare these vectors is by deciding that one place is preferred ($\leq$-smaller) over the other iff it receives equal or higher scores in all aspects. Then $\mathtt{r_1}$ is preferred over $\mathtt{c}$, while $\mathtt{r_2}$ is $\leq$-incomparable with both $\mathtt{r_1}$ and $\mathtt{c}$.

For supporting non-linear preferences, we generalize the definitions of Section 2:

• linear allocation functions are traded by allocation functions whose values need not be linearly ordered,
• numeric aggregation functions are replaced by aggregation functions that need not be numeric: their ranges are *sets* of (partially ordered) values, rather than numbers,
• a *quantitative evaluation indicator* $\dagger \in \{\exists, \forall\}$ indicates how the aggregated sets should be collectively evaluated. Accordingly, we trade linear p-attacks by $\dagger$-p-attacks.

In the following definition, as in the linear case, $v_1 < v_2$ is intuitively understood as a preference of $v_1$ over $v_2$. Thus, $v_1 \leq v_2$ means that $v_1$ is 'at least as preferred as' $v_2$.

**Definition 6** (Definition 2 extended)**.** Let $\mathbb{P} = \langle \mathbb{V}, \leq \rangle$ be a partial order.
• $v_1 \in \mathbb{V}$ is *(strictly) $\exists$-$\mathbb{P}$-stronger* than $V_2 \subseteq \mathbb{V}$ iff there is *some* $v_2 \in V_2$ such that $v_1 < v_2$.
• $v_1 \in \mathbb{V}$ is *(strictly) $\forall$-$\mathbb{P}$-stronger* than $V_2 \subseteq \mathbb{V}$ iff for *all* $v_2 \in V_2$ it holds that $v_1 < v_2$.
• A *$\mathbb{P}$-allocation function* on $Ab$ (an allocation function, when $\mathbb{P}$ is known or arbitrary) is a total function $g : Ab \to \mathbb{V}$.
• An *aggregation function* on $\mathbb{V}$ is a total function $f : \wp(\mathbb{V}) \to \wp(\mathbb{V}) \setminus \emptyset$, where $f(\mathsf{S}) = \mathsf{S}$ if $\mathsf{S}$ is a singleton.[5]
• A *preference setting* for $Ab$ is a quadruple $\mathcal{P} = \langle \mathbb{P}, g, f, \dagger \rangle$,

[5] We shall usually identify singletons with their elements.

where $g$ is a $\mathbb{P}$-allocation function on $Ab$, $f$ is an aggregation function on $\{g(\Delta) \mid \Delta \subseteq Ab\}$, and $\dagger \in \{\exists, \forall\}$.

Thus, there are two ways of comparing sets of values with a single value: by $\exists$-$\mathbb{P}$-comparison it suffices to find a single value in $V_2$ weaker than $v_1$, whereas the $\forall$-$\mathbb{P}$-comparison requires that every value in $V_2$ is weaker than $v_1$.

**Example 5.** In Example 4, $g(\mathtt{r_1}) = \langle 4,3,3\rangle$, $g(\mathtt{r_2}) = \langle 2,4,4\rangle$, and $g(\mathtt{c}) = \langle 3,3,3\rangle$ form a partial order in which $g(\mathtt{r_1}) < g(\mathtt{c})$ and the other values are incomparable. Thus, $g(\mathtt{r_1})$ is $\exists$-stronger, but not $\forall$-stronger, than $\{g(\mathtt{r_2}), g(\mathtt{c})\}$. Aggregation functions in this case (or for any complete lattice) may be, e.g., the identity, the summation $\Sigma_{x\in S}\, x$ (whenever it is defined), the least-upper-bound $\mathsf{lub}(S)$, the $<$-maximum $\mathsf{max}(S) = \{x \in S \mid \neg\exists y \in S \text{ such that } y > x\}$, the greatest lower bound $\mathsf{glb}(S)$, and the $<$-minimum $\mathsf{min}(S) = \{x \in S \mid \neg\exists y \in S \text{ such that } y < x\}$.

**Note 2.** Let $\mathbb{P} = \langle \mathbb{V}, \leq\rangle$ be a partial order.
**a)** Clearly, for every $v \in \mathbb{V}$ and $V \subseteq \mathbb{V}$, if $v$ is $\forall$-$\mathbb{P}$-stronger than $V$, then $v$ is $\exists$-$\mathbb{P}$-stronger than $V$, but not necessarily vice-versa (as Example 5 shows).
**b)** For any $\dagger \in \{\exists, \forall\}$, the relation "strictly $\dagger$-$\mathbb{P}$-stronger" preserves the relations $<$ on singletons: $v_1 < v_2$ iff $v_1$ is strictly $\dagger$-$\mathbb{P}$-stronger than $\{v_2\}$.
**c)** When $\mathbb{P}$ is linear, the claim that $v$ is strictly $\exists$-$\mathbb{P}$-stronger than $V$ means that $v < \max(V)$ and the claim that $v$ is strictly $\forall$-$\mathbb{P}$-stronger than $V$ means that $v < \min(V)$.

Next, we consider some properties of preference settings, which will later be useful in showing rationality postulates and properties of the resulting entailment relations. We start with reversibility.

**Definition 7.** Let $\mathcal{P} = \langle \mathbb{P}, g, f, \dagger\rangle$ be a preference setting for $Ab$, and $\emptyset \neq \Delta \subseteq Ab, \phi \in Ab$.
• $\phi \prec_{\mathcal{P}} \Delta$ if $f(g(\phi))$ is strictly $\dagger$-$\mathbb{P}$-stronger than $f(g(\Delta))$.
• $\mathcal{P}$ is *reversible*, if when $\phi \prec_{\mathcal{P}} \Delta$, there is a $\delta \in \Delta$ such that $\delta \not\prec_{\mathcal{P}} \Delta \cup \{\phi\} \setminus \{\delta\}$.

Thus, $\mathcal{P}$ is reversible if, whenever $\phi$ is strictly $\dagger$-$\mathbb{P}$-stronger than $\Delta$, we can substitute $\phi$ for some $\delta \in \Delta$ and end up with a set of assumptions $\Delta \cup \{\phi\} \setminus \{\delta\}$ that is not strictly $\dagger$-$\mathbb{P}$-weaker than $\delta$. It is not difficult to show that if $\mathcal{P} = \langle \mathbb{P}, g, f, \dagger\rangle$ is a preference setting in which $\mathbb{P}$ is linear, the range of $f$ is restricted to singletons (that is, $f$ is of the form $\wp(\mathbb{V}) \to \mathbb{V}$, similar to Definition 2), and $\dagger \in \{\exists, \forall\}$, then $\mathcal{P}$ is reversible according to (Arieli and Heyninck 2021, Definition 10) iff it is reversible (according to Definition 7).

We will later see that reversibility is an important condition to ensure many basic rationality postulates, such as consistency (see Proposition 1).

**Example 6.** As shown in (Heyninck and Arieli 2020), for every allocation function $g$, the linear preference settings $\langle \mathbb{N}, g, \mathsf{min}\rangle$ and $\langle \mathbb{N}, g, \mathsf{max}\rangle$ are reversible.[6] Thus, by Definition 7, for every $\dagger \in \{\exists, \forall\}$, the preference settings $\langle \mathbb{N}, g, \mathsf{min}, \dagger\rangle$ and $\langle \mathbb{N}, g, \mathsf{max}, \dagger\rangle$ are reversible as well. It is not difficult to check that this carries over to every finite partial order $\mathbb{P}$ (so every set has a minimum and a maximum). For similar reasons, for every complete lattice $\mathbb{P}$, allocation function $g$, and $\dagger \in \{\exists, \forall\}$, the preference settings $\langle \mathbb{P}, g, \mathsf{glb}, \dagger\rangle$ and $\langle \mathbb{P}, g, \mathsf{lub}, \dagger\rangle$ are reversible. Clearly, the summation function is not reversible.

[6] Here $\mathbb{N}$ denotes the linear order over the natural numbers.

The next property ensures that $f(g(\Delta))$ is a selection of values in $\{f(g(\delta)) \mid \delta \in \Delta\}$, i.e., $f(g(\Delta))$ does not introduce 'new' values other than those that are assigned to the elements in $\Delta$.

**Definition 8.** A preference setting $\mathcal{P} = \langle \mathbb{P}, g, f, \dagger\rangle$ for $Ab$ is called *selecting*, if for every nonempty set $\Delta \subseteq Ab$ it holds that $f(g(\Delta)) \subseteq \bigcup_{\delta\in\Delta} f(g(\delta))$.

**Example 7.** $\langle \mathbb{P}, g, \mathsf{min}, \dagger\rangle$ and $\langle \mathbb{P}, g, \mathsf{max}, \dagger\rangle$ are selecting for every $g$ and $\dagger \in \{\exists, \forall\}$.

**Lemma 1.** *A selecting preference setting is also reversible.*

Definitions 1 and 2 are now generalized as follows:

**Definition 9.** A *prioritized assumption-based framework* (prioritized ABF, or pABF, for short) is a pair $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P}\rangle$, where $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg\rangle$ is a simple contrapositive assumption-based argumentation framework and $\mathcal{P}$ is a preference setting for $Ab$. We shall say that $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P}\rangle$ is reversible or selecting, if so is $\mathcal{P}$.

**Definition 10** (Definition 3 extended)**.** Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P}\rangle$ be a prioritized ABF with $\mathcal{P} = \langle \mathbb{P}, g, f, \dagger\rangle$, $\Delta, \Theta \subseteq Ab$, and $\psi \in Ab$.
• $\Delta$ *attacks* $\psi$ iff $\Gamma, \Delta \vdash \neg\psi$. We say that $\Delta$ attacks $\Theta$ if $\Delta$ attacks some $\psi \in \Theta$.
• Suppose that $\Delta$ attacks $\psi$ . The $\mathcal{P}$*-attacking values* of $\Delta$ on $\psi$ are the elements of the set

$$\mathsf{val}_{f,g}(\Delta, \psi) \;=\; \{f(g(\Delta')) \mid \Delta' \text{ is a } \subseteq\text{-minimal subset of } \Delta \text{ that attacks } \psi\}.$$

• $\Delta$ $\dagger$*-p-attacks* $\psi$ iff $\Delta$ attacks $\psi$ and there is a set $V \in \mathsf{val}_{f,g}(\Delta, \psi)$ s.t. $f(g(\psi))$ is not strictly $\dagger$-$\mathbb{P}$-stronger than $V$. We say that $\Delta$ $\dagger$-p-attacks $\Theta$ if $\Delta$ $\dagger$-p-attacks some $\psi \in \Theta$.

Thus, a set $\Delta$ $\dagger$-p-attacks a formula $\psi$ if it has a subset $\Delta'$ that attacks $\psi$ and $\psi \not\prec_{\mathcal{P}} \Delta'$. The intuition behind $\dagger$-p-attacks is that an attack by $\Delta$ on the assumption $\psi$ is successful if the attacking $\Delta$ is not strictly weaker than the attacked assumption $\psi$ according to the preference setting $\mathcal{P}$.[7]

**Lemma 2.** *If $\Delta$ $\exists$-p-attacks $\psi$ then $\Delta$ $\forall$-p-attacks $\psi$.*

**Example 8.** Consider again Example 2, this time with $\mathbb{V} = \{a, b, c, d\}$ in which $a, b, c$ are $<$-incomparable and $x < d$ for every $x = a, b, c$, and where $g(p) = a$, $g(\neg p) = b$ and $g(q) = c$. Now, $\Delta = \{p, \neg p, q\}$ attacks $q$, but:
**1)** If $f(S) = \mathsf{lub}(S)$, so $\mathsf{val}_{f,g}(\Delta, q) = \{f(g(\{p, \neg p\}))\} = \{\{d\}\}$, and since $d > c = f(g(q))$ we have that $\Delta$ does *not* $\dagger$ p-attack $q$ for any $\dagger \in \{\forall, \exists\}$.
**2)** If $f(S) = \mathsf{min}(S)$, so $\mathsf{val}_{f,g}(\Delta, q) = \{f(g(\{p, \neg p\}))\} = \{\{a, b\}\}$, and $c = f(g(q))$ is not $<$-smaller than $a$ or $b$. Thus, $\Delta$ $\dagger$-p-attacks $q$ for every $\dagger \in \{\forall, \exists\}$.
**3)** Suppose that $a < c < d$, and the rest is the same as in

[7] Notice that as attacks take place from sets of assumptions to single assumptions, it is sufficient to have a way to compare a set of assumptions with a single assumption (as in Definition 6), and it is not necessary to compare two sets of assumptions.

Item 2. Then still $\mathsf{val}_{f,g}(\Delta, q) = \{\{a, b\}\}$, so this time $f(g(q))$ is not strictly $\forall$-$\mathbb{P}$-stronger than $\mathsf{val}_{f,g}(\Delta, q)$, but it *is* strictly $\exists$-$\mathbb{P}$-stronger than $\mathsf{val}_{f,g}(\Delta, q)$. Thus, $\Delta$ $\forall$-p-attacks $q$ but it does not $\exists$-p-attacks $q$.[8]

**Note 3.** Let $\dagger \in \{\exists, \forall\}$. Then $\Delta$ $\dagger$-p-attacks $\psi$ iff the following set of $\dagger$-*p-attacking subsets* of $\Delta$ on $\psi$, is not empty:

$$\dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta, \psi) = \{\Delta' \mid \Delta' \text{ is a } \subseteq\text{-minimal subset of } \Delta \text{ that attacks } \psi, \text{ and } \psi \not<_{\mathcal{P}} \Delta'\}.$$

**Note 4.** When $\mathbb{P} = \mathbb{N}$ and $f$ is a numeric aggregation function, Definitions 6 and 10 are respectively equivalent to Definitions 2 and 3 (since $f(g(\Delta)) \leq f(g(\phi))$ iff $f(g(\Delta)) \not> f(g(\phi))$ for any total order $\leq$), thus our setting indeed generalizes the linear setting.

**Lemma 3.** *If $\Delta$ $\dagger$-p-attacks $\Theta$, so does any superset of $\Delta$.*

All the other definitions (including those of the semantics and the induced entailment relations) are similar to those of linear preferences (i.e., as in the previous section) where $\dagger$-p-attacks replace linear p-attacks.

**Example 9.** Let's reconsider the prioritized ABF from Examples 4 and 5 with the setting $\mathcal{P} = \langle \mathbb{P}, g, f, \dagger \rangle$, where $f$ is either $\min$ of $\max$ and $\dagger \in \{\exists, \forall\}$. The corresponding attack diagram is presented in Figure 2.[9]

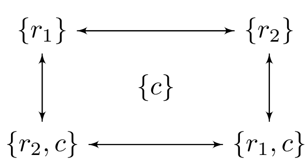



Figure 2: An attack diagram for Example 9

It follows that according to the grounded or the well-founded extension, which is $\{c\}$ in this case, the tourist will visit only the coffeehouse, while according to the preferred or the stable extensions (which are $\{r_1, c\}$ and $\{r_2, c\}$) the tourist will visit also (exactly) one of the restaurants. The scores do not dictate which restaurant should be chosen, so further considerations may be taken in this case.

To show the modularity of the framework, we conclude with a demonstration of reasoning with a pABF that is based on epistemic logic. Clearly, different epistemic logics can be incorporated for different settings.

**Example 10.** A layman $l$, believing $\neg p$, consults with two experts: one ($e_1$) thinks that $p \wedge q$ while the other ($e_2$) thinks $p \wedge \neg q$. The superiority of the experts' opinions over that of the laymen is represented by a partial order $\mathbb{P} = \langle \mathbb{V}, \leq \rangle$ in which $\mathbb{V} = \{e_1, e_2, l\}$, where $e_1 < l$ and $e_2 < l$. We want to realize the common belief (preceded by the modal operator $B$) on the basis of this scenario. For this, we incorporate modal operators $B_x$ for expressing the belief of the agents $x \in \{e_1, e_2, l\}$, and introduce strict premises by the scheme $B_x\psi \supset B\psi$ for each such $x$. This may be represented by a KD-based[10] framework $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$, in which:

- $\mathsf{ABF} = \langle \mathsf{KD}, \Gamma, Ab, \neg \rangle$, $\Gamma = \{B_x\psi \supset B\psi \mid x \in \{e_1, e_2, l\}\}$ $Ab = \{B_{e_1}(p \wedge q),\ B_{e_2}(p \wedge \neg q),\ B_l(\neg p)\}$,
- $\mathcal{P} = \langle \mathbb{P}, g, f, \dagger \rangle$, $g(B_{e_1}(p \wedge q)) = e_1$, $g(B_{e_2}(p \wedge q)) = e_2$, $g_l(B_l \neg p) = l$, $f \in \{\min, \max\}$ and $\dagger \in \{\forall, \exists\}$.

We show, for instance, that in this setting $B_{e_2}(p \wedge \neg q)$ attacks $B_l(\neg p)$: Suppose that $B_{e_2}(p \wedge \neg q)$. By Axioms K we have $B_{e_2}(p)$, and by the strict assumptions we get $B(p)$. Now, by Axiom D we infer $\neg B(\neg p)$, and since $\neg B(\neg p) \supset \neg B_l(\neg p)$ (contraposition of one of the strict assumptions), Modus Ponens gives $\neg B_l(\neg p)$, as required.

The $\dagger$-p-attack diagram is then represented in Figure 3.

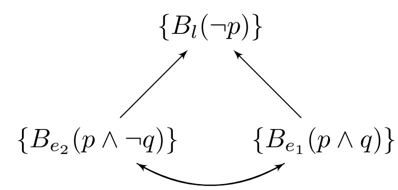



Figure 3: An attack diagram for Example 10

This results in the following preferred (and stable) extensions: $\{B_{e_1}(p \wedge q)\}$ and $\{B_{e_2}(p \wedge \neg q)\}$. The well-founded and grounded extension, on the other hand, is the emptyset in this case. We thus conclude that, e.g., $Bp$ is derived from both preferred/stable extensions (accepting the consensual part of the conflicting experts' opinions), but it is not derived from the grounded extension.[11]

## 4 Properties of Prioritized ABFs

### 4.1 Consistency of Extensions

**Proposition 1.** *If* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ *is reversible, it satisfies the following* direct consistency *postulate (Caminada and Amgoud 2007): There is no conflict-free set $\Delta \subseteq Ab$ such that $\Gamma, \Delta \vdash \neg\psi$ for some $\psi \in \Delta$.*

*Outline of proof.* The proof is based on the next lemma:

**Lemma 4.** *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ *be a reversible pABF with* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *and* $\mathcal{P} = \langle \mathbb{P}, g, f, \dagger \rangle$*, and let $\Delta \subseteq Ab$ be a conflict-free set of assumptions. If $\Delta$ attacks $\psi$ then either $\Delta$ $\dagger$-p-attacks $\psi$ or there is $\delta \in \Delta$ such that $\Delta \setminus \{\delta\} \cup \{\psi\}$ $\dagger$-p-attacks $\delta$.*

Now, suppose for a contradiction that $\Gamma, \Delta \vdash \neg\psi$ for some conflict free $\Delta \subseteq Ab$ and $\psi \in \Delta$. By Lemma 4, either $\Delta$ $\dagger$-p-attacks $\psi$ or there is a $\delta \in \Delta$ such that $\Delta \setminus \{\delta\} \cup \{\psi\}$ $\dagger$-p-attacks $\delta$. Since $\Delta \setminus \{\delta\} \cup \{\psi\} \subset \Delta$, in both cases $\Delta$ cannot be conflict-free. □

Consistency now follows from Proposition 1:

[8]This also shows that the converse of Lemma 2 does not hold.

[9]Notice, e.g., that $\{r_2, c\}$ $\dagger$-attacks $r_1$ although $g(r_1) < g(c)$, since $\mathsf{val}_{f,g}(\{r_2, c\}, r_1) = \{f(g(\{r_2\})\} = \{\langle 4, 2, 2 \rangle\}$ and $g(r_1) \not< \langle 4, 2, 2 \rangle$.

[10]For KD and other modal logics, see, e.g., (Chellas 1980).

[11]Interestingly, if the assumptions were $B_{e_1}p$, $B_{e_1}\neg q$, $B_{e_2}p$ and $B_{e_2}q$, the grounded extension would be different: $\{B_{e_1}\neg q, B_{e_2}q\}$, but still it wouldn't allow to infer $Bp$.

**Corollary 1.** *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ *be a reversible pABF with* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *and* $\mathcal{P} = \langle \mathbb{P}, g, f, \dagger \rangle$*. If* $\Delta \subseteq Ab$ *is conflict-free, then* $\Gamma \cup \Delta$ *is* $\vdash$*-consistent.*

*Proof.* If $\Gamma \cup \Delta$ is not $\vdash$-consistent, then in particular $\Gamma, \Delta \vdash \neg\psi$ for every $\psi \in \Delta$, contradicting Proposition 1. □

In (Arieli and Heyninck 2021) is it shown that the reversibility requirement in Proposition 1 is indeed necessary, even for numeric functions and linear preferences.

### 4.2 Closure of Extensions

Next, we consider the closure requirement from extensions (see Definition 4). First, we note that as shown e.g. in (Arieli and Heyninck 2021, Example 13), this requirement is in general *not* redundant in prioritized ABFs. However, as we show below, under the assumption that the aggregation function is reversible, the closure requirement may be lifted. This result generalizes similar results shown in (Heyninck and Arieli 2020) for simple contrapositive ABFs without priorities and in (Arieli and Heyninck 2021) for linearly-ordered prioritized ABFs (see also Note 1).

**Proposition 2.** *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ *be a reversible prioritized ABF. Then the closure requirement is redundant in the definition of stable extensions (Definition 4): Any conflict-free* $\Delta \subseteq Ab$ *that* $\dagger$*-p-attacks every* $A \in Ab \setminus \Delta$ *is closed.*

*Proof.* Suppose that $\Delta$ $\dagger$-p-attacks every $\psi \in Ab \setminus \Delta$, yet $\Gamma, \Delta \vdash \phi$, where $\phi \in Ab \setminus \Delta$. Since $\Delta$ $\dagger$-p-attacks $\phi$, we have $\Gamma, \Delta \vdash \neg\phi$. Thus, $\Gamma, \Delta \vdash \mathsf{F}$, contradicting Corollary 1. □

For a similar result concerning preferred extensions, we need the following lemma:

**Lemma 5.** *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ *be a selecting pABF with* $\mathcal{P} = \langle \mathbb{P}, g, f, \dagger \rangle$*, and let* $\Delta$ *be a conflict-free set in Ab. Then* $\Delta$ *is maximally admissible iff it* $\dagger$*-p-attacks any* $\psi \in Ab \setminus \Delta$*.*

**Proposition 3.** *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ *be a selecting prioritized ABF. Then the closure requirement is redundant in the definition of preferred extensions (Definition 4): Any* $\Delta \subseteq Ab$ *that is conflict free and maximally admissible is closed.*

*Proof.* Let $\Delta \subseteq Ab$ be conflict free and maximally admissible. By Lemma 5, $\Delta$ attacks every $A \in Ab \setminus \Delta$. By Proposition 2 (which holds in our case by Lemma 1), this means that $\Delta$ is closed. □

### 4.3 Existence of Extensions and Their Relations

We now examine the existence of the extensions in Definition 4, and check their relations.

**Grounded and well-founded extensions:** By its definition, the well-founded extension is always unique. Yet, as shown in (Arieli and Heyninck 2021), already in the linear case there may be several grounded extensions for a prioritized ABF. It follows, then, that in prioritized ABFs well-founded semantics and grounded semantics do not always coincide. As the next result shows, the (unique) well-founded extension of a prioritized ABF equals to the intersection of all the grounded extensions:

**Proposition 4.** *Let* $\mathsf{pABF}$ *be a prioritized ABF. Then* $\mathsf{WF}(\mathsf{pABF}) = \bigcap \mathsf{Grd}(\mathsf{pABF})$.

By Proposition 4 we thus have the following result:

**Corollary 2.** *The grounded and the well-founded semantics of* $\mathsf{pABF}$ *coincide iff* $\mathsf{pABF}$ *has a unique grounded extension.*

**Naive, stable, and preferred extensions:** In (Heyninck and Arieli 2020) it is shown that in non-prioritized simple contrapositive ABFs, the set of naive, preferred and stable extensions coincide. However, as shown in (Arieli and Heyninck 2021), when priorities are involved, this is no longer the case and the three types of semantics may yield different sets for the same pABF. Yet, it is also shown in (Arieli and Heyninck 2021) that preferred and stable extensions still coincide for what is called there 'max-bounded' linearly-ordered prioritized ABFs. We now recapture this result for the more general case where priorities may not be linearly ordered.

**Proposition 5.** *The stable and the preferred extensions of a selecting pABF coincide.*

*Proof.* By Lemma 5, using Propositions 2, 3 and Lemma 1. □

### 4.4 Representation of Extensions by Preferred Maximally Consistent Sets

Next, we represent extensions of pABFs by a generalization of the notion of preferred subtheories (Brewka 1989).

**Definition 11.** Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ be a prioritized ABF, and let $\Delta_1 \neq \Delta_2 \subseteq Ab$. We denote by $\Delta_1 \prec_{\mathcal{P}} \Delta_2$ that there is $\delta_1 \in \Delta_1 \setminus \Delta_2$ such that for every $\delta_2 \in \Delta_2 \setminus \Delta_1$, $g(\delta_1) < g(\delta_2)$.

**Definition 12.** Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ be a prioritized ABF, where $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ is a simple contrapositive ABF based on a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$.

- $\Delta \subseteq Ab$ is a *maximally consistent set* (MCS) in ABF, if $\Gamma \cup \Delta$ is $\vdash$-consistent, and $\Gamma \cup \Delta'$ is not $\mathfrak{L}$-consistent for every $\Delta \subsetneq \Delta' \subseteq Ab$. The set of the maximally consistent sets in ABF is denoted $\mathsf{MCS}(\mathsf{ABF})$.
- $\Delta \subseteq Ab$ is a *preferred (or prioritized) maximally consistent set* (pMCS) in pABF, if $\Delta \in \mathsf{MCS}(\mathsf{ABF})$ and there is no $\Theta \in \mathsf{MCS}(\mathsf{ABF})$ such that $\Theta \prec_{\mathcal{P}} \Delta$. The set of the preferred maximally consistent sets in pABF is denoted $\mathsf{MCS}_{\prec_{\mathcal{P}}}(\mathsf{ABF})$, or just $\mathsf{MCS}_{\mathcal{P}}(\mathsf{ABF})$.

**Definition 13.** A setting $\mathcal{P} = \langle \mathbb{P}, g, f, \dagger \rangle$ is *max-lower-bounded*, iff for every $\Delta \neq \emptyset$ one of the following holds: [12]
if $\dagger = \exists$:
$\forall x \in \max\{f(g(\delta)) \mid \delta \in \Delta\}\ \exists y \in f(g(\Delta))$ s.t. $x \leq y$,
if $\dagger = \forall$:
$\forall x \in \max\{f(g(\delta)) \mid \delta \in \Delta\}\ x \leq y$ for every $y \in f(g(\Delta))$.

**Proposition 6.** *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ *be a prioritized ABF where* $\mathcal{P}$ *is max lower-bounded and selecting. Then* $\Delta$ *is a stable extension of* $\mathsf{pABF}$ *iff* $\Delta \in \mathsf{MCS}_{\mathcal{P}}(\mathsf{ABF})$.

[12] Both cases in the definition are a generalization to the non-linear case of a similar property in (Arieli and Heyninck 2021, Definition 10).

We refer to (Arieli and Heyninck 2021) for a proof of the last proposition in the linear case. The proof of this proposition for partial orders and selecting ABFs is left out in view of space limitations.

**Example 11.** Let $Ab = \{q_1, q_2, p_1, p_2\}$ and $\Gamma = \{\neg(q_1 \wedge p_2), \neg(q_2 \wedge p_1)\}$ where $g(q_1) > g(p_1)$, $g(q_2) > g(p_2)$, $f(\Delta) = \{g(\delta) \mid \delta \in \Delta\}$, and $\dagger = \exists$. The stable extensions in this case are $\{p_1, p_2\}$, $\{p_1, q_1\}$, $\{p_2, q_2\}$, and $\{q_1, q_2\}$. These are also the elements of $\mathsf{MCS}_{\mathcal{P}}(\mathsf{ABF})$, as indeed the last proposition suggests.

**Note 5.** Definition 11 is a generalization to partial orders of maximally consistent sets that can be defined with respect to Brewka's *preferred subtheories* (Brewka 1989) in the linear case:

**Definition 14.** Let $Ab_i = \{\psi \in Ab \mid g(\psi) = i\}$ $(1 \leq i \leq n)$ be a stratification of $Ab$ according to a linear allocation function $g : Ab \to \mathbb{N}$ (Definition 2), and let $\Delta, \Theta \subseteq Ab$. We say that $\Delta$ is *preferred* over $\Theta$ (with respect to $g$), denoted $\Delta \sqsubset_g \Theta$, iff there is an $1 \leq i \leq n$ such that $Ab_j \cap \Delta = Ab_j \cap \Theta$ for every $1 \leq j < i$, and $Ab_i \cap \Delta \supsetneq Ab_i \cap \Theta$. The set $\mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$ is defined in a similar way to $\mathsf{MCS}_{\prec_{\mathcal{P}}}(\mathsf{ABF})$, where $\sqsubset_g$ replaces $\prec_{\mathcal{P}}$.

**Lemma 6.** *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ *be a prioritized ABF with* $\mathcal{P} = \langle \mathbb{N}, g, f, \dagger \rangle$ *where g is a linear allocation function. Then* $\mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$ *and* $\mathsf{MCS}_{\prec_{\mathcal{P}}}(\mathsf{ABF})$ *coincide.*

By Lemma 6, Proposition 6 holds also for $\sqsubset_g$-preferred maximally consistent sets in linear pABFs. This proposition thus generalizes a similar result in (Arieli and Heyninck 2021). To the best of our knowledge, this is the first argumentative characterisation of preferred maximally consistent sets using non-linear preferences.

# 5 A Postulate-Based Study

## 5.1 Postulates for pABF-based Entailments

We start by checking properties of the entailment relations that are induced by pABFs (Definition 5). The following properties were introduced in (Kraus, Lehmann, and Magidor 1990) and (Lehmann and Magidor 1992), and their formulations are adjusted to our setting. Below, for some $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ and a formula $\phi$, we let $\mathsf{ABF}^{\phi} = \langle \mathfrak{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle$.

**Definition 15.** Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic. A relation $\mid\!\sim$ between pABFs that are based on $\mathfrak{L}$ and $\mathcal{L}$-formulas is called $\vdash$*-cumulative* if the following conditions are satisfied:

- *Cautious Reflexivity* (CR):
If $\psi \in \Gamma$ is $\vdash$-consistent, then $\mathsf{ABF} \mid\!\sim \psi$.
- *Cautious Monotonicity* (CM):
If $\mathsf{ABF} \mid\!\sim \phi$ and $\mathsf{ABF} \mid\!\sim \psi$, then $\mathsf{ABF}^{\phi} \mid\!\sim \psi$.
- *Cautious Cut* (CC):
If $\mathsf{ABF} \mid\!\sim \phi$ and $\mathsf{ABF}^{\phi} \mid\!\sim \psi$, then $\mathsf{ABF} \mid\!\sim \psi$.
- *Left Logical Equivalence* (LLE):
If $\phi \vdash \psi$ and $\psi \vdash \phi$, then $\mathsf{ABF}^{\phi} \mid\!\sim \rho$ iff $\mathsf{ABF}^{\psi} \mid\!\sim \rho$.
- *Right Weakening* (RW):
If $\phi \vdash \psi$ and $\mathsf{ABF} \mid\!\sim \phi$, then $\mathsf{ABF} \mid\!\sim \psi$.

A cumulative relation is called *preferential*, if it satisfies the following condition:

- *Distribution* (OR):
If $\mathsf{ABF}^{\phi} \mid\!\sim \rho$ and $\mathsf{ABF}^{\psi} \mid\!\sim \rho$, then $\mathsf{ABF}^{\phi \vee \psi} \mid\!\sim \rho$.

**Proposition 7.** *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ *be a pABF with* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *where* $\mathcal{P}$ *is max lower-bounded and selective. Then* $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ *is preferential for every semantics* $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$.

*Outline of proof.* We first let $\mathsf{Sem} = \mathsf{Stb}$. Let $\mid\!\sim = \mid\!\sim^{\cap}_{\mathsf{Stb}}$. CR holds since a premise $\psi \in \Gamma$ cannot be attacked. CM follows from Proposition 6 and since $\mathsf{MCS}_{\prec_{\mathcal{P}}}(\mathsf{ABF}) = \mathsf{MCS}_{\prec_{\mathcal{P}}}(\mathsf{ABF}^{\phi})$ when $\Gamma, \bigcap \mathsf{MCS}_{\prec_{\mathcal{P}}}(\mathsf{ABF}) \vdash \phi$. The proof of CC is analogous to that of CM. LLE holds since by its assumptions, $\mathsf{MCS}_{\prec_{\mathcal{P}}}(\mathsf{ABF}^{\phi}) = \mathsf{MCS}_{\prec_{\mathcal{P}}}(\mathsf{ABF}^{\psi})$, thus by Proposition 6 again, also $\mathsf{Stb}(\mathsf{ABF}^{\phi}) = \mathsf{Stb}(\mathsf{ABF}^{\psi})$. For RW, suppose that $\mathsf{ABF} \mid\!\sim \phi$. Then $\Gamma, \Delta \vdash \phi$ for every $\Delta \in \mathsf{Stb}(\mathsf{pABF})$ and thus with transitivity of $\vdash$ we have that $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{Stb}(\mathsf{pABF})$, i.e., $\mathsf{ABF} \mid\!\sim \psi$. The proof of OR is left to the reader. The case $\mathsf{Sem} = \mathsf{Pref}$ follows now from Proposition 5. ☐

**Note 6.** Another property considered in (Lehmann and Magidor 1992), called *rational monotonicity* (RM), states that if $\mathsf{ABF} \mid\!\sim \phi$ and $\mathsf{ABF} \not\mid\!\sim \neg\psi$, then $\mathsf{ABF}^{\psi} \mid\!\sim \phi$. In (Heyninck and Arieli 2020, Example 11) it is shown that RM fails for $\mid\!\sim^{\cap}_{\mathsf{Prf}}$ and $\mid\!\sim^{\cap}_{\mathsf{Stb}}$ already for non-prioritized ABFs.

## 5.2 Postulates for pABF-based Extensions

Next, we consider several postulates that are concerned with the handling of preferences in prioritized ABFs and its effect on the extensions of the frameworks. The postulates below are shown to hold for prioritized frameworks $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ with linearly-ordered preferences (see (Arieli and Heyninck 2021) for proofs and discussions):

**Empty Preferences (for** $\mathsf{Sem}$**) :** [13]
If $\mathcal{P}$ is a degenerated preference setting (i.e., if $g$ is a uniform allocation function), $\mathsf{Sem}(\mathsf{pABF}) = \mathsf{Sem}(\mathsf{ABF})$.

**Extensions Selection (for** $\mathsf{Sem}$**):** [14]
If $\mathcal{E} \in \mathsf{Sem}(\mathsf{pABF})$ then $\mathcal{E} \in \mathsf{Sem}(\mathsf{ABF})$.

**Conflict Preservation (for** $\mathsf{Sem}$**):** [15]
If $\mathcal{E} \in \mathsf{Sem}(\mathsf{pABF})$ and $\Delta$ p-attacks $\Theta$, then either $\Delta \not\subseteq \mathcal{E}$ or $\Theta \not\subseteq \mathcal{E}$.

**Preferred Arguments (for** $\mathsf{Sem}$**):** [16]
For every $\mathcal{E} \in \mathsf{Sem}(\mathsf{pABF})$ it holds that $\mathsf{Min}_g(Ab) = \{\psi \in Ab \mid \neg\exists\, \phi \in Ab \text{ such that } g(\phi) < g(\psi)\} \subseteq \mathcal{E}$

**Brewka-Eiter (BE) Principle (for** $\mathsf{Sem}$**):** [17]
If both $\Delta = \Lambda \cup \{\phi\} \in \mathsf{Sem}(\mathsf{ABF})$ and $\Theta = \Lambda \cup \{\psi\} \in \mathsf{Sem}(\mathsf{ABF})$ (where $\phi, \psi \notin \Lambda$), and $g(\psi) < g(\phi)$, then $\Delta \notin \mathsf{Sem}(\mathsf{pABF})$.

[13] Amgoud and Vesic (2009) Brewka, Truszczynski, and Woltran (2010).
[14] Šimko (2014).
[15] Amgoud and Vesic (2009; 2014) Modgil (2009).
[16] Amgoud and Vesic (2014) Čyras (2017).
[17] Brewka and Eiter (2000).

**Principle of Tolerance (for Sem):**
If $\mathsf{Sem}(\mathsf{ABF}) \neq \emptyset$ then $\mathsf{Sem}(\mathsf{pABF}) \neq \emptyset$ as well.

Below, we check these postulates for partially ordered preferences and the corresponding pABFs. We start with empty preferences. The following proposition is similar to the one shown in (Arieli and Heyninck 2021):

**Proposition 8.** *Let $f$ be an aggregation function that is invariant under multiple occurrences (that is, if $V$ is a set and $V'$ is a multiset with the same elements as $V$, then $f(V) = f(V')$). Then* pABF *satisfies the empty preferences postulate for every* Sem.

*Proof.* The empty preferences postulate assumes that $g$ is uniform. Thus, under the condition on $f$, for every $\delta' \in \Delta$ we have: $f(g(\Delta)) = f(\{g(\delta) \mid \delta \in \Delta\}) = f(g(\delta')) = g(\delta')$. Again, since $g$ is uniform, we conclude that $f(g(\Delta))$ is the same for every $\Delta \subseteq Ab$. It follows that †-p-attacks coincide, for every $\dagger \in \{\forall, \exists\}$, with (standard, non-prioritized) attacks, and so $\mathsf{Sem}(\mathsf{pABF}) = \mathsf{Sem}(\mathsf{ABF})$ for every semantics Sem. □

We now turn to extension selection:

**Proposition 9.** *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ *be a selecting prioritized ABF. Then* pABF *satisfies the extensions selection postulate for* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.

*Proof.* We first show that if $\Delta \subseteq Ab$ is conflict-free in pABF then it is conflict-free in ABF. Suppose towards a contradiction that $\Delta$ attacks some $\delta \in \Delta$. This means that $\Gamma, \Delta \vdash \neg\delta$. If $\dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta, \psi) \neq \emptyset$ then by Note 3, $\Delta$ cannot be conflict-free in pABF. Suppose then that $\dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta, \psi) = \emptyset$. This means that for every minimal subset $\Delta' \subseteq \Delta$ such that $\Gamma, \Delta' \vdash \neg\delta$, it holds that $\Delta' \succ_{\mathcal{P}} \delta$. By reversibility (which, by Lemma 1 holds since pABF is selecting), for such a subset $\Delta'$, there is a $\delta' \in \Delta'$ such that $\Delta' \cup \{\delta\} \setminus \delta' \not\succ_{\mathcal{P}} \delta'$, and by contraposition, $\Gamma, \Delta' \cup \{\delta\} \setminus \delta' \vdash \neg\delta'$. Thus, $\Delta'$ p-attacks $\delta' \in \Delta'$, a contradiction to the assumption that $\Delta$ is conflict-free in pABF.

We now show that if $\Delta$ is stable in pABF then it is stable in ABF. We have already shown above that $\Delta$ is conflict-free in ABF. Now, since $\Delta$ is stable in pABF, $\Delta$ †-p-attacks every $\psi \in Ab \setminus \Delta$, which in particular means that $\Gamma, \Delta \vdash \neg\psi$ for every such $\psi$. Thus, $\Delta$ attacks every $\psi \in Ab \setminus \Delta$, and so it is stable in ABF.

We now show that if $\Delta$ is preferred in pABF then it is preferred in ABF. Indeed, suppose for a contradiction that $\Delta$ is not preferred in ABF. As is shown in (Heyninck and Arieli 2020), $\Delta$ is not stable as well. By the previous case, this means that $\Delta$ is not stable in pABF. By Proposition 5, this implies that $\Delta$ is not preferred in pABF, a contradiction.

It remains to show that if $\Delta$ is naive in pABF then it is naive in ABF. We know that $\Delta$ is conflict-free in ABF. Suppose for a contradiction that there is $\Delta \subsetneq \Delta' \subseteq Ab$ such that $\Delta'$ is conflict-free in ABF. Since $\Delta'$ is not conflict-free in pABF (due to the assumption that $\Delta$ is naive in pABF), there is some $\delta' \in \Delta'$ such that $\Gamma, \Delta' \vdash \neg\delta'$ (yet, $\dagger\text{-}\mathsf{val}^{-1}_{f,g}(\Delta', \delta') \neq \emptyset$). Thus $\Delta'$ attacks $\delta'$ in ABF, a contradiction to the assumption that $\Delta'$ is conflict-free (in ABF). □

Conflict preservation follows in our case from the fact that every $\mathcal{E} \in \mathsf{Sem}(\mathsf{pABF})$ is conflict-free. This property is not so obvious in other formalisms in which attacks are sometimes discarded due to preference over arguments (see (Čyras 2017) for some examples).

The principle of preferred arguments cannot hold in our setting unless $\mathsf{Min}_g(Ab)$ is $\vdash$-consistent (otherwise $\mathcal{E}$ is not conflict free). A sufficient condition for assuring this principle for stable semantics in max-lower-bounded and reversible pABFs is given next.

**Proposition 10.** *Let* pABF *be a max-lower-bounded and reversible pABF. If* $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_{\mathcal{P}}(\mathsf{ABF})$ *it satisfies the principle of preferred arguments for stable semantics.*

*Proof.* Let $\mathcal{E}$ be a stable extensions of pABF. By Proposition 6, $\mathcal{E} \in \mathsf{MCS}_{\mathcal{P}}(\mathsf{ABF})$. Now, since $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_{\mathcal{P}}(\mathsf{pABF})$, we get that $\mathsf{Min}_g(Ab) \subseteq \mathcal{E}$. □

Note that by Proposition 6, when pABF is selecting, the condition that $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_{\mathcal{P}}(\mathsf{ABF})$ is also necessary for assuring the satisfaction of the preferred argument postulate for stable and preferred semantics. We therefore have the following corollary:

**Corollary 3.** *Let* pABF *be a max-lower-bounded and selecting pABF. Then* pABF *satisfies the principle of preferred arguments for the stable and preferred semantics iff* $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_{\mathcal{P}}(\mathsf{ABF})$.

*Proof.* The proof for stable semantics follows from Proposition 10 and the paragraph following its proof. The result for preferred semantics then follows from Proposition 5, since pABF is selecting. □

In (Arieli and Heyninck 2021) it is shown that BE-principle doesn't hold for prioritized ABFs even for linear preference orders. However, as the next proposition shows, for selecting max-lower-bounded pABFs this postulate does hold for the stable and the preferred semantics.

**Proposition 11.** *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ *be a selecting and max-lower-bounded pABF. Then* pABF *satisfies the BE-principle for the stable and preferred semantics.*

*Proof.* Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathcal{P} \rangle$ be as in the proposition. Let $\Delta, \Theta \in \mathsf{Stb}(\mathsf{ABF})$ and $\Lambda \cup \{\phi, \psi\} \subseteq Ab$ s.t. $\phi, \psi \notin \Lambda$ and $\Delta = \Lambda \cup \{\phi\}$ and $\Theta = \Lambda \cup \{\psi\}$ and $g(\psi) < g(\phi)$. Since $\Delta, \Theta \in \mathsf{Stb}(\mathsf{ABF})$, it is shown in (Heyninck and Arieli 2020) that $\Delta, \Theta \in \mathsf{MCS}(\mathsf{ABF})$. However, $\Theta \prec_{\mathcal{P}} \Delta$ (recall Definition 11), and so $\Delta \notin \mathsf{MCS}_{\prec_{\mathcal{P}}}(\mathsf{ABF})$. By Proposition 6, $\Delta \notin \mathsf{Stb}(\mathsf{pABF})$. □

The principle of tolerance for complete and preferred semantics is clear by the fact that pABF is in particular an argumentation framework, and so $\mathsf{Cmp}(\mathsf{pABF})$ and $\mathsf{Prf}(\mathsf{pABF})$ are not empty. This principle for stable and preferred semantics holds for selecting and max-lower-bounded pABF by Corollary 3.[18]

[18] As noted in (Čyras 2017), when the prioritized assumption-based framework $\mathsf{ABA}^+$ is concerned (see (Čyras and Toni 2016)), the principle of tolerance does not hold for the stable semantics.

A summary of the conditions under which the properties and the postulates considered in this and in the previous section are satisfied with respect to the stable semantics is given in Table 1.

| Property of the pABF | Conditions on the priority setting |
|---|---|
| Extensions consistency | Reversible |
| Closure of extensions | Reversible |
| $\mathsf{Stb} = \mathsf{MCS}_{\mathcal{P}}$ | Selecting |
| $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ is preferential | Selecting & Max-lower-bounded |
| Empty preferences | Multiple-occurrences invariance[19] |
| Extension selection | Selecting |
| Conflict preservation | – |
| Preferred assumptions | Selecting & Max-lower-bounded[20] |
| Brewka-Eiter postulate | Selecting & Max-lower-bounded |
| Tolerance | Selecting & Max-lower-bounded |

Table 1: Summary of the postulates for the stable semantics

# 6 Related Work and Conclusion

Simple contrapositive assumption-based argumentation frameworks provide a robust representation and reasoning method for handling arguments and counter-arguments (see (Heyninck and Arieli 2020)). As shown in (Arieli and Heyninck 2021), the enhancement with priorities of such frameworks strengthens their expressivity and provides additional layer to their inference process. In this paper we have largely extended the range of priority settings that are integrated with these frameworks for gaining more flexibility in comparing arguments and expressing the mutual relations among them, thus making them more suitable for everyday life scenarios. The incorporation of priorities that are partially (or, more generally, non-linearly) ordered, allows us to introduce different types of attack relations (stems on existential or universal considerations over those priorities), which further extend the reasoning forms supported by the argumentation frameworks under consideration.

Non-linear preferences are very natural in many scenarios, for instance when objects are compared with respect to different aspects, as illustrated in Examples 4 and 10. Such comparisons are ubiquitous in e.g. reviewing systems, on-line marketplaces or content platforms involving different agents or sources of information. Simple contrapositive assumption-based frameworks with non-linear preferences allow to aggregate different options while respecting constraints, as shown in Examples 4, 5, 9 and 10. The principle-based study allows for the selection of the right preferential setting for a given application context. For instance, when aggregating different options in view of a set of constraints, the preferred arguments principle ensures that the maximally preferred options will be included in any selection.

The primary method of handling priorities in ABFs, used in ABA$^+$ frameworks (Čyras and Toni 2016; Čyras 2017), is different from our approach in several ways. Perhaps the most significant difference is in the interpretation of attacks: we adopt the standard approach, taken also in related argumentation-based formalisms (like ASPIC-based systems (Modgil and Prakken 2013; Modgil and Prakken 2014), sequent-based argumentation frameworks (Arieli, Borg, and Straßer 2018), and dialectical argumentation frameworks (D'Agostino and Modgil 2018a; D'Agostino and Modgil 2018b), in which for the attack to take place the attacking argument should be at least as preferred as the attacked argument. In contrast, ABA$^+$ is based on the idea of *reverse defeats*: A set of assumptions $\Delta$ reverse defeats a set of assumptions $\Theta$ if either $\Delta$ attacks $\Theta$ and $\Delta$ is not less preferred than $\Theta$, or $\Theta$ attacks $\Delta$ and $\Theta$ is (strictly) less preferred than $\Delta$. The use of reverse defeats is required for avoiding some violations of rationality postulates such as consistency (see (Čyras and Toni 2016) for more details). However, in (Heyninck 2019, Chapter 7) it is shown that such reverse defeats are actually superfluous when assuming that the deducibility relation is closed under contraposition. Also, as noted in the introduction, we allow arbitrary aggregation functions in the preference settings and so do not confine ourselves to $\max$-based attacks (reflecting only the weakest link principle).

In (Kaci et al. 2021), two other variations of reverse defeat are presented in the context of abstract argumentation. The first one, called *Reduction 3*, states that an argument $a$ successfully attacks an argument $b$, if: (1) there is an attack between $a$ and $b$, and $a$ is not worse than $b$, or (2) there is an attack between $a$ and $b$ and no attack between $b$ and $a$. This is clearly a generalization of reverse defeat, and again, since we assume contrapositive logics, any attack from $\Delta$ to $\psi$ will give rise to an attack from a set of assumptions including $\psi$ to an assumption $\Delta$. The second variation of reverse defeat presented in (Kaci et al. 2021) is called *Reduction 4* and says that an attack from $a$ to $b$ is successful, if: (1) $a$ attacks $b$ and $a$ is not worse than $b$, or (2) $b$ attacks $a$, $a$ does not attack $b$ and $b$ is worse than $a$, or (3) there is an attack between $a$ and $b$ and no attack between $b$ and $a$. Again, since we assume contrapositive logics, there is no need to consider the asymmetric cases expressed in (2) and (3).

Future work includes reformulation of pABFs under non-explosive logics, and extensions to first-order languages, including description logics. It might also be interesting to extend the representation results from Section 4.4 to other approaches to reasoning with partially ordered defeasible information, such as those in (Junker and Brewka 1991; Touazi, Cayrol, and Dubois 2015; Belabbes and Benferhat 2019; Belabbes, Benferhat, and Chomicki 2021).

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[19] That is, if $V$ is a set and $V'$ is a multiset with the same elements, then $f(V) = f(V')$.

[20] It is also assumed that $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_{\mathcal{P}}(\mathsf{pABF})$.

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