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Fuzzy Truth, Fuzzy Support and Fuzzy Information States for Inquisitive Semantics

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Abstract

In logic, the meaning of a sentence is usually reduced to its truth conditions. However, this makes sense only for declarative sentences. In order to model also the meaning of questions, inquisitive semantics replaces the truth-conditional approach relating sentences to possible worlds with a support-conditional approach that relates sentences to information states. The standard framework of inquisitive semantics is based on a crisp notion of an information state, defined as a set of possible worlds, and a crisp relation of informational support. This paper introduces and studies two refinements of the standard framework. The first refinement takes into account fuzzy information states (defined as fuzzy sets of possible worlds) and the second one introduces a notion of fuzzy informational support. The main result of this paper shows that in the resulting framework that fuzzifies inquisitive semantics in two different directions we obtain an abstract and very general version of a principle known from the basic inquisitive semantics as Truth-Support Bridge.

1 Introduction

A vast number of mathematical frameworks proposed in the logical literature are explicitly intended to represent essential aspects of information. One can distinguish two widespread types of approaches to the logical representation of information. The first one can be called *concrete* and the second one *abstract*.

The concrete approach is related to the theory of semantic information developed by Carnap and Bar-Hillel (1964). In its modern formulation it is based on the notion of a possible world which has been a prevailing tool in formal semantics at least since the work of Kripke (1959) and Hintikka (1962). A possible world can be viewed as a complete totality of facts. Every sentence of a fixed language is supposed to be true or false in a given world. A body of information is modelled as a set of possible worlds, those worlds that are compatible with the information. For this reason, the concrete approach is by van Benthem and Martinez (2008) called *Information as Range*. This notion of information is used for a semantic representation of knowledge and belief in epistemic logic (Fagin et al. 1995), including its dynamic extensions (van Ditmarsch, van der Hoek, and Kooi 2007; van Benthem 2011), and in belief revision theory (Hansson 1999). As we will see, information states, defined in this way, also play a central role in inquisitive semantics (Ciardelli, Roelofsen, and Groenendijk 2019) which allows us to model the meaning of questions.

The concrete approach to the logical modelling of information is very clear and useful. It allows one to reduce some complicated relations among sentences (e.g. entailment) to somewhat more perspicuous set-theoretic relations (e.g. inclusion). On the other hand, it is evident that the scope of application of this approach is significantly limited. It is intimately connected with classical logic with all its well-known problems and limitations.

More abstract frameworks that go beyond the concrete approach to overcome its weaknesses are related to the development of various relational semantics for non-classical logics such as intuitionistic logic, relevant logic and other substructural logics (Kripke 1965; Urquhart 1972; Anderson and Belnap 1975; Anderson, Belnap, and Dunn 1992; Wansing 1993; Mares 1997; Dunn and Hardegree 2001; Punčochář 2017; Leitgeb 2019). Semantic frameworks for such logical systems are often based on mathematical models that are interpreted as consisting of information states with respect to which sentences are evaluated. In contrast to the states of the concrete approach, these abstract states are primitive entities, they have no internal structure but they are related one to another by some external structure, for example by informational ordering determining when one state is informationally stronger than another one, which goes back to (Kripke 1965), or by an algebraic operation that allows one to combine two (abstractly conceived) pieces of information into a new piece of information, which goes back to (Urquhart 1972).

There is a big discrepancy between the concrete and the abstract logical approach to information. On the one hand, the concrete approach gives us a very clear picture of what an information state is but it is based on strong idealizations which significantly restrict its scope of application. On the other hand, the abstract approach helps us to overcome some of these limitations but it often provides only a very unclear account of what exactly the information states in an abstract semantic framework represent.

In this paper we employ a framework that overcomes some limitations of classical logic, while staying within the boundaries of the concrete approach. In particular, we refine the usual concrete approach in a way that allows us to address phenomena related to vagueness. A notion of a fuzzy information state, defined as a fuzzy set of possible worlds, is introduced and applied in a context of inquisitive semantics to model the meaning of questions involving vague vocabulary. We consider two versions of inquisitive semantics based on fuzzy information states. The first one defines a crisp notion of support of a formula by a fuzzy information state. The second one fuzzyfies inquisitive semantics in another direction and introduces a fuzzy notion of informational support. In both these modifications we formulate and prove a general version of a principle known from the standard inquisitive semantics as Truth-Support Bridge for declarative sentences, and describe some general key features of all propositions, involving the inquisitive ones.

2 Inquisitive Semantics

We start with the language $\mathcal{L}$ of propositional logic generated from a set of atomic formulas At and the contradiction constant $\bot$ by the binary connectives $\wedge$ and $\rightarrow$. Anticipating that we will be later concerned with t-norm based fuzzy logics, we can define disjunction and negation in this way: $\varphi \vee \psi =_{def} ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$ and $\neg\varphi =_{def} \varphi \rightarrow \bot$. Extending $\mathcal{L}$ with the additional binary connective $\mathbin{\vee\!\!\!\vee}$ results in the language $\mathcal{L}^{\mathbin{\vee\!\!\!\vee}}$. We adopt the convention that the Greek letters α, β will range over $\mathbin{\vee\!\!\!\vee}$-free formulas and the letters φ, ψ over formulas possibly involving $\mathbin{\vee\!\!\!\vee}$. The primitive operators of the language $\mathcal{L}$ are interpreted in the usual manner as conjunction ($\wedge$) and implication ($\rightarrow$), and the defined symbols as declarative disjunction ($\vee$) and negation ($\neg$). The connective $\mathbin{\vee\!\!\!\vee}$ is called *inquisitive disjunction* and it is viewed as an operator that allows us to form a disjunctive question. So, $\alpha \mathbin{\vee\!\!\!\vee} \beta$ is interpreted as the question *whether* α *or* β, in contrast to $\alpha \vee \beta$ which has the usual meaning as the statement *that* α *or* β. Given this reading, the formulas of the language $\mathcal{L}$, i.e. the $\mathbin{\vee\!\!\!\vee}$-free formulas of $\mathcal{L}^{\mathbin{\vee\!\!\!\vee}}$, may be called *declarative*. We also define: $?\varphi = \varphi \mathbin{\vee\!\!\!\vee} \neg\varphi$. For a declarative α, the formula $?\alpha$ can be interpreted as the polar question *whether* α.

Definition 1. *A c-model (a shorthand for "crisp model") is a pair $\mathcal{M} = \langle W, V \rangle$, where W is a non-empty set (of possible worlds) and V is a c-valuation (crisp valuation), i.e. a function assigning to each atomic formula an information c-state (crisp information state) defined as a subset of W.*

The basic semantics for classical logic can be presented in terms of a crisp relation of truth $\vDash$, relating possible worlds of a c-model and $\mathcal{L}$-formulas, which is defined in the usual recursive manner:

$w \nvDash \bot$,

$w \vDash p$ iff $w \in V(p)$, for each atomic formula p,

$w \vDash \alpha \wedge \beta$ iff $w \vDash \alpha$ and $w \vDash \beta$,

$w \vDash \alpha \rightarrow \beta$ iff $w \nvDash \alpha$ or $w \vDash \beta$.

Note that this implies the usual characterization of negation and disjunction: $w \vDash \neg\alpha$ iff $w \nvDash \alpha$; $w \vDash \alpha \vee \beta$ iff $w \vDash \alpha$ or $w \vDash \beta$.

An $\mathcal{L}$-formula α is said to be truth-conditionally valid (or tc-valid, for short) in a c-model $\mathcal{M} = \langle W, V \rangle$ if $w \vDash \alpha$, for all $w \in W$. An $\mathcal{L}$-formula α is said to be a tc-consequence (truth-conditional consequence) of a set of $\mathcal{L}$-formulas Δ if α is tc-valid in every c-model in which each formula from Δ is tc-valid. By this definition we obtain the usual consequence relation of classical logic.

Even though standard inquisitive logic (Ciardelli and Roelofsen 2011; Ciardelli, Roelofsen, and Groenendijk 2019; Ciardelli 2022) conservatively extends classical logic, it is not based on this common truth-conditional semantics for classical logic. The reason is that questions do not have truth values and thus cannot be characterized in terms of truth conditions. However, they can be characterized in terms of informational support. One can meaningfully ask whether a body of information resolves a given question. Since declarative sentences can also be characterized in terms of informational support, this notion allows us to build a uniform framework for statements and questions.

Motivated by these considerations, inquisitive semantics employs an information-based approach that replaces the notion of crisp truth $\vDash$ that relates formulas to crisp worlds with a relation of crisp support $\Vdash$ that relates formulas to crisp information states. If w and s are respectively a world and a state of a given c-model $\mathcal{M}$ then $w \vDash \alpha$ means that α is true in the world w relative to $\mathcal{M}$, while $s \Vdash \alpha$ means that α is supported by the state s relative to $\mathcal{M}$. Since, on the informational level, a semantic clause for inquisitive disjunction can be formulated, the support relation is defined for the whole language $\mathcal{L}^{\mathbin{\vee\!\!\!\vee}}$:

$s \Vdash \bot$ iff $s = \emptyset$,

$s \Vdash p$ iff $s \subseteq V(p)$, for each atomic formula p,

$s \Vdash \varphi \wedge \psi$ iff $s \Vdash \varphi$ and $s \Vdash \psi$,

$s \Vdash \varphi \rightarrow \psi$ iff $\forall t \subseteq s$, if $t \Vdash \varphi$, then $t \Vdash \psi$,

$s \Vdash \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $s \Vdash \varphi$ or $s \Vdash \psi$.

An $\mathcal{L}^{\mathbin{\vee\!\!\!\vee}}$-formula φ is said to be support-conditionally valid (or sc-valid, for short) in a c-model $\mathcal{M} = \langle W, V \rangle$ if $W \Vdash \varphi$. We say that an $\mathcal{L}^{\mathbin{\vee\!\!\!\vee}}$-formula φ is an sc-consequence (support-conditional consequence) of a set of $\mathcal{L}^{\mathbin{\vee\!\!\!\vee}}$-formulas Δ if φ is sc-valid in every c-model in which each formula from Δ is sc-valid.

If α is declarative, i.e. representing a statement, then $s \Vdash \alpha$ can be read as "the body of information s implies α". For a question $\alpha \mathbin{\vee\!\!\!\vee} \beta$, the intended meaning of $s \Vdash \alpha \mathbin{\vee\!\!\!\vee} \beta$ is that "the body of information s resolves the question *whether* α *or* β". In this case, the semantic clause for inquisitive disjunction says something very intuitive, namely that the body of information s resolves the question *whether* α *or* β if and only if s provides a direct answer to the question, i.e. s implies α or s implies β.

Note that the syntax of $\mathcal{L}^{\mathbin{\vee\!\!\!\vee}}$ allows us to embed inquisitive disjunction arbitrarily under other operators. By this we obtain some important constructions, for example conditional questions like $p \rightarrow ?q$ (i.e. *if p, is q the case?*). A crucial feature of inquisitive semantics is that the support conditions give us reasonable behaviour of the connectives also for such complex constructions, so that the semantic clauses interact in a desirable way.

As a simple but important consequence of these definitions we obtain the following principle that Ciardelli (2022) calls *Truth-Support Bridge*. We add the roman numeral one because below we will state two more abstract versions of this principle.

Proposition 1 (Truth-Support Bridge-I)**.** *For any information c-state s of any c-model, and for any $\mathcal{L}$-formula α:*

$$s \Vdash \alpha \text{ if and only if } w \vDash \alpha, \text{ for all } w \in s.$$

This principle shows that, for declarative formulas, truth and support are interdefinable. Support by s amounts to truth in all worlds of s, and truth in w amounts to support by $\{w\}$. This observation also makes clear that, for the language $\mathcal{L}$, tc-validity coincides with sc-validity. This immediately implies that in restriction to declarative formulas the information-based semantics determines the same logic as the truth-conditional semantics. Hence the standard inquisitive logic indeed extends conservatively classical logic.

Proposition 2. *An $\mathcal{L}$-formula α is an sc-consequence of a set of $\mathcal{L}$-formulas Δ if and only if α is a tc-consequence of Δ, i.e. if and only if α follows form Δ in classical logic.*

The Truth-Support Bridge-I also expresses a feature of declarative sentences that distinguishes them from questions. Questions do not have truth conditions and their support cannot be in any analogous way reduced to the notion of truth. For example, if $V(p) \nsubseteq V(q)$ and $V(q) \nsubseteq V(p)$ then it holds $V(p) \Vdash p \vee\!\!\!\vee\, q$ (intuitively, p resolves the question *whether p or q*), $V(q) \Vdash p \vee\!\!\!\vee\, q$ (also q resolves the question *whether p or q*) but $V(q) \cup V(p) \nVdash p \vee\!\!\!\vee\, q$ (the mere disjunction *p or q* does not resolve the question *whether p or q*). So, unlike in the case of declarative formulas, the set of states supporting a question does not have to be closed under union, which prevents us from having anything like the Truth-Support Bridge-I for questions. For all formulas of $\mathcal{L}^{\vee\!\!\!\vee}$ we can generally state the following weaker characteristic properties.

Proposition 3. *In any c-model and for any $\mathcal{L}^{\vee\!\!\!\vee}$-formula φ:*

(a) *$\emptyset \Vdash \varphi$ (empty-state property),*

(b) *if $s \Vdash \varphi$ and $t \subseteq s$ then $t \vDash \varphi$ (persistence property).*

It will be useful to observe that due to Proposition 3-b, i.e. the persistence property, the semantic clause for implication could be equivalently reformulated in this way:

$$s \Vdash \varphi \to \psi \text{ iff } \forall t \subseteq W, \text{ if } t \Vdash \varphi \text{ then } s \cap t \Vdash \psi.$$

We finish this section by formulating another crucial property of basic inquisitive semantics related to the traditional methodology that reduces questions to sets of possible answers. Let us assign to every $\mathcal{L}^{\vee\!\!\!\vee}$-formula φ a finite set of $\mathcal{L}$-formulas $\mathcal{R}(\varphi)$ in accordance with the following recursive clauses:

- $\mathcal{R}(\bot) = \{\bot\}$, $\mathcal{R}(p) = \{p\}$, for each atomic formula p,
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)} (\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$,
- $\mathcal{R}(\varphi \vee\!\!\!\vee\, \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$.

The formulas in $\mathcal{R}(\varphi)$ are called "resolutions of φ". Note that for any $\vee\!\!\!\vee$-free α, $\mathcal{R}(\alpha) = \{\alpha\}$. If φ represents a question, the set $\mathcal{R}(\varphi)$ contains more than one formula, and, intuitively, the formulas in $\mathcal{R}(\varphi)$ form an exhaustive set of direct answers to φ. Resolutions of φ are semantically related to φ in the following way (see, e.g., Ciardelli 2022).

Proposition 4. *Let φ be an $\mathcal{L}^{\vee\!\!\!\vee}$-formula, $\mathcal{M}$ a c-model, and s a c-state in $\mathcal{M}$. If $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$ then*

$$s \Vdash \varphi \text{ in } \mathcal{M} \text{ if and only if } s \Vdash \alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee\, \alpha_n \text{ in } \mathcal{M}.$$

So, each $\mathcal{L}^{\vee\!\!\!\vee}$-formula represents either a statement, or a question with finite set of direct answers. For example, we obtain $\mathcal{R}(p \to ?q) = \{p \to q, p \to \neg q\}$, and so the formula $p \to ?q$ (representing the conditional question *if p, is q the case?*) represents a question with two direct answers $p \to q$ and $p \to \neg q$.

3 Fuzzy Truth

Truth is not always a crisp, binary matter. Vagueness is ubiquitous in language and it leads to the phenomenon of graded truth. For example, the truth value of such a vague claim like, e.g., that a particular person is rich, might be difficult to determine. Nevertheless, it seems clear that such claim is more true for some people than for other. There are various strategies of modelling the phenomenon of vagueness (see for example (Sorensen 2022), for an overview). I will focus on the approach known as fuzzy logic in the form developed by Petr Hájek (1998).

We introduce the semantics of fuzzy logic in analogy with our presentation of the truth-conditional semantics for classical logic. Given a set of possible worlds W, an information c-state s in W is a subset of W. Each such subset can be represented by its characteristic function $g_s : W \to \{0, 1\}$, such that $g_s(w) = 1$ if $w \in s$, and $g_s(w) = 0$ if $w \notin s$. The notion of a subset of W can be generalized by replacing the set $\{0, 1\}$ with the closed real interval $[0, 1]$. Any function $g : W \to [0, 1]$ is called a *fuzzy subset* of W. The value $g(w)$, for a given world w, represents the degree to which w belongs to g. The symbols $\emptyset_f$ and W_f will denote the fuzzy sets corresponding respectively to the empty set and to the full set W. So, $\emptyset_f$ constantly assigns 0 and W_f constantly assigns 1 to the worlds of W. Fuzzy subset relation $\sqsubseteq$, fuzzy intersection $\sqcap$ and fuzzy union $\sqcup$ are defined as follows:

$$s \sqsubseteq t \text{ iff } s(w) \leq t(w), \text{ for all } w \in W,$$

$$(s \sqcap t)(w) = min\{s(w), t(w)\},$$

$$(s \sqcup t)(w) = max\{s(w), t(w)\}.$$

Note that if we restrict ourselves just to the crisp values 0 and 1 and the crisp subsets are identified with their characteristic functions then the usual subset relation, intersection and union can be defined in the same way.

Definition 2. *An f-model (fuzzy model) is a pair $\mathcal{M} = \langle W, V \rangle$, where W is a non-empty set (of possible worlds) and V is an f-valuation (fuzzy valuation), i.e. a function assigning to each atomic formula an information f-state (fuzzy information state) defined as a fuzzy subset of W.*

	w_1	w_2	w_3
the culprit is tall	0.7	0.8	0.5
the culprit is smart	0.5	0.9	0.6
the culprit knew well the victim	0.8	0.2	0.5

Table 1: An example illustrating graded states

Let us illustrate the notion of a fuzzy information state with the following simple example. Assume that a crime was committed and there are three suspects. Given this context assume that W consists of three relevant scenarios. In each of these scenarios one of the suspects committed the crime. Let an agent's information state be formed by the vague information *that the culprit is tall, smart and knew well the victim*. In each world the components of the information are evaluated as in Table 1. The worlds are more or less compatible with the information that is available to the agent (that the culprit is tall, smart and knew well the victim). The agent's information state is represented by a function that assigns to each world from W the degree to which it is compatible with the information that is available to the agent. It is not completely clear how this degree should be calculated in our example. The t-norms defined below can be viewed as alternative well-behaving strategies of calculating such a number. For example, one reasonable approach seems to be taking the minimum value. According to this strategy, the first world is compatible with the available information to the degree 0.5, the second world to the degree 0.2 and the third one to the degree 0.5. Then the agent's information state can be identified with the corresponding function: $w_1 \mapsto 0.5, w_2 \mapsto 0.2, w_3 \mapsto 0.5$.

Definition 3. *A continuous t-norm is a continuous, commutative, associative and monotone binary function $*$ on the interval $[0, 1]$ such that $1 * x = x$ and $0 * x = 0$, for each x from $[0, 1]$.*

By a t-norm, we will always mean in this paper a *continuous* t-norm. (In fact, we could be slightly more general and consider all *left-continuous* t-norms. But we will follow the classical presentation of t-norm based fuzzy logics from (Hájek 1998) which uses the notion of a continuous t-norm.) The function min, assigning to any two numbers from $[0, 1]$ their minimum, is an example of a t-norm. Another example is product $\times$, as an operation on the interval $[0, 1]$. Another important t-norm is the Łukasiewicz t-norm defined as follows:

$$x *_{Ł} y = max\{0, x + y - 1\}.$$

Fixing a t-norm $*$ there is a unique binary residual operation $\Rightarrow_*$ on $[0, 1]$ satisfying:

$$x * y \leq z \text{ iff } x \leq y \Rightarrow_* z.$$

(In fact, left-continuity of a t-norm is necessary and sufficient for the existence of the residual operation.) For example, the residual of min is the function $\Rightarrow_{min}$ defined as follows:

$$x \Rightarrow_{min} y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

The residual of the Łukasiewicz t-norm $*_{Ł}$ is the function $\Rightarrow_{Ł}$ defined in this way:

$$x \Rightarrow_{Ł} y = \begin{cases} 1 & \text{if } x \leq y \\ 1 - x + y & \text{otherwise} \end{cases}$$

If no confusion arises, we will often omit the subscript in $\Rightarrow_*$. The following properties of t-norms will be needed below. For more details, see (Hájek 1998; Cintula, Hájek, and Noguera 2011).

Proposition 5. *Let $*$ be a t-norm and $\Rightarrow$ its residual. Then the following claims hold generally for all $x, x_1, x_2, y, y_1, y_2, z \in [0, 1]$:*

(a) $x * y \leq min\{x, y\}$,
(b) $x \leq y$ *iff* $x \Rightarrow y = 1$,
(c) $(x_1 \Rightarrow y_1) * (x_2 \Rightarrow y_2) \leq (x_1 * x_2) \Rightarrow (y_1 * y_2)$,
(d) $(x \Rightarrow y) * (y \Rightarrow z) \leq x \Rightarrow z$,
(e) $0 \Rightarrow x = 1$,
(f) $1 \Rightarrow x = x$,
(g) $max\{x, y\} \Rightarrow z = min\{x \Rightarrow z, y \Rightarrow z\}$,
(h) *if* $x \leq y$ *then* $y \Rightarrow z \leq x \Rightarrow z$,
(i) *if* $x \leq y$ *then* $z \Rightarrow x \leq z \Rightarrow y$.

Different t-norms represent different ways of fusing information. As is usual in fuzzy logic, we will introduce a new *conjunction* $\&$ to be able to reflect this fusion in the object language, as a residual of implication. Negation and disjunction are defined as before. The symbols $\mathcal{L}_{\&}$ and $\mathcal{L}_{\&}^{\vee\!\!\!\vee}$ will respectively denote the extensions of the languages $\mathcal{L}$ and $\mathcal{L}^{\vee\!\!\!\vee}$ with this new conjunction. Fixing a t-norm $*$ and an f-model $\mathcal{M}$ we can define the fuzzy truth value of each $\mathcal{L}_{\&}$-formula α in each world w of $\mathcal{M}$. This value is denoted as $w_*(\alpha)$ but we will usually omit the subscript $*$. It is calculated in accordance with these recursive clauses:

$$w(\bot) = 0,$$
$$w(p) = V(p)(w), \text{ for each atomic formula } p,$$
$$w(\alpha \wedge \beta) = min\{w(\alpha), w(\beta)\},$$
$$w(\alpha \,\&\, \beta) = w(\alpha) * w(\beta),$$
$$w(\alpha \to \psi) = w(\alpha) \Rightarrow w(\beta).$$

Note that in restriction to the values $0, 1$ these clauses behave in accordance with classical logic and thus generalize the standard truth-conditional semantics for classical logic introduced in the previous section. For negation and disjunction we obtain: $w(\neg\alpha) = w(\alpha) \Rightarrow 0$; $w(\alpha \vee \beta) = max\{w(\alpha), w(\beta)\}$.

We say that an $\mathcal{L}_{\&}$-formula α is tc$_*$-valid in an f-model $\mathcal{M}$, if $w(\alpha) = 1$ in every world of $\mathcal{M}$, relative to the t-norm $*$. Let $\Delta \cup \{\alpha\}$ be any set of $\mathcal{L}_{\&}$-formulas. We say that α is a tc$_*$-consequence of Δ if α is tc$_*$-valid in every f-model $\mathcal{M}$ in which all formulas from Δ are tc$_*$-valid. This truth-conditional validity and consequence relation is relative to a given t-norm $*$. In this way, every t-norm determines a particular fuzzy logic, for example Ł determines Łukasiewicz logic, while min determines Gödel-Dummett logic (Hájek 1998).

In analogy to the crisp case, if we want to add questions and employ the language $\mathcal{L}_{\&}^{\mathbb{V}}$, we need to move to the support-conditional semantics, or more precisely, we need to formulate its fuzzyfied version. Here we get to the central topic of this paper.

4 Fuzzy Information States

Interestingly, the fuzzy version of the truth-conditional semantics that we introduced in the previous section allows us to fuzzify the support-conditional semantics in two very different ways that both make a very good sense. The first one replaces the notion of an information state, with respect to which formulas are evaluated, with the notion of a fuzzy information state. The second one replaces the crisp notion of support with a notion of fuzzy support. In this section, we will focus on the first approach, in the next section on the second one.

The generalization presented in this section is based on (Punčochář 2019), where, however, a similar framework was presented rather too abstractly, without a clear intuitive motivation. Here, we pay more attention to how the semantics is informally motivated. The framework presented in the next section is new (though there are some similarities with the approach developed in (Xie and Wu 2019) that are discussed below). It further generalizes the semantics of this section in a new dimension.

Recall from Definition 2 that an information f-state s in an f-model $\mathcal{M} = \langle W, V \rangle$ is a fuzzy subset of W, which we denote as $s \sqsubseteq W_f$, where W_f is the function assigning 1 to every element of W. Note that by this maneuver the algebra of information states is completely changed.

We can define information-based semantics relative to f-states. For the formulation of the semantic clauses it will be useful to define the operation $*$ also on the level of f-states:

$$(s * t)(w) = s(w) * t(w),$$

Now, given an f-model $\mathcal{M}$ we can define the support relation $\Vdash_*$ as a relation between f-states of $\mathcal{M}$ and $\mathcal{L}_{\&}^{\mathbb{V}}$-formulas. Again, the subscript $*$ will be omitted. The relation is defined as follows:

$s \Vdash \bot$ iff $s = \emptyset_f$,

$s \Vdash p$ iff $s \sqsubseteq V(p)$, for each atomic formula p,

$s \Vdash \varphi \wedge \psi$ iff $s \Vdash \varphi$ and $s \Vdash \psi$,

$s \Vdash \varphi \,\&\, \psi$ iff $\exists t, u \sqsubseteq W_f$: $t \Vdash \varphi$, $u \Vdash \psi$ and $s \sqsubseteq t * u$,

$s \Vdash \varphi \rightarrow \psi$ iff $\forall t \sqsubseteq W_f$, if $t \Vdash \varphi$, then $s * t \Vdash \psi$,

$s \Vdash \varphi \mathbin{\mathbb{V}} \psi$ iff $s \Vdash \varphi$ or $s \Vdash \psi$.

The support conditions have exactly the same shape as in the crisp setting, except that the clause for & is newly added, the subset relation is replaced by fuzzy subset relation, and $\cap$ is replaced with $*$ in the (modified) clause for implication stated after Proposition 3.

We say that an $\mathcal{L}_{\&}^{\mathbb{V}}$-formula φ is sc$_*$-valid in an f-model $\mathcal{M}$, if $W_f \Vdash \varphi$ in $\mathcal{M}$. Let $\Delta \cup \{\varphi\}$ be a set of $\mathcal{L}_{\&}^{\mathbb{V}}$-formulas. We say that φ is an sc$_*$-consequence of Δ if φ is sc$_*$-valid in every f-model $\mathcal{M}$ in which all formulas from Δ are sc$_*$-valid.

Propositions 1, 2, 3 and 4 can now be adapted to our generalized setting by which we obtain their more abstract analogues: Propositions 6, 7, 8 and 9. These propositions follow from the results in (Punčochář 2019).

Proposition 6 (Truth-Support Bridge-II)**.** *For any information f-state s of any f-model, and for any $\mathcal{L}_{\&}$-formula α:*

$$s \Vdash \alpha \text{ if and only if } s(w) \leq w(\alpha), \text{ for all } w \in W.$$

One can observe that if we allowed for each world w only two options, either $s(w) = 0$ (w is fully incompatible with s), or $s(w) = 1$ (w is fully compatible with s), then Truth-Support Bridge-II would express the same as Truth-Support Bridge-I, namely that for declarative formulas support by s coincides with truth in all worlds of s.

In analogy to the crisp case, Truth-Support Bridge-II implies for every declarative α that α is tc$_*$-valid in $\mathcal{M}$ if and only if α is sc$_*$-valid in $\mathcal{M}$. It follows for every t-norm that, in restriction to the declarative fragment, the support-conditional semantics determines the same consequence relation as the truth-conditional semantics.

Proposition 7. *An $\mathcal{L}_{\&}$-formula α is an sc$_*$-consequence of a set of $\mathcal{L}_{\&}$-formulas Δ if and only if α is a tc$_*$-consequence of Δ.*

For all formulas, whether declarative or inquisitive, we again obtain only some weaker properties, in particular the empty-state property and the persistence property.

Proposition 8. *In any f-model and for any $\mathcal{L}_{\&}^{\mathbb{V}}$-formula φ:*

(a) *$\emptyset_f \Vdash \varphi$ (empty-state property),*

(b) *if $s \Vdash \varphi$ and $t \sqsubseteq s$ then $t \vDash \varphi$ (persistence property).*

We also expand the notion of resolution to the language $\mathcal{L}_{\&}^{\mathbb{V}}$ in the following way. For atomic formulas and for $\bot, \wedge, \rightarrow, \mathbb{V}$, the equations defining resolutions are the same as in Section 2. We just need to add the equation for &:

$$\mathcal{R}(\varphi \,\&\, \psi) = \{\alpha \,\&\, \beta \mid \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}.$$

The crucial property relating a formula with its resolutions is preserved.

Proposition 9. *Let φ be an $\mathcal{L}_{\&}^{\mathbb{V}}$-formula, $\mathcal{M}$ an f-model, and s an f-state in $\mathcal{M}$. If $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$ then*

$$s \Vdash \varphi \text{ in } \mathcal{M} \text{ if and only if } s \Vdash \alpha_1 \mathbin{\mathbb{V}} \ldots \mathbin{\mathbb{V}} \alpha_n \text{ in } \mathcal{M}.$$

This means that the relation between the declarative part of the language and the inquisitive part remains intact by the generalization. What have changed is just the background logic of declarative sentences. Classical logic of statements can be replaced in this way with any t-norm fuzzy logic.

Even though the specification of what counts as a direct answer to a given question is the same in the crisp (classical) setting and the fuzzy setting of this section, because the resolution conditions are the same, the change of the background logic of declarative formulas has significant impact on the logic of questions. For example, in contrast to the crisp inquisitive semantics, polar questions may have nontrivial presuppositions in the fuzzy setting. Let us explain this point more carefully.

	w_1	w_2	w_3
s	0.3	0.4	0.9
p	0.6	0.7	0.89
q	0.2	0.3	0.1

Table 2: An example illustrating graded support

The presupposition of a question is what is assumed to be true when one asks the question. For example, when asking *whether* p or q (e.g. whether Greg is Ann's husband or Ann's brother), we presuppose that either p, or q is true. More formally, $p \vee q$ represents the presupposition of $p \veebar q$. If $p \vee q$ is assumed by mistake and someone asks the question $p \veebar q$ that has a false presupposition, a proper reply is a denial of the presupposition, though the negation of the presupposition does not count as a direct answer. That is exactly the meaning of $\neg(p \veebar q)$ in crisp inquisitive semantics. It is a statement denying the presupposition of the question $p \veebar q$, and thus it is equivalent to $\neg(p \vee q)$. This holds for both the crisp and the fuzzy version of inquisitive semantics.

In the crisp inquisitive semantics, the polar question *whether* p, i.e. $?p = p \veebar \neg p$, has a trivial presupposition $p \vee \neg p$. In contrast, in fuzzy logics, like Łukasiewicz or Gödel-Dummett, the principle of excluded middle does not hold and thus the presupposition becomes non-trivial. This corresponds to the natural language polar questions involving vague vocabulary. For example, a proper reply to the question *whether Greg is tall* might be *One cannot really say, he is a borderline case*.

5 Fuzzy Support

The notion of support, as we defined it in the previous section, is crisp. Given a state s, and a formula α, it either holds that s fully supports α, or s does not support α at all. According to the Truth-Support Bridge-II (Proposition 6) supporting α by s means that there is no world w that would be compatible with the state s to a degree that is bigger than the truth value of α in w. This has the following intuitive meaning. If there is $w \in W$ such that $s(w) \not\leq w(\alpha)$, i.e. $w(\alpha) < s(w)$, then α cannot be a part of the information that constitutes the state s. Otherwise, the presence of α would force the value of w in s to be smaller (in particular, smaller or equal to $w(\alpha)$). This observation is independent of the particular strategy according to which the function representing s was calculated, provided that it was calculated in accordance with a t-norm function. For it holds for every t-norm function $*$ and all $x, y \in [0,1]$ that $x * y \leq min\{x, y\}$ (Proposition 5-a), so if α is "contained" in the information constituting s then indeed $s(w)$ cannot be bigger than $w(\alpha)$.

However, intuitively, it makes sense to claim that a state s supports p a bit more than q (in analogy to the claim that p is a bit more true than q in a world). For example, consider the f-model depicted in Table 2. Even though, in this f-model, $s \nVdash p$ as well as $s \nVdash q$, we would intuitively say that s supports p more than q. Such a notion of graded support seems to be quite natural and important. Is it possible to define it in a systematic way? For this purpose, the following notational convention will be useful. For any subset X of $[0,1]$, let $\bigwedge X$ denote the infimum and $\bigvee X$ the supremum of X. For any f-state s and any $\mathcal{L}_{\&}$-formula α

$$s \rightsquigarrow_* \alpha \text{ is a shorthand for } \bigwedge_{w \in W} (s(w) \Rightarrow_* w(\alpha)).$$

Again, the subscript $*$ will usually be omitted. Recall that it holds for every t-norm and all $x, y \in [0,1]$ that $x \leq y$ iff $x \Rightarrow y = 1$ (Proposition 5-b). As a consequence of Proposition 6, for any f-state s and any $\mathcal{L}_{\&}$-formula α:

$$s \Vdash \alpha \text{ iff } s \rightsquigarrow \alpha = 1.$$

So, $s \rightsquigarrow \alpha$ reflects the support of α in the state s. However, in some cases $s \rightsquigarrow \alpha < 1$. In such a case, $s \nVdash \alpha$, regardless how far is the value $s \rightsquigarrow \alpha$ from 1. It seems that the expression $s \rightsquigarrow \alpha$ determines well the intuitive degree to which α is supported by s. In general, the value of $s \rightsquigarrow \alpha$ is dependent on the choice of the selected t-norm. As regards the notion of graded support, Łukasiewicz t-norm seems to capture the intuition particularly well. For an illustration, consider again the example from Table 2. If $*$ is the Łukasiewicz t-norm we obtain: $s \rightsquigarrow_* p = 0.99$ and $s \rightsquigarrow_* q = 0.2$.

Our aim now is to capture the notion of graded or fuzzy support that would state for any f-state s and any declarative formula α, how much s supports α. We denote this value as $s_*[\alpha]$ and we want to obtain $s_*[\alpha] = s \rightsquigarrow_* \alpha$ (if no confusion arises, the subscript $*$ will be omitted). This will be our Truth-Support Bridge in this general fuzzy setting.

One can observe that the definition of $s \rightsquigarrow_* \alpha$ is reminiscent of the way a modal box operator is defined in fuzzy modal logic (Bou et al. 2008; Rodriguez et al.). There is indeed an analogy. This is related to the fact that the clause defining $s \rightsquigarrow_* \alpha$ is a natural generalization of support in crisp inquisitive semantics and the semantic clause for box in fuzzy Kripke semantics is a natural generalization of crisp Kripke-style semantics for box. Also in the crisp setting the characterization of support (a declarative formula α is supported by s iff α is true in all worlds of s) is analogous to the Kripke style clause for box (a formula $\Box\alpha$ is true in w iff α is true in all worlds accessible from w). The difference between support of α and truth of $\Box\alpha$ is also preserved in the generalization: support is relative to states and truth to worlds. This difference is important because only the setting involving states allows us to introduce inquisitive disjunction and make a distinction between inquisitive and declarative propositions.

We will not *define* the value $s[\alpha]$ directly as $s \rightsquigarrow \alpha$. Instead, we will proceed in analogy to the previous two stages of information-based semantics in which we defined support recursively. This allowed us to include also a semantic clause for $\veebar$ and thus determine the meaning of all formulas, even those in which $\veebar$ was embedded under other operators. Only after that we stated Truth-Support Bridge as a consequence of the recursive definitions.

For each connective we are searching for a suitable recursive clause. For instance, in the case of implication we want to determine the value $s[\varphi \to \psi]$ in terms of the values $s[\varphi]$ and $s[\psi]$. For an illustration, let us show that a promising candidate, the equation

$$s[\varphi \to \psi] = s[\varphi] \Rightarrow s[\psi],$$

	v	w
s	1	0.5
p	1	0
q	0.5	0

Table 3: A counterexample to $s[\varphi \to \psi] = s[\varphi] \Rightarrow s[\psi]$

does not lead to a satisfactory solution. Consider, the f-model specified in Table 3. Let $*$ be the Łukasiewicz t-norm and assume that with respect to this t-norm we obtained the desired results: $s[p \to q] = s \rightsquigarrow (p \to q)$, $s[p] = s \rightsquigarrow p$, $s[q] = s \rightsquigarrow q$. But now we can calculate $s[p \to q] = 0.5$, while $s[p] \Rightarrow s[q] = 0.5 \Rightarrow 0.5 = 1$, and so $s[p \to q] \neq s[p] \Rightarrow s[q]$. We need to find a different recursive clause for implication.

It turns out that in order to obtain a well-behaving fuzzy support we need to translate the crisp support conditions into the language of the algebra of truth values. For example, in the previous section we used the following support condition for implication:

$$s \Vdash \varphi \to \psi \text{ iff } \forall t \sqsubseteq W_f, \text{ if } t \Vdash \varphi, \text{ then } s * t \Vdash \psi.$$

We can translate this equivalence into the algebraic language of fuzzy logic, replacing the crisp support relation with a fuzzy one, the universal quantifier with infimum, and implication with the residual of the given t-norm, obtaining this equation:

$$s[\varphi \to \psi] = \bigwedge_{t \sqsubseteq W_f} (t[\varphi] \Rightarrow s * t[\psi]).$$

This move can also be viewed as replacing the classical metalanguage in which the original clause was formulated with a fuzzy metalanguage of any given t-norm. This can be done with each of the semantic clauses from the previous section thus obtaining the following framework. For any fuzzy model $\mathcal{M} = \langle W, V \rangle$ and any t-norm $*$, we define fuzzy support in this way:

$s[\bot] = s \rightsquigarrow \bot$,
$s[p] = s \rightsquigarrow p$, for every atomic formula p,
$s[\varphi \wedge \psi] = min\{s[\varphi], s[\psi]\}$,
$s[\varphi \,\&\, \psi] = \bigvee_{t,u \sqsubseteq W_f} (t[\varphi] * u[\psi] * \bigwedge_{w \in W} (s(w) \Rightarrow (t * u(w))))$,
$s[\varphi \to \psi] = \bigwedge_{t \sqsubseteq W_f} (t[\varphi] \Rightarrow s * t[\psi])$,
$s[\varphi \vee\!\!\!\vee \psi] = max\{s[\varphi], s[\psi]\}$.

We say that an $\mathcal{L}^{\vee\!\!\!\vee}_{\&}$-formula φ is fsc$_*$-valid (fuzzy support-conditionally valid relative to the t-norm $*$) in an f-model $\mathcal{M}$, if $W_f[\varphi] = 1$ in $\mathcal{M}$. Let $\Delta \cup \{\varphi\}$ be any set of $\mathcal{L}^{\vee\!\!\!\vee}_{\&}$-formulas. We say that φ is an fsc$_*$-consequence of Δ if φ is fsc$_*$-valid in every f-model $\mathcal{M}$ in which all formulas from Δ are fsc$_*$-valid.

The semantics that we just introduced is similar in some respects to the framework from (Xie and Wu 2019). However, there are also significant differences. For example, the semantics from (Xie and Wu 2019) uses crisp information states but fuzzy support relation which is a combination that is not considered in this paper because it leads to a discrepancy between information states (as crisp sets) and propositions (as fuzzy sets). Moreover, fuzzy support is defined in (Xie and Wu 2019) by different semantic clauses and different structures of values are used (any involutive complete lattice instead of the interval $[0, 1]$). For example, a consequence of such a different setting is the validity of double negation law for atomic formulas, which is not generally valid in our setting. It would be interesting to make a more detailed comparison of the two approaches but we do not have enough space for it here.

Let us discuss the intended informal interpretation of our framework. For a declarative formula α, $s[\alpha]$ represents the degree to which the information constituting s implies α. For a question φ, $s[\varphi]$ represents the degree to which the information constituting s resolves the question φ. However, one has to be careful with this interpretation. To be clear, we can obtain a low value of both $s[\alpha]$ and $s[\neg\alpha]$ even if s is a state representing perfect knowledge of a particular world. Such a state assigns 1 to this world and 0 to all other worlds. However, if α represents a highly vague claim then s supports neither α nor $\neg\alpha$ to a high degree.

Considering questions, there are different senses in which a body of information may partially resolve a question. For example, one can reduce the possibilities without specifying completely which one is the case. The question *What is the colour of Ann's car?* is partially resolved by the reply that it is either blue, or green. But this is not the sense of a partial resolution that is captured by our model. We are here concerned rather with phenomena that are related to vagueness, and with questions that involve vague vocabulary. For an illustration, consider this question: *Is Ann a cat person or a dog person?* Assume that according to the information available to an agent, Ann is enthusiastic neither about cats nor about dogs. But if she had to decide she would prefer cats, which she likes significantly more than dogs. Then, according to our model, the agent's information state resolves the question only to a limited degree, to the degree to which she likes cats, or in other words, to which she is a cat person. There are many examples of such questions that would fall under the category of false dilemma fallacy because of vagueness. Consider the following examples: *Are you with us or against us? Is the enemy weak or strong? Do you like beer or wine? Are you an early bird or night owl?*

Returning to the technical aspects of our framework, we can now prove the main result of this paper that says that a version of Truth-Support Bridge holds also on this general level.

Proposition 10 (Truth-Support Bridge-III). *For any t-norm $*$, any information f-state $s \sqsubseteq W_f$ of any f-model and any $\mathcal{L}_{\&}$-formula α:*

$$s_*[\alpha] = s \rightsquigarrow_* \alpha.$$

Proof. We prove this claim by induction on the $\vee\!\!\!\vee$-free formulas. The inductive basis for p and $\bot$ is immediate. We need to check the inductive steps for $\wedge$, $\&$ and $\to$. The inductive step for $\wedge$ is straightforward:

$$\begin{aligned} s[\alpha \wedge \beta] &= min\{s[\alpha], s[\beta]\} \\ &= min\{s \rightsquigarrow \alpha, s \rightsquigarrow \beta\} \\ &= \bigwedge_{w \in W} (s(w) \Rightarrow min\{w(\alpha), w(\beta)\}) \\ &= \bigwedge_{w \in W} (s(w) \Rightarrow w(\alpha \wedge \beta)). \end{aligned}$$

Let us consider the inductive step for &. We have to prove:

$$\bigvee_{t,u \sqsubseteq W_f} (t[\alpha]*u[\beta]* \bigwedge_{w \in W} (s(w) \Rightarrow (t*u(w)))) = s \rightsquigarrow \alpha \,\&\, \beta.$$

Let us denote the right side of this equation as r. Using the induction hypothesis, the left side can be rewritten as

$$l = \bigvee_{t,u \sqsubseteq W_f} ((t \rightsquigarrow \alpha)*(u \rightsquigarrow \beta)* \bigwedge_{w \in W} (s(w) \Rightarrow (t*u(w))))$$

Take any $t, u \sqsubseteq W_f$. Then, using Proposition 5-c,d, we obtain

$$\begin{aligned} &(t \rightsquigarrow \alpha) * (u \rightsquigarrow \beta) * \bigwedge_{w \in W} (s(w) \Rightarrow (t * u(w))) \leq \\ &\leq \bigwedge_{w \in W} ((t(w) \Rightarrow w(\alpha)) * (u(w) \Rightarrow w(\beta))* \\ &\quad *(s(w) \Rightarrow (t * u(w)))) \leq \end{aligned}$$

$$\begin{aligned} &\leq \bigwedge_{w \in W} ((t*u(w) \Rightarrow w(\alpha)*w(\beta))*(s(w) \Rightarrow (t*u(w))) \leq \\ &\leq \bigwedge_{w \in W} (s(w) \Rightarrow (w(\alpha) * w(\beta))) = \\ &= \bigwedge_{w \in W} (s(w) \Rightarrow (w(\alpha \,\&\, \beta))). \end{aligned}$$

It follows that $l \leq r$. For the other direction take the states t, u defined as follows: for every $w \in W$, let $t(w) = w(\alpha)$ and $u(w) = w(\beta)$. Then it holds:

$$\begin{aligned} &(t \rightsquigarrow \alpha) * (u \rightsquigarrow \beta) * \bigwedge_{w \in W} (s(w) \Rightarrow (t * u(w))) = \\ &= 1 * 1 * \bigwedge_{w \in W} (s(w) \Rightarrow (w(\alpha) * w(\beta))) = \\ &= \bigwedge_{w \in W} (s(w) \Rightarrow (w(\alpha \,\&\, \beta))). \end{aligned}$$

It follows that $r \leq l$ which finishes the proof for &.

Let us consider the case of implication. We have to prove

$$\bigwedge_{t \sqsubseteq W_f} (t[\alpha] \Rightarrow t * s[\beta]) = s \rightsquigarrow (\alpha \rightarrow \beta).$$

Using the induction hypothesis, the left side of this equation boils down to

$$l = \bigwedge_{t \sqsubseteq W_f} ((t \rightsquigarrow \alpha) \Rightarrow \bigwedge_{w \in W} (t(w) \Rightarrow (s(w) \Rightarrow w(\beta)))).$$

The right side can be rewritten as

$$r = \bigwedge_{w \in W} (w(\alpha) \Rightarrow (s(w) \Rightarrow w(\beta))).$$

For every world $v \in W$ there is a corresponding state t_v defined as follows: $t_v(v) = 1$ and $t_v(w) = 0$, for all $w \neq v$. Using Proposition 5-e,f, we obtain that for every $v \in W$

$$(t_v \rightsquigarrow \alpha) \Rightarrow \bigwedge_{w \in W} (t_v(w) \Rightarrow (s(w) \Rightarrow w(\beta)))$$

is equal to

$$v(\alpha) \Rightarrow (s(v) \Rightarrow v(\beta)).$$

This implies that $l \leq r$. To show that also $r \leq l$, fix any $t \sqsubseteq W_f$ and $v \in W$. Note that $(t(v) \Rightarrow v(\alpha))*t(v) \leq v(\alpha)$. Since $t \rightsquigarrow \alpha \leq t(v) \Rightarrow v(\alpha)$, we obtain

$$(t \rightsquigarrow \alpha) * t(v) \leq v(\alpha),$$

which implies:

$$\begin{aligned} &v(\alpha) \Rightarrow (s(v) \Rightarrow v(\beta)) \leq \\ &\leq ((t \rightsquigarrow \alpha) * t(v)) \Rightarrow (s(v) \Rightarrow v(\beta)). \end{aligned}$$

From this we obtain for any any $t \sqsubseteq W_f$:

$$r \leq (t \rightsquigarrow \alpha) \Rightarrow \bigwedge_{w \in W} (t(w) \Rightarrow (s(w) \Rightarrow w(\beta))).$$

It follows that $r \leq l$ which finishes the inductive step for implication.

□

As an immediate consequence, a declarative formula is fsc$_*$-valid in an f-model if and only if it is sc$_*$- and thus also tc$_*$-valid in that model. This gives us also equivalence of the respective consequence relations.

Proposition 11. *An $\mathcal{L}_{\&}$-formula α is an fsc$_*$-consequence of a set of $\mathcal{L}_{\&}$-formulas Δ if and only if α is a tc$_*$-consequence of Δ.*

Again, the main merit of the fuzzy support-conditional semantics is that it allowed us to incorporate also questions. And again, questions cannot be reduced to truth in the style of the Truth-Support Bridge. Since we do not have truth conditions for $\mathbin{\rotatebox[origin=c]{180}{$\mathbb{\wedge}$}}$ we cannot directly say that Proposition 10 does not generally hold for inquisitive formulas. Nevertheless we can indirectly show that in a sense this is true. Note that it follows from Propositions 10 and 5-g that it holds for all f-states $s, t \sqsubseteq W_f$ and for every $\mathcal{L}_{\&}$-formula α:

$$min\{s[\alpha], t[\alpha]\} = (s \sqcup t)[\alpha].$$

This is a fuzzyfied version of the claim that, in the information-based semantics with crisp support, declarative formulas express propositions that are closed under union. This does not hold for inquisitive formulas. For example, considering the f-states s, t in the f-model in Table 4, where one can calculate $min\{s[p \vee\!\!\!\vee q], t[p \vee\!\!\!\vee q]\} = 1$ and $(s \sqcup t)[p \vee\!\!\!\vee q] = 0$.

For all $\mathcal{L}_{\&}^{\vee\!\!\!\vee}$-formulas we can again state only weaker properties that correspond to the empty-state property and the persistence property.

	v	w
s	1	0
t	0	1
p	1	0
q	0	1

Table 4: $min\{s[p \vee\!\!\!\vee q], t[p \vee\!\!\!\vee q]\} \neq (s \sqcup t)[p \vee\!\!\!\vee q]$

Proposition 12. *For every $\mathcal{L}^{\vee\!\!\!\vee}_{\&}$-formula φ:*
(a) $\emptyset_f[\varphi] = 1$ *(empty-set property),*
(b) *if $s \sqsubseteq t$ then $t[\varphi] \leq s[\varphi]$ (persistence property).*

Proof. (a) We will proceed by induction. For any atomic formula p, we have

$$\emptyset_f[p] = \bigwedge_{w \in W} (\emptyset_f(w) \Rightarrow w(p)) = \bigwedge_{w \in W} (0 \Rightarrow w(p)) = 1.$$

The case of $\bot$ is similar. As the inductive hypothesis, assume $\emptyset_f[\varphi] = 1$ and $\emptyset_f[\psi] = 1$. Then the inductive step for $\wedge$ is immediate: $\emptyset_f[\varphi \wedge \psi] = min\{\emptyset_f[\varphi], \emptyset_f[\psi]\} = 1$. For the inductive step for $\&$, note that

$$\emptyset_f[\varphi] * \emptyset_f[\psi] * \bigwedge_{w \in W} (\emptyset_f(w) \Rightarrow (\emptyset_f * \emptyset_f(w)))) = 1.$$

It follows that $\emptyset_f[\varphi \ \& \ \psi]$ is equal to

$$\bigvee_{t,u \sqsubseteq W_f} (t[\varphi] * u[\psi] * \bigwedge_{w \in W} (\emptyset_f(w) \Rightarrow (t * u(w)))) = 1.$$

We further obtain that $\emptyset_f[\varphi \to \psi]$ is equal to

$$\bigwedge_{t \sqsubseteq W_f} (t[\varphi] \Rightarrow t * \emptyset_f[\psi]) = \bigwedge_{t \sqsubseteq W_f} (t[\varphi] \Rightarrow 1) = 1.$$

Finally, $\emptyset_f[\varphi \vee\!\!\!\vee \psi] = max\{\emptyset_f[\varphi], \emptyset_f[\psi]\} = 1$.

(b) We will again proceed by induction. Assume $s \sqsubseteq t$. Then, by Proposition 5-h, $t(w) \Rightarrow w(p) \leq s(w) \Rightarrow w(p)$, for every $w \in W$, and thus

$$\bigwedge_{w \in W} (t(w) \Rightarrow w(p)) \leq \bigwedge_{w \in W} (s(w) \Rightarrow w(p)).$$

That is, $t[p] \leq s[p]$. The case of $\bot$ is similar. As the inductive hypothesis, assume that the claim holds for all states for some formulas φ, ψ. In the inductive step for $\wedge$ we can reason as follows:

$$t[\varphi \wedge \psi] = min\{t[\varphi], t[\psi]\} \leq min\{s[\varphi], s[\psi]\} = s[\varphi \wedge \psi].$$

The inductive step for $\&$ follows from Proposition 5-h that implies for all f-states u_1, u_2:

$$\bigwedge_{w \in W} (t(w) \Rightarrow u_1 * u_2(w)) \leq \bigwedge_{w \in W} (s(w) \Rightarrow u_1 * u_2(w)).$$

In the inductive step for $\to$, we can reason as follows. Take any f-state u. By monotonicity, $s \sqsubseteq t$ implies $s * u \sqsubseteq t * u$, which, by the inductive hypothesis, implies $t * u[\psi] \leq s * u[\psi]$, and form this it follows, using Proposition 5-i, that $u[\varphi] \Rightarrow t * u[\psi] \leq u[\varphi] \Rightarrow s * u[\psi]$. Since this holds for every f-state u, we obtain $t[\varphi \to \psi] \leq s[\varphi \to \psi]$. Finally, in the inductive step for $\vee\!\!\!\vee$ we can reason as follows:

$$t[\varphi \vee\!\!\!\vee \psi] = max\{t[\varphi], t[\psi]\} \leq max\{s[\varphi], s[\psi]\} = s[\varphi \vee\!\!\!\vee \psi].$$

□

Setting	Truth-Support Bridge
crisp states, crisp support	$s \Vdash \alpha$ iff $\forall w \in s, w \vDash \alpha$
fuzzy states, crisp support	$s \Vdash \alpha$ iff $\forall w \in W, s(w) \leq w(\alpha)$
fuzzy states, fuzzy support	$s[\alpha] = \bigwedge_{w \in W} (s(w) \Rightarrow w(\alpha))$

Table 5: Truth-Support Bridge in different versions of inquisitive semantics

Setting	Propositions expressed by formulas
crisp states, crisp support	downward closed crisp sets of crisp information states
fuzzy states, crisp support	downward closed crisp sets of fuzzy information states
fuzzy states, fuzzy support	antitone fuzzy sets of fuzzy information states

Table 6: Propositions in different versions of inquisitive semantics

6 Conclusion

To sum up, we have introduced three different versions of inquisitive semantics. The first one is based on crisp information states and crisp support relation, the second one on fuzzy information states and crisp support relation, and the third one (which is the main novel contribution of this paper) on fuzzy information states and fuzzy support relation. Among the four possibilities, we have not considered the version of the semantics that would be based on crisp states but fuzzy support. This option, though technically also interesting, is conceptually somewhat problematic because in the related framework information states (crisp sets) would be entities of a different kind than propositions (fuzzy sets).

For declarative formulas we have formulated in each of these frameworks a connection between truth and support in the form of a Truth-Support Bridge. The three versions of this principle are summarized in Table 5.

Truth-Support Bridge does not apply generally to formulas involving inquisitive disjunction. We have also formulated the key general features of all propositions, i.e. semantic contents expressed in a given model by formulas that may possibly involve inquisitive disjunction. These features amount to the three versions of the empty-set property and the persistence property and they are summarized in Table 6.

In future research, we will explore the relation between a question and its resolutions in the context of fuzzy support. We also plan to extend the semantics with epistemic modalities and we will explore the resulting logic of questions.

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