Weak-Ensconcement for Shielded Base Contraction

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Abstract

In this article, we provide the weak version of ensconcement which characterizes an interesting family of Shielded base contractions. In turn, this characterization induces a class of AGM contractions satisfying certain postulates that we reveal here. Finally, we show a connection among the class of contractions given by our weak ensconcement and other kinds of base contraction operators. In doing so, we also point out a flaw in the original theorems that link the epistemic entrenchment with ensconcement (which are well established in the literature), and then we provide two possible solutions.

1 Introduction

Nowadays, any interesting intelligent agent should be capable of adapting its beliefs to the new information constantly provided by its environment. It is then crucial to develop belief models that are useful for the agent's decision-making task. The one which is currently considered the standard model in the belief change literature is known as the AGM model (Alchourrón, Gärdenfors, and Makinson 1985) which has different, but equivalent, presentations. The main one is a set of rational postulates that point out natural properties that any change operator should satisfy. In (Gärdenfors and Makinson 1988), the authors proposed a constructive approach based on these postulates, called "epistemic entrenchment". This approach involves establishing an ordering between the facts in the agent's knowledge, based on their epistemic value. Such ordering then determines the priority of the facts when revising and contracting. However, when the agent's beliefs are represented by a belief set (i.e. a set closed under logical consequences) like in the AGM model, not only it could be necessary to take into account an infinite set of formulas (since, for example, all tautologies are in the belief set), but also the epistemic entrenchment construction requires explicitly ordering all logical consequences of the known facts.

As an example, consider the following situation: Asimov is a robot that interacts with humans, and his beliefs are *p: "I must not injure human beings"* and *q: "Alex is a human"*. However, the epistemic entrenchment model not only requires to order epistemically p and q, but also beliefs such as *"I must not injure human beings or the earth is flat"* which are logical consequences of her original beliefs.

To overcome this drawback, one solution is to work with belief bases, which are more suitable for computer-based implementations (Hansson 1992). In (Williams 1994), an adaptation of epistemic entrenchment called *ensconcement relation* was proposed. This is a total preorder on a belief base B that can be "blown up" to a full epistemic entrenchment $\leq$ related to $Cn(B)$. The formal axiomatic characterization of the ensconcement was given in (Fermé, Garapa, and Reis 2017), while also proving that the ensconcement contractions are a generalization of entrenchment contractions.

However, the AGM model still has another drawback very criticized in the literature: every non-tautological sentence is always removed when contracting by it. When this happens, due to the success postulate, it is said that the new observation is prioritized. Going back to our example, if Asimov was asked by a user to attack Alex, the robot must contract "*I must not hurt Alex*", a consequence of its beliefs. Thus Asimov would have to give up either p or q. However, both beliefs should not be available for contraction: Alex is indeed a human, and Asimov must avoids hurting humans. Thus, both beliefs p and q should be *shielded* from any possible contraction.

To model this behaviour, shielded contractions were introduced in (Fermé and Hansson 2001). This operator considers a set of *retractable observations* under which the contraction is performed as usual, and otherwise, the knowledge is left unchanged.

In (Fermé and Hansson 2001), the authors defined a *shielded* non-prioritized contraction for belief sets, by using an alternative epistemic entrenchment first defined in (Hansson et al. 2001) (which is obtained by eliminating the maximality of the tautologies postulate). Later, (Garapa, Fermé, and Reis 2018) provided the axiomatic characterization of the shielded contraction for belief bases.

In this work, we generalize that shielded entrenchment construction by defining a *weak ensconcement*, achieved by removing the maximality of the tautologies postulate for ensconcements. We then present the contraction operator based on this ensconcement and provide its axiomatic characterization and its standard shielded form. Finally, we establish the links between the shielded epistemic entrenchment and our new construction, as it has been done with their classical counterparts. In doing so, we also point out a flaw in the original theorems of (Williams 1994) that link the epistemic entrenchment with ensconcement (which has been reproduced by other authors (Peppas 2008; Williams et al. 1995; Fermé, Krevneris, and Reis 2008; Fermé, Garapa, and Reis 2017; Garapa, Fermé, and Reis 2017)), and then we provide two possible solutions.

The beliefs of an agent will be represented by a belief set K or a belief base A, both finite or infinite sets of sentences from a (possibly infinite) propositional logic $\mathcal{L}$. The main difference between them is that belief sets are closed under logical consequences. We will also add $\perp$ and $\top$ as formulas of $\mathcal{L}$ denoting an arbitrary contradiction and an arbitrary tautology, respectively.

2 Belief Base Contraction

This section will be devoted to prioritized belief operators. We shall briefly review the axiomatic approach of the belief base paradigm, as well as its explicit construction for this process based on preorders over sentences known as ensconcement. The following are well-known postulates for belief contraction over every belief base A, and every formula $\alpha, \beta \in \mathcal{L}$:

(Success) If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin Cn(A - \alpha)$

(Inclusion) $A - \alpha \subseteq A$.

(Failure) If $\vdash \alpha$ then $A - \alpha = A$.

(Vacuity) If $A \nvdash \alpha$, then $A \subseteq A - \alpha$.

(Relative Closure) $A \cap Cn(A - \alpha) \subseteq A - \alpha$.

(Core retainment) If $\beta \in A$ and $\beta \notin A - \alpha$ then there is some set A' such that $A' \subseteq A$ and $\alpha \notin Cn(A')$ but $\alpha \in Cn(A' \cup \{\beta\})$.

(Relevance) If $\beta \in A$ and $\beta \notin A - \alpha$ then there is some set A' such that $A - \alpha \subseteq A' \subseteq A$ and $\alpha \notin Cn(A')$ but $\alpha \in Cn(A' \cup \{\beta\})$.

(Recovery) $A \subseteq Cn((A - \alpha) \cup \{\alpha\})$.

(Disjunctive Elimination) If $\beta \in A$ and $\beta \notin A - \alpha$ then $A - \alpha \nvdash \alpha \vee \beta$.

(Extensionality) If $\vdash \alpha \leftrightarrow \beta$, then $A - \alpha = A - \beta$.

(Uniformity) If it holds for all subsets $A' \subseteq A$ that $\alpha \in Cn(A')$ if and only if $\beta \in Cn(A')$, then $A - \alpha = A - \beta$.

(Closure) If A is logically closed, then so is $A - \alpha$.

If A is logically closed, then the postulates of closure, inclusion, vacuity, success, extensionality, and recovery correspond to the commonly called *basic AGM postulates*. It is worth noticing that some postulates are implied by combinations of others as the following properties show:

Lemma 1. *(Garapa, Fermé, and Reis 2018) Let A be a belief base and $-$ an operator on A. Then:*

(a) *If $-$ satisfies relevance, then it satisfies relative closure and core-retainment.*

(b) *If $-$satisfies inclusion and core-retainment, then it satisfies failure and vacuity.*

(c) *If $-$ satisfies uniformity, then it satisfies extensionality.*

(d) *If $-$ satisfies disjunctive elimination, then it satisfies relative closure. If also satisfies inclusion, then it satisfies failure.*

(e) *If $-$ satisfies relevance, then it satisfies disjunctive elimination.*

We also introduce other postulates used in the characterization of ensconcement and their relations (Fermé, Garapa, and Reis 2017):

(Conjunctive Factoring)

$$A - \alpha \wedge \beta = \begin{cases} A - \alpha \text{ or} \\ A - \beta \text{ or} \\ A - \alpha \cap A - \beta \end{cases}$$

This postulate is equivalent to the well-known *supplementary AGM postulates* when A is logically closed.

(Transitivity) If $\beta \in A$, $\alpha \notin A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$, then $\alpha \notin A - \alpha \wedge \delta$.

(ST) If $\delta \in A$, $\beta \in A - \alpha \wedge \beta$ and $\beta \notin A - \beta \wedge \delta$, then $\delta \in A - \alpha \wedge \delta$.

(SST) If $\alpha \in A$, $\beta \in A - \alpha \wedge \beta$ and $\delta \in A - \beta \wedge \delta$, then $\delta \in A - \alpha \wedge \delta$.

(EB1) If $\beta \in A$ and $\{\gamma \in A : \beta \notin A - \beta \wedge \gamma\} \nvdash \alpha$, then $\beta \in A - \alpha$.

(EB2) If $\beta \in A - \alpha$ then $\{\gamma \in A : \gamma \in A - \gamma \wedge \alpha\} \vdash \alpha \vee \beta$.

Lemma 2. *(Fermé, Garapa, and Reis 2017) Let A be a belief base and $-$ be an operator on A. Then:*

(a) *If $-$ satisfies extensionality, then $-$ satisfies transitivity if and only if $-$ satisfies ST.*

(b) *If $-$ satisfies success, inclusion, extensionality, relative closure, and transitivity, then it satisfies SST.*

2.1 Epistemic Entrenchment

Epistemic entrenchment is a very flexible constructive approach for change operators. For example, given a total preorder on $\mathcal{L}$ satisfying certain properties then, we shall see next, it can uniquely determine a contraction operator that satisfies both basic and complementary AGM postulates.

Definition 1. *An ordering of epistemic entrenchment (in short* **EE***) with respect to a belief set K is a binary relation $\leq$ on $\mathcal{L}$ which satisfies the following properties:*

(EE1) *For all $\alpha, \beta, \delta \in \mathcal{L}$, if $\alpha \leq \beta$ and $\beta \leq \delta$ then $\alpha \leq \delta$.*
(EE2) *For all $\alpha, \beta \in \mathcal{L}$, if $\alpha \vdash \beta$ then $\alpha \leq \beta$.*
(EE3) *For all $\alpha, \beta \in \mathcal{L}$, $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$.*
(EE4) *When $K \nvdash \perp$, $\alpha \notin K$ iff $\alpha \leq \beta$ for all $\beta \in \mathcal{L}$.*
(EE5) *If $\beta \leq \alpha$ for all $\beta \in \mathcal{L}$, then $\vdash \alpha$.*

Each one of these properties receives an alternative name: **(EE1)** *Transitivity*, **(EE2)** *Dominance*, **(EE3)** *Conjunctiveness*, **(EE4)** *K-minimality*, **(EE5)** *Maximality*.

Definition 2. *Let K be a belief set and $\leq$ be an* **EE** *relation with respect to K. The $\leq$-based contraction on K is the contraction operation $\div_{\leq}$ defined, for any $\alpha \in \mathcal{L}$, by:*

$$K \div_{\leq} \alpha = \begin{cases} \{\beta \in K : \alpha < \alpha \vee \beta\} & \text{if } \nvdash \alpha \\ K & \text{if } \vdash \alpha \end{cases}$$

An operation $\div$ on K is an **EE***-based contraction on K if and only if there is an* **EE** *relation with respect to K such that, for all sentences $\alpha \in \mathcal{L}$, $K \div \alpha = K \div_{\leq} \alpha$.*

The following theorem shows the characterizations of these contractions in terms of a subset of the postulates presented before.

Theorem 1. *(Gärdenfors and Makinson 1988) Let K be a belief set and $\div$ be a contraction function on K. Then $\div$ is an* **EE***-based contraction if and only if it satisfies both the basic and the supplementary AGM postulates for contraction.*

The proof of the *only if* part of this theorem is straightforward. In order to complete the *if* direction of the proof, one shall construct the **EE** ordering by defining **(C$_{\leq}$)** as follows:

$$\alpha \leq \beta \iff \alpha \notin K \div (\alpha \wedge \beta) \text{ or } \vdash \beta \qquad (\mathrm{C}_{\leq})$$

The last definition captures the idea that a sentence α is epistemically less entrenched according to a knowledge state A than another sentence β if and only if an agent in a knowledge state A who is forced to give up either α or β will give up α and possibly preserves β.

2.2 Ensconcement

Mary-Anne Williams introduced a Belief Base Contraction operator called *Ensconcement* (Williams 1994), which generalizes the **EE** relation to belief bases. In particular, it paves the way for computer implementations of **EE**. The following definitions were provided in (Fermé, Krevneris, and Reis 2008).

Definition 3. *Define an ensconcement to be a set of formulas A together with a total and transitive preorder $\preceq$ of A satisfying the following conditions:*

($\preceq$ 1) *If $\beta \in A$ and $\nvdash \beta$, then $\{\alpha \in A \mid \beta \prec \alpha\} \nvdash \beta$.*
($\preceq$ 2) *If $\nvdash \alpha$ and $\vdash \beta$, then $\alpha \prec \beta$, for all $\alpha, \beta \in A$.*
($\preceq$ 3) *If $\vdash \alpha$ and $\vdash \beta$ then $\alpha \preceq \beta$, for all $\alpha, \beta \in A$.*

According to the first of these conditions, the formulas that are strictly more ensconced than β do not (even conjointly) imply β. According to the last two conditions, if there are any tautologies in a belief base A, then they are its most ensconced formulas.

In order to contract A by α given an ensconcement operator, a subset of A is used, called *cut*.

Definition 4. *Let A be a belief base and $\alpha \in \mathcal{L}$. Then $cut_{\prec}(\alpha)$ and $cut_{\preceq}(\alpha)$ are defined as follows:*

$$cut_{\prec}(\alpha) = \{\beta \in A : \{\gamma \in A : \beta \preceq \gamma\} \nvdash \alpha\}$$

$$cut_{\preceq}(\alpha) = \{\beta \in A : \{\gamma \in A : \beta \prec \gamma\} \nvdash \alpha\}$$

Then, the contraction operator is defined as follows:

Definition 5. *Let A be a belief base, and $\preceq$ an ensconcement defined over A. Then the ensconcement-based contraction operator for A $(-)$ is defined as follows:*

$$A - \alpha = \{\beta \in A : cut_{\prec}(\alpha) \vdash \alpha \vee \beta\}$$

In (Fermé, Garapa, and Reis 2017) the axiomatic characterization of the ensconcement-based contraction was proven in the following theorem.

Theorem 2. *Let A be a belief base. An operator $-$ of A is an ensconcement-based contraction on A if and only if it satisfies success, inclusion, vacuity, extensionality, conjunctive factoring, disjunctive elimination, transitivity, EB1 and EB2.*

The proof of this theorem requires constructing an associated ensconcement which is given, again, by using (C$_{\leq}$) considering A instead of K (i.e. considering a belief base instead of a belief set).

2.3 Relationship Between Epistemic Entrenchment and Ensconcement

In her original paper (Williams 1994), where she presented the ensconcement operator, she also provided the following definitions and properties:

Definition 6. *Let $(A, \preceq)$ be an ensconcement. For $\alpha, \beta \in \mathcal{L}$, we define $\leq_{\preceq}$ to be given by $\alpha \leq_{\preceq} \beta$ if and only if either*

(i) *$\alpha \notin Cn(A)$, or*
(ii) *$\alpha, \beta \in Cn(A)$ and $cut_{\preceq}(\beta) \subseteq cut_{\preceq}(\alpha)$.*

Lemma 3. *(Williams 1994, Theorem 4)*
$\leq_{\preceq}$ is an **EE** *order relative to $Cn(A)$.*

However, we shall see next that Lemma 3 does not hold in general by providing a counterexample. Fortunately, this flaw can be overcome in a very natural way.

Example 1. *Let $A = \{(p_0 \wedge p_i) \mid i \in \mathbb{N}\} \subseteq L$ be our set of formulas. Let $\preceq$ be an ensconcement defined over A such that:*

$$(p_0 \wedge p_k) \prec (p_0 \wedge p_j) \iff k < j$$

Then Lemma 3 does not hold.

Proof. First, we shall see that $cut_{\preceq}(p_0) = \emptyset$. Let us assume $cut_{\preceq}(p_0) \neq \emptyset$. Then there is a formula $(p_0 \wedge p_k) \in A$ with $k \in \mathbb{N}_{>0}$ such that: $\{\gamma \in A : (p_0 \wedge p_k) \prec \gamma\} \nvdash p_0$. But $(p_0 \wedge p_{k+1}) \in A$, and $(p_0 \wedge p_k) \prec (p_0 \wedge p_{k+1})$. Thus $\{(p_0 \wedge p_{k+1})\} \subset \{\gamma \in A : (p_0 \wedge p_k) \prec \gamma\} \vdash p_0$, resulting in a contradiction. Thus $cut_{\preceq}(p_0) = \emptyset$.

Next, using this result with Definition 6, we have that $\alpha \leq_{\preceq} p_0$ for all $\alpha \notin Cn(A)$, and $\alpha \leq_{\preceq} p_0$ for all $\alpha \in Cn(A)$ since $cut_{\preceq}(p_0) = \emptyset \subseteq cut_{\preceq}(\alpha)$. Thus, $\alpha \leq_{\preceq} p_0$ for all $\alpha \in \mathcal{L}$.

Interestingly, this means that Lemma 3 is not true, since the entrenchment postulate **EE5** would imply that $\vdash p_0$ which leads to a contradiction. □

This flaw has not been pointed out before, and Lemma 3 is used throughout literature to link **EE** and ensconcement (Peppas 2008; Fermé, Garapa, and Reis 2017). Luckily, there are different ways in which to solve this problem, and we shall present two different approaches. The first one[1] is to change Definition 6 and make $\alpha \leq_{\preceq} \beta$ iff one of the following is satisfied:

(i) $\vdash \beta$;

[1] Which was proposed by Garapa in a personal communication.

(ii) $\alpha \notin Cn(A)$;
(iii) $\alpha, \beta \in Cn(A) \setminus Cn(\emptyset)$ and $cut_{\preceq}(\beta) \subseteq cut_{\preceq}(\alpha)$.

Another possible approach is to add the condition that every belief base A should satisfy $\top \in A$, where $\top$ is a tautology. This would mean that every non-tautological sentence would have $\top$ in their $cut_{\preceq}$, so it would never be empty, and the only ones with empty $cut_{\preceq}$ would be the tautologies. This last approach will become more relevant when the shielded version of ensconcement is presented.

We shall continue this section with the properties that relate the **EE** and ensconcement which are well known in the literature. However, we will restate them including the condition $\top \in A$. It is relatively easy to check that each of the original proofs in (Williams 1994) can be extended to take into consideration this new requirement.

Definition 7. *Let $K \subseteq \mathcal{L}$ be a belief set, $\leq$ an* **EE** *defined over K and $A \subseteq K$ a belief base. We define $\leq_{|A}$ to be the restriction of $\leq$ over A, that is:*

$$\textit{For every } \alpha, \beta \in A,\ \ \alpha \leq_{|A} \beta \iff \alpha \leq \beta$$

The next result confirms that it is always possible to obtain an ensconcement from an epistemic entrenchment under certain natural conditions.

Lemma 4. *Let K be a belief set and $\leq$ an* **EE** *ordering relative to K. Let $A \subseteq K$ a belief base such that $\top \in A$, and let $\leq_{|A}$ be the restriction of $\leq$ over A. Then $\leq_{|A}$ is an ensconcement ordering on A.*

Moreover, under stricter conditions, it is possible to recover the original **EE** ordering using Definition 6 as it is shown in the following theorem.

Theorem 3. *Let K be a belief set, and $A \subseteq K$ a belief base such that $\top \in A$. Let $\leq$ be an* **EE** *related to K. If A is such that for all $\alpha \in K$, $\{\beta \in A : \alpha \leq \beta\} \vdash \alpha$, then $\leq_{|A}$ is an ensconcement ordering on A such that $\leq_{(\leq_{|A})} = \leq$*

The last theorem implies that an **EE** ordering over a belief set K can be used to define both an **EE** contraction operator (according to Definition 2) and an ensconcement one.

The following result shows the connection between an ensconcement-based contraction (using Definition 5) and its corresponding **EE**-based contraction (according to Lemma 3 with $\top \in A$ and Definition 2).

Theorem 4. *Let A be a belief base such that $\top \in A$ and let $\preceq$ be an ensconcement over A. Let $-_{\preceq}$ be the belief base contraction operator as in* Definition 5. *Let $-_{\leq_{\preceq}}$ be the contraction operator for $Cn(A)$ uniquely determined by* Definitions 2 *and* 6. *Then $A -_{\preceq} \alpha = (Cn(A) -_{\leq_{\preceq}} \alpha) \cap A$*

This theorem sheds light on several properties that link **EE** with ensconcement. First, we can see that for every **EE** contraction operator, there is an equivalent ensconcement contraction operator. To see that, let $\leq$ be an **EE** over K. Then we can trivially define an ensconcement as the restriction of $\leq$ over the whole set K ($\leq_{|K}$), which would be as thinking the **EE** ordering as an ensconcement. Then $\leq$ clearly satisfies conditions of Theorem 3, so Theorem 4 implies:

$$K -_{\leq_{|K}} \alpha = K -_{\leq} \alpha$$

Furthermore, Theorems 3 and 4 make explicit the link between **EE** and ensconcement. The relationship can be established in both ways: on one hand, given an ensconcement over a belief base A (satisfying $\top \in A$) one can build an **EE** over $Cn(A)$. Then perform the contraction over $Cn(A)$ with this last operator, and finally intersect the result with A to obtain the same outcome as if contracting A with the operator determined by our original ensconcement, as Theorem 4 shows. In addition, by also taking into consideration Theorem 3, we can deduce the following result.

Theorem 5. *Let K be a belief set and $\leq$ an* **EE** *ordering relative to K. Let $A \subseteq K$ be a belief base satisfying conditions of Theorem 3. Then, defining the ensconcement ($\leq_{|A}$) over A, we have:*

$$Cn(A -_{\leq_{|A}} \alpha) = K -_{\leq} \alpha$$

This theorem shows the other way **EE** and ensconcement can relate: given an **EE** $\leq$ relative to a belief set K, we can focus over a particular subset A and the restriction over A of $\leq$. Then performing the ensconcement contraction over A by a formula α and then closing the result under its logical consequences, yields the same result as performing the **EE** contraction over the whole belief set.

All the results in this section show how ensconcement is a natural adaptation of epistemic entrenchment over bases. In fact, this link is also preserved between their respective order-based contractions, which satisfy the success postulate. In the next sections, we are going to show how this link could be preserved in a non-prioritized context by considering some non-trivial adaptations.

3 Shielded Base Contraction

The success postulate for contraction requires that all non-tautological beliefs should be retractable. This is not a fully realistic requirement, since agents are known to have beliefs of a non-logical nature, which should not be given up under any circumstance. To meet this requirement, the success postulate is not generally taken into account by shielded contraction. In this way, some non-tautological beliefs can be shielded from contraction, and thus cannot be given up.

In this section, we will introduce the formalities behind Shielded Base Contraction. Bear in mind that Shielded Contraction for theories is just a special case where the belief base is closed under logical consequence.

In (Fermé and Hansson 2001), a shielded contraction is defined by means of an AGM contraction and a set of sentences R satisfying certain properties, called the set of retractable sentences, which models the set of sentences that the agent is willing to give up (if needed).

Definition 8. *Let $-$ be a contraction operator on a belief base A (i.e. an operator that satisfies at least success and inclusion). Let R be a set of sentences (the associated set of retractable sentences). Then $\sim$ is the shielded base contraction induced by $-$ and R if and only if:*

$$A \sim \alpha = \begin{cases} A - \alpha & \textit{if } \alpha \in R \\ A & otherwise \end{cases}$$

Besides the already specified postulates for belief base contraction operators, the following are some postulates that $\sim$ might satisfy:

(Relative Success) $A \sim \alpha = A$ or $\alpha \notin Cn(A \sim \alpha)$

(Success Propagation) If $A \sim \beta \vdash \beta$ and $\vdash \beta \rightarrow \alpha$ then $A \sim \alpha \vdash \alpha$

(Conjunctive Constancy) If $A \sim \alpha = A \sim \beta = A$, then $A \sim \alpha \wedge \beta = A$

(Persistence) If $\beta \in Cn(A \sim \beta)$, then $\beta \in Cn(A \sim \alpha)$

In (Fermé, Mikalef, and Taboada 2003) the properties of the set of retractable sentences that were considered as properties that may be desirable from a set R of retractable sentences were the following:

(Non-retractability Propagation) If $\alpha \notin R$, then $Cn(\alpha) \cap R = \emptyset$.

(Conjunctive Completeness) If $\alpha \wedge \beta \in R$, then $\alpha \in R$ or $\beta \in R$.

(Non-retractability Preservation) $\mathcal{L} \setminus R \subseteq Cn(A \sim \alpha)$.

(Non-retractability of Tautology) $R \cap Cn(\emptyset) = \emptyset$.

It is also possible to define relations between some usual base contraction operators and shielded base contraction operators.

Observation 1. *(Fermé, Mikalef, and Taboada 2003; Fermé, Garapa, and Reis 2017) Let A be a belief base and $\sim$ an operator on A. Then the following conditions are equivalent:*

(a) $\sim$ *satisfies relative success, persistence, inclusion, relevance and uniformity.*

(b) $\sim$ *is an operator of shielded contraction induced by a contraction operator satisfying the basic AGM postulates, and an associated retractable set $R \subseteq \mathcal{L}$ that satisfies non-retractability propagation and non-retractability preservation.*

This observation presents an interesting fact: some shielded base contractions can be directly represented by a base contraction.

3.1 Shielded Epistemic-Entrenchment

In (Fermé and Hansson 2001, Section 3), the authors show how to build shielded contractions based on an **EE** ordering. Essentially, they point out that **EE5** establishes that only tautologies can be maximally entrenched. Therefore, they consider a new entrenchment-based contraction by just withdrawing this property in the following way:

Definition 9. *Let K be a belief set. A relation $\leq_K$ defined over K satisfying **EE1-EE4** is called a Shielded Epistemic-Entrenchment (Shielded* **EE***). Then $\sim_{\leq}$ is the entrenchment-based shielded contraction based on $\leq_K$ if and only if:*

$$K \sim_{\leq} \alpha = \begin{cases} \{\beta \in K : \alpha <_K \alpha \vee \beta\} & \textit{if } \alpha <_K \top \\ K & otherwise \end{cases}$$

Bear in mind that Definition 9 is not presented in the way as the standard shielded contraction in Definition 8. However, the following characterization is given in (Garapa 2017):

Theorem 6. *Let K be a consistent belief set. Let $-$ be a contraction operator defined over K, then the following are equivalent:*

- *$-$ is an entrenchment-based shielded contraction operator.*
- *$-$ satisfies closure, inclusion, vacuity, extensionality, recovery, relative success, persistence and conjunctive factoring.*
- *$-$ is an operator of shielded contraction induced by a contraction operator for K that satisfies both the basic and complementary AGM postulates, and a set $R \subseteq \mathcal{L}$ that satisfies non-retractability propagation, conjunctive completeness and non-retractability preservation.*

4 Weak Ensconcement

To generalize the shielded **EE**, as (Williams 1994) did with the *classic* one, we defined the *weak* version of ensconcement. The intuition behind the construction is the following: analogously to the shielded **EE**, the *irretractable* beliefs should be the most *ensconced*, a place that ($\preceq$ 2) reserves only for tautologies. Thus, it seems natural to remove ($\preceq$ 2) to weaken the ensconcement. Moreover, if present, such formulas at the top are the only ones with empty cut. So the contraction by any formula with empty cut should yield no change in the belief base. This intuition motivated the following definitions.

Definition 10. *Define a w-ensconcement to be a set of formulas A together with a total preorder $\preceq$ of A satisfying the following conditions:*

($\preceq_{\mathbf{w}}$ 1) *If $\beta \in A$ and $\nvdash \beta$, then $\{\alpha \in A \mid \beta \prec \alpha\} \nvdash \beta$.*

($\preceq_{\mathbf{w}}$ 2) *If $\beta \in A$ and $\vdash \beta$, then for every $\alpha \in A : \alpha \preceq \beta$*

Intuitively, ($\preceq_w$ 1) says that the formulas which are strictly more ensconced than an arbitrary formula β do not entail β. If there are any tautologies in A, then ($\preceq_w$ 2) says they are the most ensconced formulas but they are not necessarily unique. It is worth noticing that ($\preceq_w$ 1) is the same as ($\preceq$ 1) in the Definition 3 of (prioritized) ensconcement, and that ($\preceq_w$ 2) is weaker than ($\preceq$ 2) and ($\preceq$ 3) together because the former implies that tautologies are on top of the order and equally ensconced, but that they are not necessarily the only ones.

Now, we are in a position to introduce the contraction operator based on this new order.

Definition 11. *Let A be a belief base and $\preceq$ be a weak ensconcement (*Definition 10*) over A. Then $-_{\preceq}$ is the weak-ensconcement contraction based on $\preceq$ if and only if:*

$$A -_{\preceq} \alpha = \begin{cases} \{\beta \in A : & \textit{if } cut_{\prec}(\alpha) \neq \emptyset \\ cut_{\prec}(\alpha) \vdash \alpha \vee \beta\} & \\ A & otherwise \end{cases}$$

Now we will resume the discussion about the condition which requires $\top \in A$ for every belief base. In Subsection 2.3 we showed that there were two possible ways to solve the flaw in Lemma 3. We opted for the solution with requires $\top \in A$ and the reason why will become clearer with the following example.

As a weak ensconcement is, precisely, a weakened version of ensconcement, then, for every contraction operator defined over an ensconcement there should be an equivalent contraction operator defined over a weak-ensconcement. However, let us consider the following example (assuming that the condition $\top \in A$ is not required):

Example 2. *Let $A = \{\alpha\}$ with $\nvdash \alpha$. There is just one ensconcement possible over A, where $\alpha \preceq \alpha$. The same goes for a weak-ensconcement, where $\alpha \preceq_w \alpha$. We clearly have $cut_{\prec}(\alpha) = cut_{\prec_w}(\alpha) = \emptyset$. Thus, $A -_{\preceq} \alpha = \emptyset$ since $\emptyset \nvdash \alpha$, but $A -_{\preceq_w} \alpha = A = \{\alpha\}$. So there is no weak-ensconcement possible that could yield $A -_{\preceq} \alpha = A -_{\preceq_w} \alpha$*

This problem arises from the fact that every maximal sentence (if there is one) will always have an empty cut, so, by our construction, it would be irretractable. More precisely, the problem is that there is no way for distinguishing irretractable sentences from just maximal ones. An elegant (and computationally cheap) solution is to always have a maximal irretractable element in the belief base, $\top$.

This is not much to ask, as intuitively this can be interpreted as having an *intrinsic truth* in our beliefs to which compare other beliefs. This way, if they are equally *ensconced*, they must be irretractable too. In previous example, considering $A = \{\alpha, \top\}$, we have two possible ensconcements: $\{\alpha \prec \top\}$ and $\{\alpha \equiv \top\}$. The second one corresponds to shielding α from contraction.

It is worth noticing that adding $\top$ to the belief base A will not change the belief set it represents, since $\top \in Cn(A)$ always holds. Finally, this new definition of belief base would solve the problem in Lemma 3 (from Section 2). So from now on we have:

Definition 12. *Define a belief base to be a set of formulas $A \subseteq \mathcal{L} \cup \{\top\}$ such that $\top \in A$, where $\vdash \top$*

Also, it is important to notice that the converse does not hold in the general case. That is, not every weak-ensconcement-based contraction can be thought of as an ensconcement-based contraction (even with the new Definition 12), as we shall see in the following example:

Example 3. *Let $A = \{\alpha, \top\}$ with $\nvdash \alpha$. Let $\preceq_w$ be a w-ensconcement defined as $\alpha \equiv \top$. Let $-_{\preceq_w}$ be the w-ensconcement contraction operator defined over $\preceq_w$. Let $\preceq$ be an ensconcement (prioritized). In fact, there is only one possible ensconcement over A, where $\alpha \prec \top$, since any other ordering would violate ($\preceq 2$).*

On one hand we have $cut_{\prec_w}(\alpha) = \emptyset$, so $A -_{\preceq_w} \alpha = A$. On the other hand, $cut_{\prec}(\alpha) = \{\top\}$, so $A -_{\preceq} \alpha = \{\top\}$ since $\{\top\} \nvdash \alpha$. Thus $A -_{\preceq} \alpha \neq A -_{\preceq_w} \alpha$. Since $\preceq$ is the only ensconcement possible over A, this means there is no equivalent contraction operator defined over an ensconcement, to the one defined over $\preceq_w$.

Using the new definition of Belief Base in Definition 12, the way of defining a w-ensconcement contraction operator equivalent to an ensconcement contraction operator is straightforward:

Lemma 5. *Let $\preceq$ be an ensconcement operator and $-_{\preceq}$ the corresponding ensconcement based contraction operator. Since $\preceq$ also satisfies ($\preceq_w 1$) and ($\preceq_w 2$), it is also a w-ensconcement. Let $\sim_{\preceq}$ be the w-ensconcement operator defined by $\preceq$ using Definition 11, then $-_{\preceq} = \sim_{\preceq}$*

4.1 Representation Theorem

In order to prove the Representation Theorem, the following properties are useful to us. The first set of properties is already known in the literature. Although they have been presented for the prioritized ensconcement, they can easily be shown to hold for the weak ensconcement case (even when Definition 12 is used).

Lemma 6. *(Fermé, Krevneris, and Reis 2008, Lemma 11) Let $A \subseteq \mathcal{L}$ a belief set and $\preceq$ an ensconcement over A. Let $\alpha, \beta \in \mathcal{L}$, then the following properties hold:*

1. *If $\vdash \alpha$, then $cut_{\prec}(\alpha) = \emptyset$.*
2. *If $\nvdash \alpha$, then $cut_{\prec}(\alpha) \nvdash \alpha$.*
3. *If $A \nvdash \alpha$, then $cut_{\prec}(\alpha) = A$.*
4. *If $\beta \vdash \alpha$, then $cut_{\prec}(\alpha) \subseteq cut_{\prec}(\beta)$.*
5. *If $\vdash \alpha \leftrightarrow \beta$, then $cut_{\prec}(\alpha) = cut_{\prec}(\beta)$.*
6. *$\forall \alpha, \beta \in A$, if $\alpha \preceq \beta$, then $cut_{\prec}(\beta) \subseteq cut_{\prec}(\alpha)$.*
7. *$\forall \alpha, \beta \in A$, if $\alpha \prec \beta$, then $cut_{\prec}(\alpha) \vdash \beta$ and $cut_{\prec}(\beta) \nvdash \alpha$.*
8. *$\forall \alpha, \beta \in A$, if $\alpha \prec \beta$, then $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$.*
9. *$\forall \alpha, \beta \in A$, if $\alpha \preceq \beta$ and $\beta \preceq \alpha$, then $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha) = cut_{\prec}(\beta)$.*
10. *If $cut_{\prec}(\alpha) \vdash \beta$, then $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha)$.*
11. *If $cut_{\prec}(\alpha) \nvdash \beta$, then $cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\beta)$.*

The next new sets of properties expand the previous ones and are necessary for proving the representation theorem that will be presented later.

Lemma 7. *Let $A \subseteq \mathcal{L}$ and $\preceq$ be a w-ensconcement defined over A, the following holds:*

1. *For all $\alpha, \beta \in \mathcal{L}$, then either $cut_{\prec}(\alpha) \subseteq cut_{\prec}(\beta)$ or $cut_{\prec}(\beta) \subseteq cut_{\prec}(\alpha)$.*
2. *If $cut_{\prec}(\alpha) = \emptyset$ then $A \vdash \alpha$*
3. *Let $\beta \in cut_{\prec}(\alpha)$. Then $\{\gamma \in A : \beta \preceq \gamma\} \subseteq cut_{\prec}(\alpha)$*
4. *Let $cut_{\prec}(\alpha) \subset cut_{\prec}(\beta)$. Then $cut_{\prec}(\beta) \vdash \alpha$*
5. *If $cut_{\prec}(\alpha) = \emptyset$ and $cut_{\prec}(\beta) \neq \emptyset$ then $cut_{\prec}(\beta) \vdash \alpha$ for all α, β.*
   Collolaries:
   - *If $cut_{\prec}(\alpha) = \emptyset$ and $cut_{\prec}(\beta) \nvdash \alpha$ then $cut_{\prec}(\beta) = \emptyset$*
   - *If $cut_{\prec}(\beta) \neq \emptyset$ and $cut_{\prec}(\beta) \nvdash \alpha$ then $cut_{\prec}(\alpha) \neq \emptyset$*
6. *$cut_{\prec}(\alpha \wedge \beta) = cut_{\prec}(\alpha) \cup cut_{\prec}(\beta)$ for all α, β*
7. *If $\gamma \in A$, then $cut_{\prec}(\gamma) = \emptyset \iff \beta \preceq \gamma$ for all $\beta \in A$*
8. *$cut_{\prec}(\alpha) = \emptyset \iff \{\gamma \in A : cut_{\prec}(\gamma) = \emptyset\} \vdash \alpha$*
9. *If $\nvdash \beta$ and $cut_{\prec}(\alpha) \vdash \beta$, then $cut_{\prec}(\beta) \subset cut_{\prec}(\alpha)$*

Lastly, the following set of properties is related to the w-ensconcement order and its contraction operator.

Lemma 8. *Let $A \subseteq \mathcal{L}$, $\preceq$ a w-ensconcement ordering and $-$ a w-ensconcement based contraction based on $\preceq$. Then:*

1. *If $cut_{\prec}(\alpha) \neq \emptyset$, then $A - \alpha \nvdash \alpha$*
2. *$A \vdash \alpha \wedge A - \alpha = A \iff cut_{\prec}(\alpha) = \emptyset$*
3. *$cut_{\prec}(\alpha) \subseteq A - \alpha$ for all α*

4. *If $A - \alpha \nvdash \beta$, then $cut_{\prec}(\alpha) \nvdash \beta$.*
5. *If $A - \alpha \vdash \beta$ and $cut_{\prec}(\alpha) \neq \emptyset$ then $cut_{\prec}(\alpha) \vdash \alpha \vee \beta$.*
6. *If $cut_{\prec}(\alpha) = \emptyset$ then $A - \alpha \wedge \beta = A - \beta$ for all α, β*
7. *If $cut_{\prec}(\alpha) \vdash \beta$, then $A - \alpha \wedge \beta \vdash \beta$*
 Collorary: *If $\beta \in cut_{\prec}(\alpha)$ then $\beta \in A - \alpha \wedge \beta$*

With these properties, we shall see the axiomatic characterization of the w-ensconcement contraction operator in the following theorem:

Theorem 7. *Let A be a belief base (as in Definition 12). An operator $-$ is a w-ensconcement-based contraction on A if and only if it satisfies* relative success, inclusion, vacuity, extensionality, disjunctive elimination, conjunctive factoring, transitivity, EB1, EB2, persistence and success propagation.

The difference with the original ensconcement representation theorem is that the *success* postulate is weakened for *relative success* and that both persistence and success propagation are added. It is worth noticing that when success and failure hold, persistence and success propagation are redundant. Thus, the set of postulates that characterize the w-ensconcement implies those of the ensconcement. This means that every ensconcement operator is, indeed, a weak ensconcement operator.

Once again, due to space restrictions, we provide the proof in a separate appendix. However, it is worth noticing that in order to prove the only if, the construction of the w-ensconcement using $C_{\leq}$ does not work. In the proof, the following construction is used:

$$\alpha \leq \beta \iff \alpha \notin A \div (\alpha \wedge \beta) \text{ or } A - \beta \vdash \beta \qquad (\mathbf{WC}_{\leq})$$

Instead of requiring $\vdash \beta$ as in $C_{\leq}$, the condition $A - \beta \vdash \beta$ places all irretractable sentences on top of the ordering. In particular, as Failure holds, this condition places all tautologies on top.

4.2 Relationship Between Shielded Epistemic Entrenchment and w-Ensconcement

This section is devoted to relating the Shielded **EE** with the w-ensconcement. Intuitively, one would hope that applying the same definitions from Section 2.3, analogous to the same theorems that relate **EE** with ensconcement would hold, but for their shielded counterparts.

The following definition is the adaptation of Definition 6 given by Williams and presented in Section 2.3.

Definition 13. *Let $K \subseteq \mathcal{L}$ be a belief set, $\leq$ a shielded **EE** relative to K and $A \subseteq K$ a belief base. We define $\leq_{|A}$ to be the restriction of $\leq$ over A; that is for every $\alpha, \beta \in A$, $\alpha \leq \beta \iff \alpha \leq_{|A} \beta$*

Continuing with the adaptation, the following lemma is a reformulation of Lemma 4 for our proposal.

Lemma 9. *Let K be a belief set and $\leq$ a shielded **EE** ordering over K. Let $A \subseteq K$ a belief base such that $\top \in A$, and let $\leq_{|A}$ be the restriction of $\leq$ over A.*

Then $\leq_{|A}$ is a w-ensconcement ordering on A which satisfies ($\preceq_w$ 1), ($\preceq_w$ 2).

We would like for a property such as Theorem 3 to hold for shielded **EE** and weak ensconcement. However, we shall see that this will not work if we use the same construction as in Definition 6 for building an **EE** $\leq_{(\preceq_w)}$ from a w-ensconcement $\preceq_w$. To avoid confusion, bear in mind that Definition 6 uses $cut_{\preceq}$, as opposed to the contraction operator which is defined using $cut_{\prec}$.

Example 4. *Let $K = Cn(\alpha)$ with $\nvdash \alpha$ be a belief set, and let $\leq$ be a shielded* **EE** *defined as: $\beta \equiv \gamma$ for all $\beta, \gamma \in K$, $\beta \equiv \gamma$ for all $\beta, \gamma \notin K$, and $\beta < \gamma$ if $\beta \notin K$ and $\gamma \in K$. Let $A \subseteq K$ a belief base defined as $A = \{\alpha, \top\}$. By Definition 13 we have $\leq_{|A}$ a weak ensconcement such that $\alpha \leq_{|A} \top$ and $\top \leq_{|A} \alpha$. Intuitively, with this w-ensconcement, we are establishing α as an irretractable sentence. Notice that $\{\beta \in A : \lambda \leq \beta\} \vdash \lambda$ for all $\lambda \in K$, because $\lambda \leq \alpha$ for all $\lambda \in K$, and $K = Cn(\alpha)$.*

We then have $cut_{\leq_{|A}}(\top) = \emptyset$, and $cut_{\leq_{|A}}(\alpha) = \{\alpha, \top\}$. If we use Definition 6 to obtain a shielded **EE** *$\leq_{(\leq_{|A})}$, we will get $\alpha <_{(\leq_{|A})} \top$, since $cut_{\leq_{|A}}(\top) \subset cut_{\leq_{|A}}(\alpha)$. This means that shielded version for Theorem 3 does not hold using Definition 6, because we have $\leq \neq \leq_{(\leq_{|A})}$*

The problem we have here is that if a belief base has a maximal equivalence class, then $cut_{\preceq}(\alpha)$ is empty if and only if $\vdash \alpha$. Thus, even if there are irretractable sentences β such that $\nvdash \beta$, we will have $cut_{\preceq}(\beta) \neq \emptyset$, so when applying Definition 6 we will have $\beta <_{\preceq} \alpha$ if $\vdash \alpha$. This can be thought of as if $cut_{\preceq}$ distinguishes between maximal formulas which are tautologies and those that are not.

This problem can simply be solved by changing $cut_{\preceq}$ by $cut_{\prec}$, yielding the following definition:

Definition 14. *Let $(A, \preceq)$ be a w-ensconcement. For $\alpha, \beta \in \mathcal{L}$, we define $\leq_{\preceq}$ to be given by $\alpha \leq_{\preceq} \beta$ if and only if either*

(i) *$\alpha \notin Cn(A)$, or*

(ii) *$\alpha, \beta \in Cn(A)$ and $cut_{\prec}(\beta) \subseteq cut_{\prec}(\alpha)$.*

Luckily, with this new definition, the analogous theorems to those for the prioritized operators hold:

Lemma 10. *$\leq_{\preceq}$ is a shielded* **EE** *relative to $Cn(A)$.*

In fact, it is easy to prove that Definition 14 produces an **EE** (which satisfies **EE1**-**EE5**) if we use an ensconcement (instead of a w-ensconcement).

Theorem 8. *Let K be a belief set in $\mathcal{L}$, and $A \subseteq K$ a belief base. Let $\leq$ be a shielded* **EE** *related to K. If A is such that for all $\alpha \in K$, $\{\beta \in A : \alpha \leq \beta\} \vdash \alpha$, then $\leq_{|A}$ is a weak ensconcement ordering on A such that $\leq_{(\leq_{|A})} = \leq$*

Theorem 9. *Let A be a belief base and let $\preceq$ be a w-ensconcement over A. Let $-$ be the w-ensconcement contraction operator as in Definition 10. Let $\div$ be the entrenchment-based shielded contraction operator for $Cn(A)$ uniquely determined by $\leq_{\preceq}$ according to Definition 9. Then $A - \alpha = (Cn(A) \div \alpha) \cap A$*

Equivalently to its prioritized counterpart (Theorem 4), this theorem shows how a w-ensconcement contraction operator can be represented as a shielded epistemic entrenchment contraction over the logical consequences of the belief base, and then intersecting it with the original belief base.

Furthermore, it can also be used to show, in analogous way, that every shielded epistemic entrenchment can be thought as a w-ensconcement, yielding the same contraction operators. Finally, the following Theorem shows the other direction in the link between both orderings.

Theorem 10. *Let K be a belief set and $\leq$ a shielded* **EE** *ordering relative to K. Let $A \subseteq K$ a belief base satisfying conditions of Theorem 8. Then, defining the w-ensconcement ($\leq_{|A}$) given by Definition 13, we have:*

$$Cn(A -_{\leq_{|A}} \alpha) = K -_{\leq} \alpha$$

As Theorem 5 shows for their prioritized counterparts, here we can see how the behavior of a shielded **EE** contraction over a belief set K can be replicated by performing a w-ensconcement contraction over a particular subset A and then closing the result under its logical consequences. This result is very useful when looking for a computational approach for shielded **EE** contractions, since we can restrict ourselves to a possibly finite subset of our belief set and obtain the same results.

For finishing this section, we would like to summarize that for establishing the connection between shielded **EE** and w-ensconcement we should use both a new notion of *belief base* given in Definition 12 and an alternative way to obtain a shielded **EE** from a w-ensconcement through Definition 14 instead of Definition 6 (i.e. by using $cut_{\prec}$ instead of $cut_{\preceq}$).

4.3 Standard Shielded Form

The construction that has been presented for w-ensconcement is not in the Standard Shielded Form (Definition 8). However, in this section, we shall see that w-ensconcement characterizes a proper subclass of shielded contractions. First, we will introduce two new properties for shielded operators:

(Weak Non-Retractability Preservation):
$Cn(A) \cap (\mathcal{L} \setminus R) \subseteq Cn(A \sim \alpha)$

(Non-Retractability Inclusion):
$Cn(A) \cap (\mathcal{L} \setminus R) \subseteq Cn(A \cap (\mathcal{L} \setminus R))$

Weak Non-Retractability Preservation means that all irretractable formulas that are consequences of the belief base A are also consequences of contracting A by any formula (i.e.: irretractable sentences are *preserved* during contraction). Weak Non-Retractability Preservation is indeed a weaker version of the known postulate of Non-Retractability Preservation if $\sim$ satisfies Relative Closure and Inclusion.

Lemma 11. *Let A be a set of formulas, and $\sim$ a contraction operator defined over A. If $\sim$ satisfies Relative Closure, Inclusion and Non-Retractability Preservation, then $\sim$ satisfies Weak Non-Retractability Preservation.*

This property is introduced to take advantage of the following fact: Let γ be a formula and let $\sim$ be an operator satisfying Vacuity and Inclusion. If $\gamma \notin R$ and $A \nvdash \gamma$, then $A \sim \gamma = A$. This means that sentences not in A behave as irretractable when A do not imply them, and if inclusion holds, will continue behaving like that after a contraction.

However, the already known postulate of Non-Retractability Preservation implies that $A \vdash \beta$ for all $\beta \notin R$, which we think is too strong. Weak Non-Retractability Preservation requires the Non-Retractability Preservation only for irretractable sentences *already* in A. This allows for the agent to behave the following way: it may not have a particular belief, but if that belief becomes part of its belief base, then it will be as entrenched as its most entrenched belief, thus it will become irretractable.

Non-Retractability Inclusion simply says that the irretractable formulas that are consequences of A, are in particular consequences of the irretractable formulas in A

With these postulates, we can formulate the following representation theorems for w-ensconcement in its shielded standard form. First, we will construct a shielded operator given a w-ensconcement:

Theorem 11. *Let $\preceq_w$ be a weak ensconcement and $-$ the weak-ensconcement contraction operator related to $\preceq_w$. There is a set $R \subseteq \mathcal{L}$ and a (prioritized) ensconcement $\preceq$ with its related contraction operator $\div$ such that defining:*

$$A \sim \alpha = \begin{cases} A \div \alpha & \textit{if } \alpha \in R \\ A & otherwise \end{cases}$$

$\sim$ satisfies Non-Retractability Propagation, Weak Non-Retractability Preservation *and* Non-Retractability Inclusion*, and $A - \alpha = A \sim \alpha$ for all $\alpha \in \mathcal{L}$*

The idea behind the proof is to define $R = \{\alpha \in \mathcal{L} : cut_{\prec}(\alpha) \neq \emptyset\}$. Then, let the prioritized ensconcement ordering be such that for all $\alpha, \beta \in A$:

(i) If $cut_{\prec_w}(\beta) \neq \emptyset$ then $\alpha \preceq \beta \iff \alpha \preceq_w \beta$

(ii) If $\vdash \beta$ then $\alpha \preceq \beta$

(iii) If $cut_{\prec_w}(\beta) = \emptyset$ and $\nvdash \beta$ then $\alpha \preceq \beta$ for all $\alpha \in A \setminus Cn(\emptyset)$

This ordering basically places tautologies strictly on top, then non-tautological formulas with empty cut and, strictly below, the rest of the formulas in A maintain their order.

Next, we will see that it is possible to build a w-ensconcement given a Shielded Operator.

Theorem 12. *Let A a set of formulas, $\preceq$ an ensconcement ordering over A and $\div$ the contraction operator defined by $\preceq$ over A. Let $R \subseteq \mathcal{L}$. Let $\sim$ be a contraction operator over A defined as follow:*

$$A \sim \alpha = \begin{cases} A \div \alpha & \textit{if } \alpha \in R \\ A & otherwise \end{cases}$$

If $\sim$ satisfies Non-Retractability Propagation, Weak Non-Retractability Preservation *and* Non-Retractability Inclusion*, then there is a weak ensconcement $\preceq_w$ over A such that its contraction operator $-$ verifies: $A - \alpha = A \sim \alpha$ for all $\alpha \in \mathcal{L}$*

The idea behind this proof is to build the w-ensconcement ordering as follows: for all $\alpha, \beta \in A$, $\alpha \preceq_w \beta$ if and only if either:

(i) α and $\beta \in R$ and $\alpha \preceq \beta$

(ii) $\beta \notin R$

This yields an ordering where formulas not in R are strictly on top, and the rest remain the same. In particular, since $\sim$ satisfies Non-retractability Propagation, tautologies are placed on top. The postulates that $\sim$ satisfies guarantee that placing the irretractable sentences on top does not affect conditions ($\preceq_w 1$) and ($\preceq_w 2$).

5 Conclusion and Future Works

We introduced the concept of weak-ensconcement, an order that merges the ideas of ensconcement (Williams 1994) and non-prioritized shielded epistemic entrenchment operator (Fermé and Hansson 2001), and we also defined its associated base contraction operator. In doing so, we had to provide a new definition of *belief base* for solving a flaw we discovered in Theorem 4 of (Williams 1994). Then, we provided the axiomatic characterization of our proposal, after introducing new ensconcement properties relevant to the proof. Furthermore, we described its relationship to a shielded epistemic entrenchment in the same way ensconcements and epistemic entrenchment are related. Then, we showed that the original definition for obtaining an **EE** from an ensconcement given by Williams in that same paper does not work for establishing the relationship between weak-ensconcement and shielded epistemic entrenchment, so we provided an alternative definition. Finally, we showed that the w-ensconcement contraction operators correspond to a subfamily of the Shielded base contractions family by providing its standard shielded form.

We consider that our characterization of a new non-prioritized contraction operator in terms of an order among formulas allow us to introduce a new perspective for future computer-based implementations.

We finish this section by mentioning some ideas for working in the near future.

- An important challenge faced when implementing a contraction operator based on a w-ensconcement is how to construct the order between formulas. We have begun experimenting with AI language models to extract formulas that represent the beliefs of an agent from text in natural language (descriptions, dialogues, etc.). Additionally, such AI language models are also capable of providing and explaining a possible epistemic order among the agent's beliefs. However, these orders do not necessarily satisfy the postulates of the w-ensconcement. Consequently, a promising line of work would be to integrate these tools with logical processing software in order to automate the creation of correct w-ensconcement orders.
- In (Rott 2003), the author introduces the concept of basic entrenchment which is weaker than the original notion of epistemic entrenchment given by Gärdenfors and Makinson in (Gärdenfors and Makinson 1988). In this way, Rott is able to construct partial meet base and safe contractions using this new concept. We want to apply Rott's approach considering more flexible and adaptable notions of w-ensconsement relations that exactly fit the different classes of shielded base contractions.
- In (Garapa, Fermé, and Reis 2017), the authors present an axiomatic characterization for brutal contractions based on a general ensconcement relation. We want to extend this result for brutal shielded base contraction.

Ethical Statement

There are no ethical issues.

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References

Alchourrón, C.; Gärdenfors, P.; and Makinson, D. 1985. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic* 50:510–530.

Fermé, E. L., and Hansson, S. O. 2001. Shielded contraction. In Williams, M.-A., and Rott, H., eds., *Frontiers in Belief Revision*. Dordrecht: Springer Netherlands. 85–107.

Fermé, E.; Garapa, M.; and Reis, M. D. L. 2017. On ensconcement and contraction. *Journal of Logic and Computation* 27(7):2011–2042.

Fermé, E.; Krevneris, M.; and Reis, M. 2008. An axiomatic characterization of ensconcement-based contraction. *Journal of Logic and Computation* 18(5):739–753.

Fermé, E.; Mikalef, J.; and Taboada, J. 2003. Credibility-limited Functions for Belief Bases. *Journal of Logic and Computation* 13(1):99–110.

Garapa, M.; Fermé, E.; and Reis, M. D. L. 2017. Studies on brutal contraction and severe withdrawal. *Studia Logica* 105(2):331–360.

Garapa, M.; Fermé, E.; and Reis, M. D. 2018. Shielded base contraction. *Artificial Intelligence* 259:186–216.

Garapa, M. P. F. 2017. *Advances on Belief Base Dynamics*. Ph.D. Dissertation, Universidade Da Madeira, Funchal, Portugal.

Gärdenfors, P., and Makinson, D. 1988. Revisions of knowledge systems using epistemic entrenchment. In *Proceedings of the 2nd Conference on Theoretical Aspects of Reasoning about Knowledge*, TARK '88, 83–95. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc.

Gärdenfors, P., and Makinson, D. 1988. Revisions of knowledge systems using epistemic entrenchment. In Vardi, M., ed., *Proceedings of the Second Conference on Theoretical Aspects of Reasoning About Knowledge*, 83–95. Morgan Kaufmann.

Hansson, S. O.; Fermé, E.; Cantwell, J.; and Falappa, M. 2001. Credibility limited revision. *The Journal of Symbolic Logic* 66(4):1581–1596.

Hansson, S. O. 1992. In defense of base contraction. *Synthese* 91(3):239–245.

Peppas, P. 2008. Chapter 8 belief revision. In van Harmelen, F.; Lifschitz, V.; and Porter, B., eds., *Handbook of Knowledge Representation*, volume 3 of *Foundations of Artificial Intelligence*. Elsevier. 317–359.

Rott, H. 2003. Basic entrenchment. *Studia Logica* 73:257–280.

Williams, M.-A.; Pagnucco, M.; Foo, N.; and Sims, B. 1995. Determining explanations using transmutations. *Proceedings of the Fourteenth International Joint Conference on Artificial Intelligence* 822–827.

Williams, M. 1994. On the logic of theory base change. In MacNish, C.; Pearce, D.; and Pereira, L. M., eds., *Logics in Artificial Intelligence, European Workshop, JELIA '94, York, UK, September 5-8, 1994, Proceedings*, volume 838 of *Lecture Notes in Computer Science*, 86–105. Springer.