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Definitions and (Uniform) Interpolants in First-Order Modal Logic

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Abstract

We first consider two decidable fragments of quantified modal logic $\mathsf{S5}$: the one-variable fragment $\mathsf{Q}^1\mathsf{S5}$ and its extension $\mathsf{S5}_{\mathcal{ALC}^u}$ that combines $\mathsf{S5}$ and the description logic $\mathcal{ALC}$ with the universal role. As neither of them enjoys Craig interpolation or projective Beth definability, the existence of interpolants and explicit definitions of predicates—which is crucial in many knowledge engineering tasks—does not directly reduce to entailment. Our concern therefore is the computational complexity of deciding whether (uniform) interpolants and definitions exist for given input formulas, signatures and ontologies. We prove that interpolant and definition existence in $\mathsf{Q}^1\mathsf{S5}$ and $\mathsf{S5}_{\mathcal{ALC}^u}$ is decidable in coN2ExpTime, being 2ExpTime-hard, while uniform interpolant existence is undecidable. Then we show that interpolant and definition existence in the one-variable fragment $\mathsf{Q}^1\mathsf{K}$ of quantified modal logic K is nonelementary decidable, while uniform interpolant existence is undecidable.

1 Introduction

Decidable fragments of first-order modal logics have been a well-established KR formalism for many decades, e.g., in the form of epistemic, temporal, or standpoint description logics (Donini et al. 1998; Lutz, Wolter, and Zakharyaschev 2008; Artale et al. 2017; Álvarez, Rudolph, and Strass 2022), spatio-temporal logics (Kontchakov et al. 2007), and logics of knowledge and belief (Belardinelli and Lomuscio 2009; Wang 2017; Liu et al. 2022; Wang, Wei, and Seligman 2022). While significant progress has been made in understanding the computational complexity of entailment in these 'two-dimensional' logics, little is known about the algorithmic properties of logic-based support mechanisms for engineering knowledge bases or specifications in these logics. Important examples of relevant problems are:

- **(*definition existence*)** Given a knowledge base (KB), a predicate P, and a signature σ, is it possible to give a definition of P in terms of σ-predicates modulo the KB?
- **(*forgetting/uniform interpolants*)** Given a KB and a signature σ, is it possible to 'forget σ', i.e., find a new KB without σ-predicates that says the same about non-σ-symbols as the original KB?
- **(*conservative extensions*)** Given a KB and a set of additional axioms, is it the case that the expanded KB does not entail new relationships between the original predicates?

These and related problems have been studied extensively for many KR formalisms (Eiter and Kern-Isberner 2019) including propositional logic (Lang and Marquis 2008), answer set programming (Gonçalves, Knorr, and Leite 2023), and description logics (Konev et al. 2009; Botoeva et al. 2016) but investigating them for first-order modal logics (FOMLs) poses particular challenges. In contrast to many other KR formalisms, FOMLs used in KR typically do not enjoy the Craig interpolation property (CIP) as $\models \varphi \to \psi$ does not necessarily entail the existence of an *interpolant* χ whose predicate symbols occur in both φ and ψ, with $\models \varphi \to \chi$ and $\models \chi \to \psi$. Nor do they enjoy the projective Beth definability property (BDP) according to which implicit definability of a predicate in a given signature (which can be reduced to entailment) implies its explicit definability as required in definition existence. Forgetting and conservative extensions in FOMLs become dependent on predicates that do not occur in the original KB. In fact, Fine (1979) showed that no FOML with constant domains (the standard assumption in KR applications) between the basic quantified modal logics K and $\mathsf{S5}$ enjoys CIP or BDP.

Example 1 (based on (Fine 1979))**.** Interpreting $\Box$ as the $\mathsf{S5}$-modality 'always', let a KB contain the axioms

$$\mathsf{rep} \to \Diamond\forall x \big(\mathsf{inPower}(x) \to \Box(\mathsf{rep} \to \neg\mathsf{inPower}(x))\big),$$

$$\neg\mathsf{rep} \to \Box\exists x \big(\mathsf{inPower}(x) \land \Box(\neg\mathsf{rep} \to \mathsf{inPower}(x))\big),$$

where rep stands for the proposition 'replaceable'. Then rep is true at a world w satisfying the KB iff there is a world w' where all those who were in power at w lose it. Thus, rep is implicitly defined via $\mathsf{inPower}$. However, there is no explicit definition of rep via $\mathsf{inPower}$ in FOML (see Example 8).

Fine's example shows that CIP/BDP fail already in typical decidable fragments of FOML lying between the one-variable fragment FOM^1 and full FOML. Because of their wide use, 'repairing' CIP/BDP has become a major research challenge. For instance, it is shown in (Fitting 2002; Areces, Blackburn, and Marx 2003) that by adding second-order quantifiers or the machinery of hybrid logic constructors to FOML, one obtains logics with CIP and BDP. The price, however, is that these extensions are undecidable even if applied to decidable fragments of FOML.

In this paper, we take a fundamentally different, non-uniform approach. Instead of repairing CIP/BDP by enriching the language, we stay within its original boundaries and explore if it is possible to check the existence of interpolants/definitions even though the reduction to entailment via CIP/BDP is blocked. We conjecture that, in real-world applications, interpolants and definitions often do exist, so the failure of CIP/BDP will have a limited effect on the users.

We first focus on two decidable fragments of quantified S5: its one-variable fragment $\mathsf{Q}^1\mathsf{S5}$ illustrated in Example 1 and $\mathsf{S5}_{\mathcal{ALC}^u}$, the FOML obtained by combining S5 and the description logic (DL) $\mathcal{ALC}^u$ extending the basic DL $\mathcal{ALC}$ with the universal role. In $\mathsf{S5}_{\mathcal{ALC}^u}$, we admit the application of modal operators to concepts and concept inclusions but not to roles, and so consider a typical *monodic* fragment of FOML, in which modal operators are only applied to formulas with at most one free variable (Hodkinson, Wolter, and Zakharyaschev 2000; Wolter and Zakharyaschev 2001). $\mathsf{Q}^1\mathsf{S5}$ is a fragment of $\mathsf{S5}_{\mathcal{ALC}^u}$, and satisfiability is NEXPTIME-complete for both languages (Gabbay et al. 2003).

We chose S5 as our starting point as it is widely used in Knowledge Representation, underpinning fundamental modalities such as necessity and agents' knowledge (Fagin et al. 1995). Combined with DLs, it has also been proposed as a logic of change interpreting $\Box$ as 'always' (Artale, Lutz, and Toman 2007) and can naturally encode a rather expressive version of standpoint logic (Álvarez, Rudolph, and Strass 2022); see the full paper for details.

Example 2. In $\mathsf{S5}_{\mathcal{ALC}^u}$, we can encode different standpoints s_i by concept names S_i that hold everywhere in the domain of any world conceivable by s_i. That $C \sqsubseteq D$ holds according to s_i can then be represented as $\Box(S_i \sqcap C \sqsubseteq D)$ or $\Box_{s_i}(C \sqsubseteq D)$ for short. Suppose that our KB, K, contains the axioms

$\Box(\top \sqsubseteq S_1 \sqcup S_2)$,
$\Box(\mathsf{KR} \sqcup \mathsf{Databases} \sqcup \mathsf{Verification} \equiv \mathsf{CS} \sqcap \exists \mathsf{uses.Logic})$,
$\Box(\mathsf{Databases} \sqcup \mathsf{Verification} \sqsubseteq \neg\exists \mathsf{historicAreaOf.AI})$,
$\Box_{s_1}(\mathsf{KR} \equiv \mathsf{CS} \sqcap \exists \mathsf{areaOf.AI} \sqcap \exists \mathsf{uses.Logic})$,
$\Box_{s_2}(\mathsf{KR} \sqsubseteq \exists \mathsf{historicAreaOf.AI})$,
$\Box_{s_2}(\exists \mathsf{areaOf.AI} \sqsubseteq \neg\exists \mathsf{uses.Logic})$.

The first says that every world in any model of the KB is regarded as possible from either standpoint s_1 or s_2. According to the next two, it is generally agreed that the areas of CS that use Logic are KR, Databases and Verification, while Verification and Databases are not historic areas of AI. The last three express the s_i's diverging views on KR. Then

$$\mathsf{KR} \equiv \mathsf{CS} \sqcap \exists \mathsf{uses.Logic} \sqcap (\exists \mathsf{areaOf.AI} \sqcup \exists \mathsf{historicAreaOf.AI})$$

is entailed by K, and so explicitly defines KR modulo K without referring to S_i, KR, Databases, and Verification. $\dashv$

Our main result is that interpolant and definition existence in $\mathsf{Q}^1\mathsf{S5}$ and $\mathsf{S5}_{\mathcal{ALC}^u}$ is decidable in CON2EXPTIME, being 2EXPTIME-hard. The proof is based on novel 'componentwise' bisimulations that replace standard FOML bisimulations in our characterisation of interpolant/definition existence. For the upper bound, we show that there are bisimilar models witnessing non-existence of interpolants/definitions of double-exponential size. The proof is inspired by the recent upper bound proofs of interpolant existence in the two-variable first-order logic FO^2 (Jung and Wolter 2021) but requires a novel use of types. The lower bound proof combines the interpolation counterexample of (Marx and Areces 1998), the exponential grid generation from (Hodkinson et al. 2003; Göller, Jung, and Lohrey 2015), and the representation of exponentially space bounded ATMs from (Jung and Wolter 2021). As a corollary, we obtain a 2EXPTIME lower bound for FO^2 without equality, answering an open question of (Jung and Wolter 2021).

We then consider uniform interpolant existence and conservative extension and show that both problems are undecidable for $\mathsf{Q}^1\mathsf{S5}$ and $\mathsf{S5}_{\mathcal{ALC}^u}$. The proof extends a reduction proving undecidability of conservative extensions for FO^2 (with and without equality) from (Jung et al. 2017). As a corollary of our proof, we obtain that uniform interpolant existence is undecidable for FO^2 (with and without equality), settling an open problem from (Jung et al. 2017).

Finally, we consider the one-variable fragment $\mathsf{Q}^1\mathsf{K}$ of quantified K and prove a non-elementary upper bound for interpolant/definition existence using the fact that $\mathsf{Q}^1\mathsf{K}$ has finitely many non-equivalent formulas of bounded modal depth. To our surprise, conservative extensions and uniform interpolant existence are still undecidable in $\mathsf{Q}^1\mathsf{K}$, which is proved by adapting the undecidability proof for $\mathsf{Q}^1\mathsf{S5}$.

Related Work on Interpolant Existence. Except for work on linear temporal logic *LTL* by (Henkell 1988; Henkell et al. 2010; Place and Zeitoun 2016), the non-uniform approach to Craig interpolants has only very recently been studied by (Jung and Wolter 2021) for the guarded and two-variable fragment, by (Artale et al. 2021; Jung, Mazzullo, and Wolter 2022) for classical DLs, and by (Benedikt et al. 2016; Fortin, Konev, and Wolter 2022) for Horn logics. The non-uniform investigation of uniform interpolants started with complexity results by (Lutz, Seylan, and Wolter 2012; Lutz and Wolter 2011) and upper bounds on their size in (Nikitina and Rudolph 2014). The practical computation of uniform interpolants is an active research area for many years (Konev, Walther, and Wolter 2009; Koopmann and Schmidt 2015; Zhao and Schmidt 2016); see (Zhao et al. 2018; Koopmann 2020) for recent system descriptions.

Omitted proofs and definitions can be found in the full arXiv paper (Kurucz, Wolter, and Zakharyaschev 2023).

2 Preliminaries

Logics. The formulas of the *one-variable fragment* FOM^1 *of first-order modal logic* are built from unary predicate symbols $\boldsymbol{p} \in \mathcal{P}$ in a countably-infinite set $\mathcal{P}$ and a single variable x using $\top$, $\neg$, $\wedge$, $\exists x$, and the possibility operator $\Diamond$ via which the other Booleans, $\forall x$, and the necessity operator $\Box$ are standardly definable. A *signature* is any finite set $\sigma \subseteq \mathcal{P}$; the signature $sig(\varphi)$ of a formula φ comprises the predicate symbols in φ. If $sig(\varphi) \subseteq \sigma$, we call φ a *σ-formula*. By $sub(\varphi)$ we denote the closure under single negation of the set of subformulas of φ, and by $|\varphi|$ the cardinality of $sub(\varphi)$.

We interpret FOM^1 in (*Kripke*) *models* with *constant domains* of the form $\mathfrak{M} = (W, R, D, I)$, where $W \neq \emptyset$ is a set of *worlds*, $R \subseteq W \times W$ an *accessibility relation* on W, $D \neq \emptyset$ an (FO-)*domain* of $\mathfrak{M}$, and $I(w)$, for each $w \in W$, is an *interpretation* of the $\boldsymbol{p} \in \mathcal{P}$ over D, that is, $\boldsymbol{p}^{I(w)} \subseteq D$.

The *truth-relation* $\mathfrak{M}, w, d \models \varphi$, for any $w \in W$, $d \in D$ and FOM^1-formula φ, is defined inductively by taking

- $\mathfrak{M}, w, d \models \boldsymbol{p}(x)$ iff $d \in \boldsymbol{p}^{I(w)}$, for $\boldsymbol{p} \in \mathcal{P}$,
- $\mathfrak{M}, w, d \models \exists x\, \varphi$ iff there is $d' \in D$ with $\mathfrak{M}, w, d' \models \varphi$,
- $\mathfrak{M}, w, d \models \Diamond\varphi$ iff there is $w' \in W$ with $R(w, w')$ and $\mathfrak{M}, w', d \models \varphi$,

and the standard clauses for $\top$, $\neg$, $\wedge$. If φ is a sentence (i.e., every occurrence of x in φ is in the scope of $\exists$), then $\mathfrak{M}, w, d \models \varphi$ iff $\mathfrak{M}, w, d' \models \varphi$, for any $d, d' \in D$, and so we can omit d and write $\mathfrak{M}, w \models \varphi$. In a similar way, we can use $\mathfrak{M}, d \models \psi$ if every $\boldsymbol{p}$ in ψ is in the scope of $\Diamond$.

The set of formulas φ with $\mathfrak{M}, w, d \models \varphi$, for all $\mathfrak{M}$, w, d, is denoted by $\mathsf{Q}^1\mathsf{K}$; it is the FOM^1-extension of the modal logic K. Those φ that are true everywhere in all models $\mathfrak{M}$ with $R = W \times W$ comprise $\mathsf{Q}^1\mathsf{S5}$, the FOM^1-extension of the modal logic $\mathsf{S5}$. Let L be one of these two logics.

A *knowledge base* (KB), K, is any finite set of sentences. We say that K (*locally*) *entails* φ *in* L and write $K \models_L \varphi$ if $\mathfrak{M}, w \models K$ implies $\mathfrak{M}, w, d \models \varphi$, for any L-model $\mathfrak{M}$ and any w and d in it. Shortening $\emptyset \models_L \varphi$ to $\models_L \varphi$ (i.e., $\varphi \in L$), we note that $K \models_L \varphi$ iff $\models_L (\bigwedge_{\psi \in K} \psi \to \varphi)$, which reduces KB-entailment in L to L-validity which is known to be coNExpTime-complete (Marx 1999).

Bisimulations. Given two models $\mathfrak{M} = (W, R, D, I)$ with w, d and $\mathfrak{M}' = (W', R', D', I')$ with w', d', we write $\mathfrak{M}, w, d \equiv_\sigma \mathfrak{M}', w', d'$, for a signature σ, if the same σ-formulas are true at w, d in $\mathfrak{M}$ and at w', d' in $\mathfrak{M}'$. We characterise $\equiv_\sigma$ using bisimulations. Namely, a relation

$$\boldsymbol{\beta} \subseteq (W \times D) \times (W' \times D')$$

is called a σ-*bisimulation* between models $\mathfrak{M}$ and $\mathfrak{M}'$ if the following conditions hold for all $((w, d), (w', d')) \in \boldsymbol{\beta}$ and $\boldsymbol{p} \in \sigma$:

(a) $\mathfrak{M}, w, d \models \boldsymbol{p}$ iff $\mathfrak{M}', w', d' \models \boldsymbol{p}$;

(w) if $(w, v) \in R$, then there is v' such that $(w', v') \in R'$ and $((v, d), (v', d')) \in \boldsymbol{\beta}$, and the other way round;

(d) for every $e \in D$, there is $e' \in D'$ such that $((w, e), (w', e')) \in \boldsymbol{\beta}$, and the other way round.

We say that $\mathfrak{M}, w, d$ and $\mathfrak{M}', w', d'$ are σ-*bisimilar* and write $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$ if there is a σ-bisimulation $\boldsymbol{\beta} \ni ((w, d), (w', d'))$ between $\mathfrak{M}$ and $\mathfrak{M}'$. The next characterisation is proved in a standard way using ω-saturated models (Chang and Keisler 1998; Goranko and Otto 2007):

Lemma 3. *For any signature σ and any ω-saturated models $\mathfrak{M}$ with w, d and $\mathfrak{M}'$ with w', d', we have:*

$$\mathfrak{M}, w, d \equiv_\sigma \mathfrak{M}', w', d' \quad \textit{iff} \quad \mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'.$$

The direction from right to left holds for arbitrary models.

Modal products and succinct notation. As observed by (Wajsberg 1933), $\mathsf{S5}$ is a notational variant of the one-variable fragment FO^1 of FO: just drop x from $\exists x$ and $\boldsymbol{p}(x)$ in FO^1-formulas, treating $\exists$ as a possibility operator and $\boldsymbol{p}$ as a *propositional variable*. The same operation transforms FOM^1-formulas into more succinct *bimodal formulas* with $\Diamond$ interpreted over the (W, R) 'dimension' and $\exists$ over the $(D, D \times D)$ 'dimension'. This way we view the FOM^1-extensions of $\mathsf{S5}$ and K as two-dimensional *products* of modal logics: $\mathsf{S5} \times \mathsf{S5}$ and $\mathsf{K} \times \mathsf{S5}$. The former is known to be the 'equality and substitution-free' fragment of two-variable FO-logic FO^2 (Gabbay et al. 2003); the latter is embedded into FO by the *standard translation* *:

$$\boldsymbol{p}^* = \boldsymbol{q}(z, x), \;\; (\neg\varphi)^* = \neg\varphi^*, \;\; (\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*,$$
$$(\exists\varphi)^* = \exists x\, \varphi^*, \;\; (\Diamond\varphi)^* = \exists y\, (R(z, y) \wedge \varphi^*\{y/z\}),$$

where y is a fresh variable not occurring in φ^* and $\{y/z\}$ means a substitution of y in place of z.

From now on, we write FOM^1-formulas as *bimodal* ones: for example, $\exists\Box\boldsymbol{p}$ instead of $\exists x\, \Box\boldsymbol{p}(x)$. By a formula we mean an FOM^1-formula unless indicated otherwise; a logic, L, is one of $\mathsf{Q}^1\mathsf{S5}$ and $\mathsf{Q}^1\mathsf{K}$, again unless stated otherwise.

3 Main Notions and Characterisations

We now introduce the main notions studied in this paper and provide their model-theoretic characterisations.

Craig interpolants. A formula χ is an *interpolant* of formulas φ and ψ *in a logic* L if $sig(\chi) \subseteq sig(\varphi) \cap sig(\psi)$, $\models_L \varphi \to \chi$ and $\models_L \chi \to \psi$. L enjoys the *Craig interpolation property* (*CIP*) if an interpolant for φ and ψ exists whenever $\models_L \varphi \to \psi$. One of our main concerns here is the *interpolant existence problem* (*IEP*) for L: decide if given φ and ψ have an interpolant in L. For logics with CIP, IEP reduces to entailment, and so is not interesting. This is the case for many logics including propositional $\mathsf{S5}$ and K, but not for FOMLs with constant domain between $\mathsf{Q}^1\mathsf{K}$ and $\mathsf{Q}^1\mathsf{S5}$ (Fine 1979; Marx and Areces 1998).

Explicit definitions. Given formulas φ, ψ and a signature σ, an *explicit σ-definition of ψ modulo φ in L* is a σ-formula χ with $\models_L \varphi \to (\psi \leftrightarrow \chi)$. The *explicit σ-definition existence problem* (*EDEP*) for L is to decide, given φ, ψ and σ, whether there exists an explicit σ-definition of ψ modulo φ in L. EDEP reduces trivially to entailment for logics enjoying the *projective Beth definability property* (*BDP*) according to which ψ is explicitly σ-definable modulo φ in L iff it is implicitly σ-definable modulo φ in the sense that $\{\varphi, \varphi'\} \models_L \psi \leftrightarrow \psi'$, where φ', ψ' result from φ, ψ by uniformly replacing all non-σ-symbols with fresh ones. Again, many logics including $\mathsf{S5}$ and K enjoy BDP while FOMLs with constant domains between $\mathsf{Q}^1\mathsf{K}$ and $\mathsf{Q}^1\mathsf{S5}$ do not.

Note that, in typical KR applications, φ in our formulation of EDEP corresponds to a KB K and ψ is a predicate $\boldsymbol{p}$. Then the problem whether there exists an explicit σ-definition of $\boldsymbol{p}$ modulo K is the problem of deciding whether there is χ with $sig(\chi) \subseteq \sigma$ and $K \models_L \forall x(\boldsymbol{p}(x) \leftrightarrow \chi(x))$. This problem trivially translates to EDEP using our discussion of KBs above. In more detail, this view of EDEP is discussed in Section 6 in the context of $\mathsf{S5}_{\mathcal{ALC}^u}$.

IEP and EDEP are closely related (Gabbay and Maksimova 2005). In this paper, we only require the following:

Theorem 4. *For any $L \in \{\mathsf{Q}^1\mathsf{S5}, \mathsf{Q}^1\mathsf{K}\}$, EDEP for L and IEP for L are polynomially reducible to each other.*

The proof, given in the full version, is based on a characterisation of IEP and EDEP using bisimulations.

Lemma 3 together with the fact that FOM^1 is a fragment of FO are used to obtain, again in a standard way, the following criterion of interpolant existence. We call formulas φ and ψ *σ-bisimulation consistent* in L if there exist L-models $\mathfrak{M}$ with w, d and $\mathfrak{M}'$ with w', d' such that $\mathfrak{M}, w, d \models \varphi$, $\mathfrak{M}', w', d' \models \psi$ and $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$.

Theorem 5. *For any φ and ψ, the following are equivalent*:

– *there does not exist an interpolant of φ and ψ in L;*

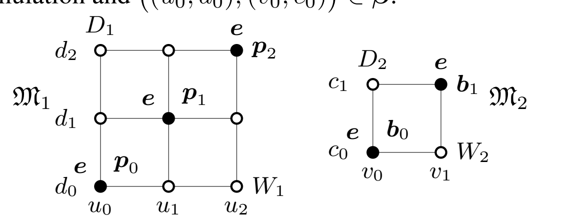
– *φ, $\neg\psi$ are $sig(\varphi) \cap sig(\psi)$-bisimulation consistent in L.*

Proof. Suppose φ and ψ do not have an interpolant in L and $\sigma = sig(\varphi) \cap sig(\psi)$. Consider the set Ξ of σ-formulas χ with $\models_L \varphi \to \chi$. By compactness, we have an ω-saturated model $\mathfrak{M}$ of L with w and d such that $\mathfrak{M}, w, d \models \chi$, for all $\chi \in \Xi$, and $\mathfrak{M}, w, d \models \neg\psi$. Take the set Ξ' of σ-formulas χ with $\mathfrak{M}, w, d \models \chi$ and an ω-saturated model $\mathfrak{M}'$ with $\mathfrak{M}', w', d' \models \Xi'$ and $\mathfrak{M}', w', d' \models \varphi$, for some w' and d'. Then $\mathfrak{M}, w, d \equiv_\sigma \mathfrak{M}', w', d'$, and so $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$ by Lemma 3. The converse implication is straightforward.⊣

Example 6. For every $n < \omega$, Marx and Areces (1998) constructed FOM^1-formulas φ and ψ with $\models_{\mathsf{Q}^1\mathsf{S5}} \varphi \to \psi$ and $sig(\varphi) \cap sig(\psi) = \{\boldsymbol{e}\}$ that have no interpolant in the n-variable $\mathsf{Q}^n\mathsf{S5}$. For $n = 1$, φ and ψ look as follows:

$$\begin{aligned}\varphi \;=\;& \boldsymbol{p}_0 \wedge \Diamond\exists(\boldsymbol{p}_1 \wedge \Diamond\exists \boldsymbol{p}_2) \wedge \\ & \Box\forall\big[(\boldsymbol{e} \leftrightarrow \boldsymbol{p}_0 \vee \boldsymbol{p}_1 \vee \boldsymbol{p}_2) \wedge \textstyle\bigwedge_{i \neq j}(\boldsymbol{p}_i \to \neg \boldsymbol{p}_j) \wedge \\ & \qquad \textstyle\bigwedge_i (\boldsymbol{p}_i \to \Box(\boldsymbol{e} \to \boldsymbol{p}_i) \wedge \forall(\boldsymbol{e} \to \boldsymbol{p}_i))\big], \\ \psi \;=\;& \Box\forall(\boldsymbol{e} \leftrightarrow \boldsymbol{b}_0 \vee \boldsymbol{b}_1) \to \\ & \Diamond\exists\big(\boldsymbol{b}_0 \wedge \Diamond(\neg \boldsymbol{e} \wedge \exists \boldsymbol{b}_0)\big) \vee \Diamond\exists\big(\boldsymbol{b}_1 \wedge \Diamond(\neg \boldsymbol{e} \wedge \exists \boldsymbol{b}_1)\big).\end{aligned}$$

To see that φ, $\neg\psi$ are $\{\boldsymbol{e}\}$-bisimulation consistent in $\mathsf{Q}^1\mathsf{S5}$, take the models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ below with $\mathfrak{M}_1, u_0, d_0 \models \varphi$ and $\mathfrak{M}_2, v_0, c_0 \models \neg\psi$. (In our pictures, the possible worlds are always shown along the horizontal axis and the domain elements along the vertical one, giving points of the form (w, d).) The relation $\boldsymbol{\beta}$ connecting each $\boldsymbol{e}$-point in $\mathfrak{M}_1$ with each $\boldsymbol{e}$-point in $\mathfrak{M}_2$, and similarly for $\neg\boldsymbol{e}$-points, is an $\{\boldsymbol{e}\}$-bisimulation and $\big((u_0, d_0), (v_0, c_0)\big) \in \boldsymbol{\beta}$.



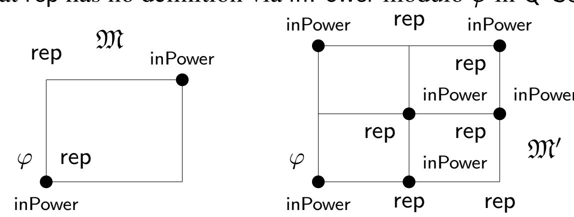
Similarly to Theorem 5 we obtain the following criterion of explicit definition existence:

Theorem 7. *For any φ, ψ, σ, the following are equivalent*:

– *there is no explicit σ-definition of ψ modulo φ in L;*

– *$\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ are σ-bisimulation consistent in L.*

Example 8. Let φ be the conjunction of the two KB axioms from Example 1, $\sigma = \{\mathsf{inPower}\}$, and let $\psi = \mathsf{rep}$.[1] Then the second condition of Theorem 7 holds for the $\mathsf{Q}^1\mathsf{S5}$-models shown below, in which (w, d) in $\mathfrak{M}$ is bisimilar to (w', d') in $\mathfrak{M}'$ iff (w, d) and (w', d') agree on σ. It follows that rep has no definition via inPower modulo φ in $\mathsf{Q}^1\mathsf{S5}$. ⊣



We next define conservative extensions, an important notion in the context of ontology modules and modularisation (Grau et al. 2008; Botoeva et al. 2016).

Conservative extensions. Given formulas φ and ψ, we call φ an *L-conservative extension of ψ* if (a) $\models_L \varphi \to \psi$ and (b) $\models_L \varphi \to \chi$ implies $\models_L \psi \to \chi$, for any χ with $sig(\chi) \subseteq sig(\psi)$. In typical KR applications, ψ is given by a KB K and φ is obtained by adding fresh axioms to K. (The translation of our results to the language of KBs is obvious.) The next example shows that this notion of conservative extension is syntax-dependent in the sense that it is not robust under the addition of fresh predicates.

Example 9. For the formulas

$\varphi = \mathsf{rep} \wedge \Diamond\forall\big(\mathsf{inPower} \to \Box(\mathsf{rep} \to \neg\mathsf{inPower})\big)$,

$\psi = \Box\forall(\Diamond\mathsf{inPower} \wedge \Diamond\neg\mathsf{inPower} \wedge \exists\mathsf{inPower} \wedge \exists\neg\mathsf{inPower})$

$\varphi \wedge \psi$ is a conservative extension of ψ in $\mathsf{Q}^1\mathsf{S5}$ (as all models of ψ are $\{\mathsf{inPower}\}$-bisimilar to $\mathfrak{M}$ in Example 8). Now, let $\psi' = \psi \wedge (\mathsf{p} \vee \neg\mathsf{p})$, for a fresh proposition p. Then $\varphi \wedge \psi'$ is not a conservative extension of ψ' as witnessed by the formula $\chi = \neg\big(\mathsf{p} \wedge \Box\exists(\mathsf{inPower} \wedge \Box(\mathsf{p} \to \mathsf{inPower}))\big)$. ⊣

If in the previous definition we require (b) to hold for all χ with $sig(\chi) \cap sig(\varphi) \subseteq sig(\psi)$, then φ is called a *strong L-conservative extension of ψ*. As observed by (Jung et al. 2017), the difference between conservative and strong conservative extensions is closely related to the failure of CIP: if L enjoys CIP, then L-conservative extensions coincide with strong L-conservative extensions. The problem of deciding whether a given φ is a (strong) conservative extension of a given ψ will be referred to as (S)CEP. The study of the complexity of (S)CEP for DLs and modal logics started with (Ghilardi, Lutz, and Wolter 2006) and (Ghilardi et al. 2006); see (Botoeva et al. 2019; Jung, Lutz, and Marcinkowski 2022) for more recent work.

Uniform interpolants. Given a formula φ and a signature σ, we call a formula ψ a *σ-uniform interpolant of φ in L* if $sig(\psi) = \sigma$ and φ is a strong L-conservative extension of ψ.

A logic L has the *uniform interpolation property* (*UIP*) if, for any φ and σ, there is a σ-uniform interpolant of φ in L. UIP entails CIP but not the other way round. For example, modal logic S4 and $\mathcal{ALC}^u$ enjoy CIP but not UIP (Ghilardi and Zawadowski 1995; Lutz and Wolter 2011). This leads to the *uniform interpolant existence problem* (*UIEP*): given φ and σ, decide whether φ has a uniform σ-interpolant in L. Uniform interpolants are closely related to forgetting introduced by (Lin and Reiter 1994). SCEP is equivalent to verifying whether a given formula is a uniform interpolant.

[1] As our FOM^1 has no 0-ary predicates, the *proposition* rep is given as $\forall x\, \mathsf{rep}(x)$ assuming that $\models_L \forall x\, \mathsf{rep}(x) \vee \forall x\, \neg\mathsf{rep}(x)$.

4 Deciding IEP and EDEP for $\mathsf{Q}^1\mathsf{S5}$

In this section, we first give a simpler—yet equivalent—definition of bisimulation between $\mathsf{Q}^1\mathsf{S5}$-models and then use it to show that, when checking bisimulation consistency in $\mathsf{Q}^1\mathsf{S5}$, it is enough to look for bisimilar models of double-exponential size in the size of the given formulas.

As $R = W \times W$ in any $\mathsf{Q}^1\mathsf{S5}$-model $\mathfrak{M} = (W, R, D, I)$, we drop R and write simply $\mathfrak{M} = (W, D, I)$. Given a signature σ and $(w, d) \in W \times D$, the *literal σ-type* $\ell^\sigma_\mathfrak{M}(w, d)$ of (w, d) in $\mathfrak{M}$ is the set

$$\{\boldsymbol{p} \in \sigma \mid \mathfrak{M}, w, d \models \boldsymbol{p}\} \cup \{\neg\boldsymbol{p} \mid \boldsymbol{p} \in \sigma,\ \ \mathfrak{M}, w, d \not\models \boldsymbol{p}\}.$$

A pair $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ of relations $\boldsymbol{\beta}_1 \subseteq W \times W'$ and $\boldsymbol{\beta}_2 \subseteq D \times D'$ is called a σ-$\mathsf{S5}$-*bisimulation* between $\mathfrak{M} = (W, D, I)$ and $\mathfrak{M}' = (W', D', I')$ when the following conditions hold:

($\mathsf{s5}_1$) if $(w, w') \in \boldsymbol{\beta}_1$ then, for any $d \in D$, there is $d' \in D'$ such that $(d, d') \in \boldsymbol{\beta}_2$ and $\ell^\sigma_\mathfrak{M}(w, d) = \ell^\sigma_{\mathfrak{M}'}(w', d')$, and the other way round;

($\mathsf{s5}_2$) if $(d, d') \in \boldsymbol{\beta}_2$ then, for any $w \in W$, there is $w' \in W'$ such that $(w, w') \in \boldsymbol{\beta}_1$ and $\ell^\sigma_\mathfrak{M}(w, d) = \ell^\sigma_{\mathfrak{M}'}(w', d')$, and the other way round.

We say that $\mathfrak{M}, w, d$ and $\mathfrak{M}', w', d'$ are σ-$\mathsf{S5}$-*bisimilar* and write $\mathfrak{M}, w, d \sim^{\mathsf{S5}}_\sigma \mathfrak{M}', w', d'$ if there is a σ-$\mathsf{S5}$-bisimulation $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ with $(w, w') \in \boldsymbol{\beta}_1$, $(d, d') \in \boldsymbol{\beta}_2$ and $\ell^\sigma_\mathfrak{M}(w, d) = \ell^\sigma_{\mathfrak{M}'}(w', d')$. Note that in this case we have $\mathrm{dom}(\boldsymbol{\beta}_1) = W$, $\mathrm{ran}(\boldsymbol{\beta}_1) = W'$, $\mathrm{dom}(\boldsymbol{\beta}_2) = D$, and $\mathrm{ran}(\boldsymbol{\beta}_2) = D'$.

Theorem 10. *$\mathfrak{M}, w, d \sim^{\mathsf{S5}}_\sigma \mathfrak{M}', w', d'$ if and only if $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$.*

Proof. If $\mathfrak{M}, w, d \sim^{\mathsf{S5}}_\sigma \mathfrak{M}', w', d'$ is witnessed by $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$, then $\boldsymbol{\beta}$ defined by setting $((v, e), (v', e')) \in \boldsymbol{\beta}$ iff $(v, v') \in \boldsymbol{\beta}_1$, $(e, e') \in \boldsymbol{\beta}_2$ and $\ell^\sigma_\mathfrak{M}(v, e) = \ell^\sigma_{\mathfrak{M}'}(v', e')$ satisfies **(a)**, **(w)** and **(d)**. Conversely, if $\mathfrak{M}, w, d \sim_\sigma \mathfrak{M}', w', d'$ is witnessed by $\boldsymbol{\beta}$, then $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ below satisfies ($\mathsf{s5}_1$), ($\mathsf{s5}_2$)

$$\boldsymbol{\beta}_1 = \{(v, v') \mid \exists e, e' \, ((v, e), (v', e')) \in S\},$$
$$\boldsymbol{\beta}_2 = \{(e, e') \mid \exists v, v' \, (v, e), (v', e) \in S\},$$

$(w, w') \in \boldsymbol{\beta}_1$, $(d, d') \in \boldsymbol{\beta}_2$, and $\ell^\sigma_\mathfrak{M}(w, d) = \ell^\sigma_{\mathfrak{M}'}(w', d')$. ⊣

In this section, we only deal with σ-$\mathsf{S5}$-bisimulations, and so omit explicit $\mathsf{S5}$ from the relevant notations. We write $\mathfrak{M}_1, w_1 \sim_\sigma \mathfrak{M}_2, w_2$ if there is a σ-bisimulation $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ with $(w_1, w_2) \in \boldsymbol{\beta}_1$. By ($\mathsf{s5}_1$), $\mathfrak{M}_1, w_1 \sim_\sigma \mathfrak{M}_2, w_2$ entails that the interpretations $I_1(w_1)$ in $\mathfrak{M}_1$ and $I_2(w_2)$ in $\mathfrak{M}_2$ are *globally σ-bisimilar* in the sense that, for any $d_1 \in D_1$, there exists $d_2 \in D_2$ satisfying the same $\boldsymbol{p} \in \sigma$ in $I_1(w_1)$ and $I_2(w_2)$, and the other way round. Similarly, we write $\mathfrak{M}_1, d_1 \sim_\sigma \mathfrak{M}_2, d_2$ if there is a σ-bisimulation $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ with $(d_1, d_2) \in \boldsymbol{\beta}_2$. We omit $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and write simply $(w_1, d_1) \sim_\sigma (w_2, d_2)$, $w_1 \sim_\sigma w_2$, $d_1 \sim_\sigma d_2$ if understood.

Observe that σ-bisimulations between the same models are preserved under set-theoretic union: if Γ is a set of σ-bisimulations, then $(\bigcup_{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) \in \Gamma} \boldsymbol{\beta}_1, \bigcup_{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) \in \Gamma} \boldsymbol{\beta}_2)$ is a σ-bisimulation too. It follows that $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ defined by taking $(w_1, w_2) \in \boldsymbol{\beta}_1$ if $w_1 \sim_\sigma w_2$ and $(d_1, d_2) \in \boldsymbol{\beta}_2$ if $d_1 \sim_\sigma d_2$ is the maximal σ-bisimulation between the given models.

Example 11. Consider $\mathfrak{M}_1$, $\mathfrak{M}_2$, and $\sigma = \{\boldsymbol{e}\}$ from Example 6. Then $(W_1 \times W_1, D_1 \times D_1)$ is a σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_1$ witnessing $(u_i, d_i) \sim_\sigma (u_j, d_j)$ and $(u_k, d_l) \sim_\sigma (u_m, d_n)$, for $i, j, k, l, m, n \in \{0, 1, 2\}$, $k \neq l$, $m \neq n$. The pair $(W_1 \times W_2, D_1 \times D_2)$ is a σ-bisimulation between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ witnessing $(u_i, d_i) \sim_\sigma (v_j, c_j)$ and $(u_k, d_l) \sim_\sigma (v_m, c_n)$, for $i, k, l \in \{0, 1, 2\}$, $k \neq l$, and $j, m, n \in \{0, 1\}$, $m \neq n$ (cf. $\boldsymbol{\beta}$ in Example 6).

We now use σ-bisimulations to develop an algorithm deciding IEP for $\mathsf{Q}^1\mathsf{S5}$ in coN2ExpTime. Suppose we want to check whether φ and ψ have an interpolant in $\mathsf{Q}^1\mathsf{S5}$. By Theorem 5, this is not the case iff there are $\mathsf{Q}^1\mathsf{S5}$-models $\mathfrak{M}_1$ with w_1, d_1 and $\mathfrak{M}_2$ with w_2, d_2 such that $\mathfrak{M}_1, w_1, d_1 \models \varphi$, $\mathfrak{M}_2, w_2, d_2 \not\models \psi$, and $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$. We are going to show that if such $\mathfrak{M}_i$ do exist, they can be chosen to be of double-exponential size in $|\varphi|$ and $|\psi|$.

Fix φ, ψ and $\sigma = \mathit{sig}(\varphi) \cap \mathit{sig}(\psi)$. Denote by $\mathit{sub}_\exists(\varphi, \psi)$ the closure under single negation of the set of formulas of the form $\exists\xi$ in $\mathit{sub}(\varphi, \psi) = \mathit{sub}(\varphi) \cup \mathit{sub}(\psi)$. The *world-type* of $w \in W$ in a model $\mathfrak{M} = (W, D, I)$ is defined as

$$\mathsf{wt}_\mathfrak{M}(w) = \{\rho \in \mathit{sub}_\exists(\varphi, \psi) \mid \mathfrak{M}, w \models \rho\}.$$

A *world-type*, wt, in $\mathfrak{M}$ is the world-type of some $w \in W$.

Similarly, let $\mathit{sub}_\Diamond(\varphi, \psi)$ be the closure under single negation of the set of formulas of the form $\Diamond\xi$ in $\mathit{sub}(\varphi, \psi)$. The *domain-type* of $d \in D$ in $\mathfrak{M}$ is the set

$$\mathsf{dt}_\mathfrak{M}(d) = \{\rho \in \mathit{sub}_\Diamond(\varphi, \psi) \mid \mathfrak{M}, d \models \rho\}.$$

A *domain-type*, dt, in $\mathfrak{M}$ is the domain-type of some $d \in D$.

The *full type* of $(w, d) \in W \times D$ in $\mathfrak{M}$ is the set

$$\mathsf{ft}_\mathfrak{M}(w, d) = \{\rho \in \mathit{sub}(\varphi, \psi) \mid \mathfrak{M}, w, d \models \rho\}.$$

A *full type*, ft, in $\mathfrak{M}$ is the full type of some (w, d) in $\mathfrak{M}$.

The main result of this section generalises the following construction that shows how, given any $\mathsf{Q}^1\mathsf{S5}$-model $\mathfrak{M}$ satisfying a formula φ, we can construct from the world and domain types in $\mathfrak{M}$ a model $\mathfrak{M}'$ satisfying φ and having exponential size in $|\varphi|$. Intuitively, as a first approximation, we could start by taking the worlds W' (domain D') in $\mathfrak{M}'$ to comprise all the world- (domain-) types in $\mathfrak{M}$. But then we might have w, w' and d, d' with $\mathsf{wt}_\mathfrak{M}(w) = \mathsf{wt}_\mathfrak{M}(w')$, $\mathsf{dt}_\mathfrak{M}(d) = \mathsf{dt}_\mathfrak{M}(d')$ and different truth-values of some variables $\boldsymbol{p}$ at (w, d) and (w', d') in $\mathfrak{M}$. To deal with this issue, we introduce, as shown in the example below, sufficiently many copies of each world- and domain-type so that we can accommodate *all* possible truth-values in $\mathfrak{M}$ of the $\boldsymbol{p}$ in φ.

Example 12. Let $\mathfrak{M}, w, d \models \varphi$, for $\mathfrak{M} = (W, D, I)$, and let n be the number of full types in $\mathfrak{M}$ (over $\mathit{sub}(\varphi)$) and $[n] = \{1, \dots, n\}$. Define D' to be a set that contains n distinct copies of each dt in $\mathfrak{M}$ over $\mathit{sub}_\Diamond(\varphi)$, denoting the kth copy by dt^k. For any wt and dt in $\mathfrak{M}$, let $\pi_{\mathsf{wt},\mathsf{dt}}$ be a function from $[n]$ *onto* the set of full types ft in $\mathfrak{M}$ with $\mathsf{wt} = \mathsf{ft} \cap \mathit{sub}_\exists(\varphi)$ and $\mathsf{dt} = \mathsf{ft} \cap \mathit{sub}_\Diamond(\varphi)$. Let Π be a smallest set of sequences π of such $\pi_{\mathsf{wt},\mathsf{dt}}$ for which the following condition holds: for any $\mathsf{ft} = \mathsf{ft}_\mathfrak{M}(u, e)$ and $k \in [n]$, there exists $\pi \in \Pi$ with $\pi_{\mathsf{wt}_\mathfrak{M}(u), \mathsf{dt}_\mathfrak{M}(e)}(k) = \mathsf{ft}$. We then set $W' = \{\mathsf{wt}^\pi_\mathfrak{M}(u) \mid u \in W,\ \pi \in \Pi\}$, treating each $\mathsf{wt}^\pi_\mathfrak{M}(u)$ as a fresh π-copy of $\mathsf{wt}_\mathfrak{M}(u)$. As $|\Pi| \leq n^2$, both $|W'|$ and $|D'|$

are exponential in $|\varphi|$. Define a model $\mathfrak{M}' = (W', D', I')$ by taking $\mathfrak{M}', \mathsf{wt}^\pi, \mathsf{dt}^k \models \boldsymbol{p}$ iff $\boldsymbol{p} \in \pi_{\mathsf{wt},\mathsf{dt}}(k)$. One can show by induction that $\mathfrak{M}', \mathsf{wt}^\pi, \mathsf{dt}^k \models \rho$ iff $\rho \in \pi_{\mathsf{wt},\mathsf{dt}}(k)$, for any $\rho \in \mathit{sub}(\varphi)$; see the full paper for details. $\dashv$

We now introduce more complex 'data structures' that allow us to extend the construction above from satisfiability to σ-bisimulation consistency. Let $\mathfrak{M}_i = (W_i, D_i, I_i)$, for $i = 1, 2$, be two $\mathsf{Q^1S5}$-models with pairwise disjoint W_i and D_i. For any $w \in W_1 \cup W_2$ and $i \in \{1, 2\}$, we set

$$T_i(w) = \{\mathsf{wt}_{\mathfrak{M}_i}(v) \mid v \in W_i,\ v \sim_\sigma w\} \tag{1}$$

and call $\mathsf{wm}(w) = (T_1(w), T_2(w))$ the *world mosaic* of w in $\mathfrak{M}_1, \mathfrak{M}_2$. The pair $\mathsf{wp}_i(w) = (\mathsf{wt}_{\mathfrak{M}_i}(w), \mathsf{wm}(w))$, for $w \in W_i$, is called the *i-world point* of w in $\mathfrak{M}_1$, $\mathfrak{M}_2$. A *world mosaic*, wm, and an *i-world point*, wp_i, in $\mathfrak{M}_1, \mathfrak{M}_2$ are defined as the world mosaic and i-world point of some $w \in W_1 \cup W_2$ in $\mathfrak{M}_1, \mathfrak{M}_2$ (in the latter case, $w \in W_i$).

Similarly, for any $d \in D_1 \cup D_2$ and $i \in \{1, 2\}$, we set

$$S_i(d) = \{\mathsf{dt}_{\mathfrak{M}_i}(e) \mid e \in D_i,\ e \sim_\sigma d\} \tag{2}$$

and call $\mathsf{dm}(d) = (S_1(d), S_2(d))$ the *domain mosaic* of d in $\mathfrak{M}_1$, $\mathfrak{M}_2$. If $d \in D_i$, the pair $\mathsf{dp}_i(d) = (\mathsf{dt}_{\mathfrak{M}_i}(d), \mathsf{dm}(d))$ is called the *i-domain point* of d in $\mathfrak{M}_1$, $\mathfrak{M}_2$. A *domain mosaic*, dm, and an *i-domain point*, dp_i, in $\mathfrak{M}_1$, $\mathfrak{M}_2$ are defined as the domain mosaic and i-domain point of some $d \in D_1 \cup D_2$. As follows from the definitions and Lemma 3 coupled with Theorem 10,

(wm) $u \sim_\sigma v$ implies $\mathsf{wm}(u) = \mathsf{wm}(v)$,

(dm) $d \sim_\sigma e$ implies $\mathsf{dm}(d) = \mathsf{dm}(e)$.

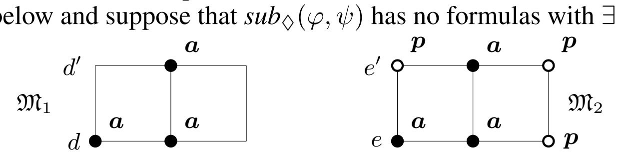
Observe that the number of distinct wp_i and dp_i is at most double-exponential in $|\varphi|$ and $|\psi|$.

Example 13. (a) Consider models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ from Example 6, $\sigma = \{\boldsymbol{e}\}$ and $\tau = \{\boldsymbol{e}, \boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{b}_0, \boldsymbol{b}_1\}$. Then $\mathsf{wt}_{\mathfrak{M}_1}(u_i)$ and $\mathsf{dt}_{\mathfrak{M}_2}(c_i)$ contain, respectively, the sets

$$\{\exists(\boldsymbol{p}_i \wedge \boldsymbol{e})\} \cup \{\exists\neg\boldsymbol{p} \mid \boldsymbol{p} \in \tau\} \cup \{\neg\exists\boldsymbol{p}_j \mid j \neq i\},$$
$$\{\Diamond(\boldsymbol{p}_i \wedge \boldsymbol{e})\} \cup \{\Diamond\neg\boldsymbol{p} \mid \boldsymbol{p} \in \tau\} \cup \{\neg\Diamond\boldsymbol{p}_j \mid j \neq i\}.$$

From the σ-bisimulations shown in Example 11 we obtain $\mathsf{wm}(u_0) = \mathsf{wm}(u_1) = \mathsf{wm}(u_2) = \mathsf{wm}(v_0) = \mathsf{wm}(v_1)$, and so $\mathfrak{M}_1, \mathfrak{M}_2$ have only one world mosaic: $\mathsf{wm} = (\{\mathsf{wt}_{\mathfrak{M}_1}(u_i) \mid i = 0,1,2\}, \{\mathsf{wt}_{\mathfrak{M}_2}(v_i) \mid i = 0,1\})$. $\mathfrak{M}_1$ has three distinct 1-world points $(\mathsf{wt}_{\mathfrak{M}_1}(u_i), \mathsf{wm})$, for $i = 0, 1, 2$; $\mathfrak{M}_2$ has two 2-world points. Similarly, $\mathfrak{M}_1, \mathfrak{M}_2$ define one domain mosaic, $\mathfrak{M}_1$ has three distinct 1-domain points and $\mathfrak{M}_2$ has two 2-domain points.

(b) It can happen that non-bisimilar domain elements give the same domain-point. Consider the models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ below and suppose that $\mathit{sub}_\Diamond(\varphi, \psi)$ has no formulas with $\exists$



in the scope of $\Diamond$, $\sigma = \{\boldsymbol{a}\}$ and $\mathit{sig}(\varphi, \psi) = \{\boldsymbol{a}, \boldsymbol{p}\}$. Then $\mathsf{dt}_{\mathfrak{M}_1}(d) = \mathsf{dt}_{\mathfrak{M}_1}(d')$ but $d \not\sim_\sigma d'$ as $\Diamond(\boldsymbol{a} \wedge \exists\neg\boldsymbol{a})$ is true at d and false at d'; likewise, $\mathsf{dt}_{\mathfrak{M}_2}(e) = \mathsf{dt}_{\mathfrak{M}_2}(e')$ but $e \not\sim_\sigma e'$. Since $d \sim_\sigma e$ and $d' \sim_\sigma e'$, we have $\mathsf{dm}(d) = (\{\mathsf{dt}_{\mathfrak{M}_1}(d)\}, \{\mathsf{dt}_{\mathfrak{M}_2}(e)\}) = \mathsf{dm}(e)$, $\mathsf{dm}(d') = (\{\mathsf{dt}_{\mathfrak{M}_1}(d')\}, \{\mathsf{dt}_{\mathfrak{M}_2}(e')\}) = \mathsf{dm}(e')$, $\mathsf{dp}_1(d) = \mathsf{dp}_1(d')$, and $\mathsf{dp}_2(e) = \mathsf{dp}_2(e')$. $\dashv$

Suppose $\mathfrak{M}_1, w_1, d_1 \sim_\sigma \mathfrak{M}_2, w_2, d_2$, $\mathfrak{M}_1, w_1, d_1 \models \varphi$ and $\mathfrak{M}_2, w_2, d_2 \not\models \psi$. We construct $\mathfrak{M}'_i = (W'_i, D'_i, I'_i)$, $i = 1, 2$, witnessing σ-bisimulation consistency of φ and $\neg\psi$ and having at most double-exponential size in $|\varphi|$ and $|\psi|$. Intuitively, W'_i and D'_i consist of copies of the i-world and, respectively, i-domain points in $\mathfrak{M}_1, \mathfrak{M}_2$ rather than copies of the world- and domain-types as in Example 12. Then we obtain the required σ-bisimulation $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ by including in $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ exactly those 1/2-world and, respectively, 1/2-domain points that share the same world and domain mosaic.

Let n be the number of full types over $\mathit{sub}(\varphi, \psi)$ and let $[n] = \{1, \dots, n\}$. For $i = 1, 2$, we set

$$D'_i = \{\mathsf{dp}_i^k \mid \mathsf{dp}_i \text{ an } i\text{-domain point in } \mathfrak{M}_1, \mathfrak{M}_2,\ k \in [n]\},$$

treating dp_i^k as the kth copy of dp_i and assuming all of the copies to be distinct. Next, we define W'_i, $i = 1, 2$, using *surjective* functions of the form

$$\pi_{\mathsf{wp}_i,\mathsf{dp}_i} \colon [n] \to \{\mathsf{ft}_{\mathfrak{M}_i}(w, d) \mid (w, d) \in W_i \times D_i,\ \mathsf{wp}_i = \mathsf{wp}_i(w), \mathsf{dp}_i = \mathsf{dp}_i(d)\}.$$

Observe that, for any $\mathsf{wp}_i = (\mathsf{wt}, \mathsf{wm})$, $\mathsf{dp}_i = (\mathsf{dt}, \mathsf{dm})$, and $k \in [n]$, we have $\mathsf{wt} = \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) \cap \mathit{sub}_\exists(\varphi, \psi)$ and $\mathsf{dt} = \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) \cap \mathit{sub}_\Diamond(\varphi, \psi)$.

Let Π be a smallest set of sequences π of $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}$ such that, for any $\mathsf{ft} = \mathsf{ft}_{\mathfrak{M}_i}(w, d)$, $\mathsf{wp}_i = \mathsf{wp}_i(w)$, $\mathsf{dp}_i = \mathsf{dp}_i(d)$ with (w, d) in $\mathfrak{M}_i$ and any $k \in [n]$, there is $\pi \in \Pi$ with $\pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) = \mathsf{ft}$. Clearly, $|\Pi| \leq n^2$. Then we set

$$W'_i = \{\mathsf{wp}_i^\pi \mid \mathsf{wp}_i \text{ an } i\text{-world point in } \mathfrak{M}_1, \mathfrak{M}_2,\ \pi \in \Pi\},$$

treating wp_i^π as a fresh π-copy of wp_i. Clearly, both $|D'_i|$ and $|W'_i|$ are double-exponential in $|\varphi|$, $|\psi|$. Finally, we set

$$\mathfrak{M}'_i, \mathsf{wp}_i^\pi, \mathsf{dp}_i^k \models \boldsymbol{p} \quad \text{iff} \quad \boldsymbol{p} \in \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k) \tag{3}$$

and define $\boldsymbol{\beta}_1 \subseteq W'_1 \times W'_2$ and $\boldsymbol{\beta}_2 \subseteq D'_1 \times D'_2$ by taking $\boldsymbol{\beta}_1(\mathsf{wp}_1^{\pi^1}, \mathsf{wp}_2^{\pi^2})$ iff $\mathsf{wm}_1 = \mathsf{wm}_2$, where $\mathsf{wp}_i = (\mathsf{wt}_i, \mathsf{wm}_i)$, for $i = 1, 2$; and $\boldsymbol{\beta}_2(\mathsf{dp}_1^{k_1}, \mathsf{dp}_2^{k_2})$ iff $\mathsf{dm}_1 = \mathsf{dm}_2$, where $\mathsf{dp}_i = (\mathsf{dt}_i, \mathsf{dm}_i)$, for $i = 1, 2$.

Lemma 14. *(i) $\mathfrak{M}'_i, \mathsf{wp}_i^\pi, \mathsf{dp}_i^k \models \rho$ iff $\rho \in \pi_{\mathsf{wp}_i,\mathsf{dp}_i}(k)$, for every $\rho \in \mathit{sub}(\varphi, \psi)$. (ii) The pair $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ is a σ-bisimulation between $\mathfrak{M}'_1$ and $\mathfrak{M}'_2$.*

The construction and lemmas above yield the following:

Theorem 15. *Any formulas φ and ψ are $\mathit{sig}(\varphi) \cap \mathit{sig}(\psi)$-bisimulation consistent in $\mathsf{Q^1S5}$ iff there are witnessing $\mathsf{Q^1S5}$-models of size double-exponential in $|\varphi|$ and $|\psi|$.*

Theorems 15, 5 and 4 give the upper bound of

Theorem 16. *(i) Both IEP and EDEP for $\mathsf{Q^1S5}$ are decidable in* coN2ExpTime.

(ii) IEP and EDEP for $\mathsf{Q^1S5}$ are both 2ExpTime-hard.

The lower bound results hold even if we want to decide, for any FOM^1-formulas φ and ψ, whether an interpolant or an explicit definition exists not only in $\mathsf{Q}^1\mathsf{S5}$ but in any finite-variable fragment of quantified $\mathsf{S5}$.

The lower bounds are established in the full paper. Here, we only comment on the intuition behind the proof. Given a 2^n-space bounded alternating Turing Machine M and an input word $\overline{a}$ of length n, we construct in polytime formulas φ and ψ such that $\models_{\mathsf{Q}^1\mathsf{S5}} \varphi \to \psi$ and M accepts $\overline{a}$ iff $\varphi, \neg\psi$ are σ-bisimulation consistent, where $\sigma = sig(\varphi) \cap sig(\psi)$.

One aspect of our construction is similar to that of (Artale et al. 2021; Jung and Wolter 2021): we also represent accepting computation-trees as binary trees whose nodes are coloured by predicates in σ. However, unlike the formalisms in the cited work, $\mathsf{Q}^1\mathsf{S5}$ cannot express the uniqueness of properties, and so the remaining ideas are novel. One part of φ 'grows' 2^n-many copies of σ-coloured binary trees, using a technique from 2D propositional modal logic (Hodkinson et al. 2003; Göller, Jung, and Lohrey 2015). Another part of φ colours the tree-nodes with non-σ-symbols to ensure that, in the mth tree, for each $m < 2^n$, the content of the mth tape-cell is properly changing during the computation. Then we use ideas from Example 6 to make sure that the generated 2^n-many trees are all σ-bisimilar, and so represent the same accepting computation-tree.

The following corollary is also proved in the full paper:

Corollary 17. *IEP and EDEP for* FO^2 *without equality are both* 2EXPTIME*-hard.*

5 (S)CEP and UIEP in $\mathsf{Q}^1\mathsf{S5}$: Undecidability

We now turn to the (strong) conservative extension and uniform interpolant existence problems, which, in contrast to interpolant existence, turn out to be undecidable.

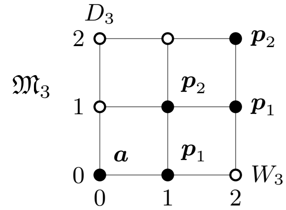
Theorem 18. (i) *(S)CEP in* $\mathsf{Q}^1\mathsf{S5}$ *is undecidable.*
(ii) *UIEP in* $\mathsf{Q}^1\mathsf{S5}$ *is undecidable.*

The undecidability proof for CEP is by adapting an undecidability proof for CEP of FO^2 in (Jung et al. 2017). The main new idea is the generation of arbitrary large binary trees within $\mathsf{Q}^1\mathsf{S5}$-models that can then be forced to be grids in case one does not have a (strong) conservative extension. The undecidability proof for UIEP merges a counterexample to UIP with the formulas constructed to prove undecidability of CEP. Here we provide the counterexample to UIP, details of the proofs are given in the full paper.

Example 19. Let $\sigma = \{\boldsymbol{a}, \boldsymbol{p}_1, \boldsymbol{p}_2\}$ and

$$\varphi_0 = \Box\forall\big(\boldsymbol{a} \to \Diamond(\boldsymbol{p}_1 \wedge \boldsymbol{b})\big) \wedge \Box\forall\big(\boldsymbol{p}_1 \wedge \boldsymbol{b} \to \exists(\boldsymbol{p}_2 \wedge \boldsymbol{b})\big) \wedge \Box\forall\big(\boldsymbol{p}_2 \wedge \boldsymbol{b} \to \Diamond(\boldsymbol{p}_1 \wedge \boldsymbol{b})\big).$$

To show that $\boldsymbol{a} \wedge \varphi_0$ has no σ-uniform interpolant in $\mathsf{Q}^1\mathsf{S5}$, for every positive $r < \omega$, we define a formula χ_r inductively by taking $\chi_0 = \top$ and $\chi_{r+1} = \boldsymbol{p}_1 \wedge \exists(\boldsymbol{p}_2 \wedge \Diamond\chi_r)$. Then $\models_{\mathsf{Q}^1\mathsf{S5}} \boldsymbol{a} \wedge \varphi_0 \to \Diamond\chi_r$ for all $r > 0$. Thus, if ϱ were a σ-uniform interpolant of $\boldsymbol{a} \wedge \varphi_0$, then $\models_{\mathsf{Q}^1\mathsf{S5}} \varrho \to \Diamond\chi_r$ would follow for all $r > 0$. Consider a model $\mathfrak{M}_r = (W_r, D_r, I_r)$ with $W_r = D_r = \{0, \dots, r-1\}$, in which $\boldsymbol{a}$ is true at $(0, 0)$, $\boldsymbol{p}_1$ at $(k, k-1)$, and $\boldsymbol{p}_2$ at (k, k), for $0 < k < r$, as illustrated in the picture below.



$\mathfrak{M}_3, 0, 0 \models \chi_2$
$\mathfrak{M}_3, 0, 0 \not\models \chi_3$

Then $\mathfrak{M}_r, 0, 0 \not\models \Diamond\chi_r$, for any $r > 0$, and so $\mathfrak{M}_r, 0, 0 \not\models \varrho$. On the other hand, $\mathfrak{M}_r, 0, 0 \models \Diamond\chi_{r'}$ for all $r' < r$. Now consider the ultraproduct $\prod_U \mathfrak{M}_r$ with a non-principal ultrafilter U on $\omega \setminus \{0\}$. As each $\Diamond\chi_{r'}$ is true at $(0, 0)$ in almost all $\mathfrak{M}_r$, it follows from the properties of ultraproducts (Chang and Keisler 1998) that, for a suitable $\overline{0}$ and all $r > 0$, we have $\prod_U \mathfrak{M}_r, \overline{0}, \overline{0} \models \boldsymbol{a} \wedge \neg\varrho \wedge \Diamond\chi_r$. One can interpret $\boldsymbol{b}$ in $\prod_U \mathfrak{M}_r$ so that $\mathfrak{M}, \overline{0}, \overline{0} \models \varphi_0$ for the resulting model $\mathfrak{M}$. Then $\mathfrak{M} \models \boldsymbol{a} \wedge \varphi_0 \wedge \neg\varrho$, contrary to the fact that $\models_{\mathsf{Q}^1\mathsf{S5}} \boldsymbol{a} \wedge \varphi_0 \to \varrho$ for any uniform interpolant ϱ of $\boldsymbol{a} \wedge \varphi_0$.

Remark 20. Example 19 can be translated into FO^2 to prove that the latter does not have UIP. It can then be merged with the proof of undecidability of CEP in FO^2 from (Jung et al. 2017)—in the same way as we combined Example 19 with the undecidability proof for UIEP in $\mathsf{Q}^1\mathsf{S5}$—to show that UIEP is undecidable in FO^2 (with and without $=$). The latter problem has so far remained open.

6 Modal Description Logic $\mathsf{S5}_{\mathcal{ALC}^u}$

Next, we extend the results of Sections 4, 5 to the description modal logic $\mathsf{S5}_{\mathcal{ALC}^u}$, where $\mathcal{ALC}^u$ is the basic description logic $\mathcal{ALC}$ with the universal role (Baader et al. 2017), which is a notational variant of multimodal K with the universal modality (and can be regarded as a fragment of FO^2).

The *concepts* of $\mathsf{S5}_{\mathcal{ALC}^u}$ are constructed from *concept names* $A \in \mathcal{C}$, *role names* $R \in \mathcal{R}$, for some countably-infinite and disjoint sets $\mathcal{C}$ and $\mathcal{R}$, and a distinguished *universal role* $U \in \mathcal{R}$ by means of the following grammar:

$$C, D \;:=\; A \mid \top \mid C \sqcap D \mid \neg C \mid \exists R.C \mid \exists U.C \mid \Diamond C.$$

A *signature* σ is any finite set of concept and role names. The signature $sig(C)$ of a concept C comprises the concept and role names in C. We interpret $\mathsf{S5}_{\mathcal{ALC}^u}$ in models $\mathfrak{M} = (W, \Delta, I)$, where $I(w)$ is an interpretation of the concept and role names at each world $w \in W$ over domain $\Delta \neq \emptyset$: $A^{I(w)} \subseteq \Delta$, $R^{I(w)} \subseteq \Delta \times \Delta$, and $U^{I(w)} = \Delta \times \Delta$. The *truth-relation* $\mathfrak{M}, w, d \models C$ is defined by taking

- $\mathfrak{M}, w, d \models \top$, $\quad \mathfrak{M}, w, d \models A$ iff $d \in A^{I(w)}$,
- $\mathfrak{M}, w, d \models \exists R.C$ iff there is $(d, d') \in R^{I(w)}$ such that $\mathfrak{M}, w, d' \models C$,
- $\mathfrak{M}, w, d \models \Diamond C$ iff there is $w' \in W$ with $\mathfrak{M}, w', d \models C$,

and standard clauses for Boolean $\sqcap$, $\neg$. We sometimes use more conventional $C^{I(w)} = \{d \in \Delta \mid \mathfrak{M}, w, d \models C\}$, writing $\mathfrak{M}, w \models C \sqsubseteq D$ if $C^{I(w)} \subseteq D^{I(w)}$, and $\models C \sqsubseteq D$ if $\mathfrak{M}, w \models C \sqsubseteq D$ for all $\mathfrak{M}$ and w. The problem of deciding if $\models C \sqsubseteq D$, for given C and D, is CONEXPTIME-complete (Gabbay et al. 2003).

Typical applications of description logics use reasoning modulo *ontologies*—finite sets $\mathcal{O}$ of *concept inclusions*

(CIs) $C' \sqsubseteq D'$ regarded as axioms—by taking $\mathcal{O} \models C \sqsubseteq D$ iff whenever $\mathfrak{M}, w \models \alpha$ for all $\alpha \in \mathcal{O}$ then $\mathfrak{M}, w \models C \sqsubseteq D$. Reasoning modulo ontologies is reducible to the ontology-free case by the following equivalence: $\mathcal{O} \models C \sqsubseteq D$ iff $\models \top \sqsubseteq \bigsqcup_{C' \sqsubseteq D' \in \mathcal{O}} \exists U.(C' \sqcap \neg D') \sqcup \forall U.(\neg C \sqcup D)$.

An *interpolant for* $C \sqsubseteq D$ *in* $\mathsf{S5}_{\mathcal{ALC}^u}$ is a concept E such that $sig(E) \subseteq sig(C) \cap sig(D)$, $\models C \sqsubseteq E$, and $\models E \sqsubseteq D$. The IEP for $\mathsf{S5}_{\mathcal{ALC}^u}$ is to decide whether a given concept inclusion $C \sqsubseteq D$ has an interpolant in $\mathsf{S5}_{\mathcal{ALC}^u}$. The following related problems can easily be reduced to IEP in polytime:

(IEP modulo ontologies) Given an ontology $\mathcal{O}$, a signature σ, and a CI $C \sqsubseteq D$, does there exist a σ-concept E such that $\mathcal{O} \models C \sqsubseteq E$ and $\mathcal{O} \models E \sqsubseteq D$?

(ontology interpolant existence, OIEP) Given an ontology $\mathcal{O}$, a signature σ, and a CI $C \sqsubseteq D$, is there an ontology $\mathcal{O}'$ with $sig(\mathcal{O}') \subseteq \sigma$, $\mathcal{O} \models \mathcal{O}'$, and $\mathcal{O}' \models C \sqsubseteq D$?

(EDEP modulo ontologies) Given an ontology $\mathcal{O}$, a signature σ, and a concept name A, does there exist a concept C such that $sig(C) \subseteq \sigma$ and $\mathcal{O} \models A \equiv C$?

(See Example 2 for an illustration.) Explicit definitions have been proposed for query rewriting in ontology-based data access (Franconi, Kerhet, and Ngo 2013; Toman and Weddell 2021), developing and maintaining ontology alignments (Geleta, Payne, and Tamma 2016), and ontology engineering (ten Cate et al. 2006). IEP is fundamental for robust modularisations and decompositions of ontologies (Konev et al. 2009; Botoeva et al. 2016).

Our main result in this section is the following:

Theorem 21. *IEP, EDEP* (*modulo ontologies*)*, and OIEP are decidable in* coN2ExpTime*, being* 2ExpTime*-hard.*

The detailed proof is given in the full paper. Here, we only formulate a model-theoretic characterisation of interpolant existence in $\mathsf{S5}_{\mathcal{ALC}^u}$ in terms of the following generalisation of σ-bisimulations for $\mathsf{Q}^1\mathsf{S5}$ from Section 4.

A σ-*bisimulation* between models $\mathfrak{M}_i = (W_i, \Delta_i, I_i)$, $i = 1, 2$, is any triple $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta})$ with $\boldsymbol{\beta}_1 \subseteq W_1 \times W_2$, $\boldsymbol{\beta}_2 \subseteq \Delta_1 \times \Delta_2$, and $\boldsymbol{\beta} \subseteq (W_1 \times \Delta_1) \times (W_2 \times \Delta_2)$ if

(w) for any $(w_1, w_2) \in \boldsymbol{\beta}_1$ and $d_1 \in \Delta_1$, there is $d_2 \in \Delta_2$ with $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$ and similarly for $d_2 \in \Delta_2$,

(d) for any $(d_1, d_2) \in \boldsymbol{\beta}_2$ and $w_1 \in W_1$, there is $w_2 \in W_2$ with $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$ and similarly for $w_2 \in W_2$,

(c) $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$ implies both $(w_1, w_2) \in \boldsymbol{\beta}_1$ and $(d_1, d_2) \in \boldsymbol{\beta}_2$,

and the following hold for all $((w_1, d_1), (w_2, d_2)) \in \boldsymbol{\beta}$:

(a) $\mathfrak{M}_1, w_1, d_1 \models A$ iff $\mathfrak{M}_2, w_2, d_2 \models A$, for all $A \in \sigma$;

(r) if $(d_1, e_1) \in R^{I(w_1)}$ and $R \in \sigma$, then there is $e_2 \in \Delta_2$ with $(d_2, e_2) \in R^{I(w_2)}$ and $((w_1, e_1), (w_2, e_2)) \in \boldsymbol{\beta}$, and the other way round.

The criterion below—in which σ-bisimulation consistency is defined as in Section 3 with concepts C, D in place of formulas φ, ψ—is an $\mathsf{S5}_{\mathcal{ALC}^u}$-analogue of Theorem 5:

Theorem 22. *The following conditions are equivalent for any concept inclusion* $C \sqsubseteq D$:

– *there does not exist an interpolant for* $C \sqsubseteq D$ *in* $\mathsf{S5}_{\mathcal{ALC}^u}$;

– C *and* $\neg D$ *are* $sig(C) \cap sig(D)$*-bisimulation consistent.*

We then extend the 'filtration' construction of Section 4 from $\mathsf{Q}^1\mathsf{S5}$ to $\mathsf{S5}_{\mathcal{ALC}^u}$. In contrast to $\mathsf{Q}^1\mathsf{S5}$, we now have to deal with non-trivial σ-bisimulations between the respective $\mathcal{ALC}$-models $I(w_1)$ and $I(w_2)$ (satisfying conditions $(\mathbf{a})$ and $(\mathbf{r})$). To this end we introduce *full mosaics* (sets of full types realised in σ-bisimilar pairs (w, d)) and *full points* (full mosaics with a distinguished full type). The range of the surjections π used to construct W' and D' then consists of full points rather than full types. This provides us with the data structure to define σ-bisimilar $\mathcal{ALC}$-models $I(w)$ when required. This construction establishes an upper bound on the size of models witnessing bisimulation consistency:

Theorem 23. *Any concepts C and D do not have an interpolant in* $\mathsf{S5}_{\mathcal{ALC}^u}$ *iff there are witnessing* $\mathsf{S5}_{\mathcal{ALC}^u}$*-models of size double-exponential in* $|C|$ *and* $|D|$.

This result gives the upper bound of Theorem 21. The lower one follows from Theorem 16 (ii) as, treating FOM^1-formulas φ, ψ as role-free $\mathsf{S5}_{\mathcal{ALC}^u}$-concepts and using Theorems 5 and 22, one can readily show that φ and ψ have an interpolant in $\mathsf{Q}^1\mathsf{S5}$ iff they have an interpolant in $\mathsf{S5}_{\mathcal{ALC}^u}$.

The (strong) conservative extension problem, (S)CEP, and the uniform interpolant existence problem, UIEP, in $\mathsf{S5}_{\mathcal{ALC}^u}$ are defined in the obvious way. Using the same argument as for interpolation, the undecidability of (S)CEP and UIEP in $\mathsf{S5}_{\mathcal{ALC}^u}$ follows directly from the undecidability of both problems for $\mathsf{Q}^1\mathsf{S5}$. Note that, for the component logics—propositional $\mathsf{S5}$ and $\mathcal{ALC}^u$—CEP is coNExpTime and 2ExpTime-complete, respectively (Ghilardi et al. 2006; Jung et al. 2017).

7 Quantified Modal Logic $\mathsf{Q}^1\mathsf{K}$

Finally, we consider the one-variable quantified modal logic $\mathsf{Q}^1\mathsf{K}$. By the *modal depth* $md(\varphi)$ of a FOM^1-formula φ we mean the maximal number of nestings of $\Diamond$ in φ; if φ has no modal operators, then $md(\varphi) = 0$. Formulas of modal depth k can be characterised using a finitary version of bisimulations, called k-bisimulations, defined below.

For a signature σ and two models $\mathfrak{M} = (W, R, D, I)$ and $\mathfrak{M}' = (W', R', D', I')$, a sequence $\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_k$ of relations $\boldsymbol{\beta}_i \subseteq (W \times D) \times (W' \times D')$ is a σ-k-*bisimulation* between $\mathfrak{M}$ and $\mathfrak{M}'$ if the following conditions hold for all $\boldsymbol{p} \in \sigma$ and $((w, d), (w', d')) \in \boldsymbol{\beta}_i$: **(a)**, **(d)** from Section 2 as well as

(w′) if $i > 0$, $(w, v) \in R$, then there is v' with $(w', v') \in R'$ and $((v, d), (v', d')) \in \boldsymbol{\beta}_{i-1}$, and the other way round.

We say that $\mathfrak{M}, w, d$ and $\mathfrak{M}', w', d'$ are σ-k-*bisimilar* and write $\mathfrak{M}, w, d \sim_\sigma^k \mathfrak{M}', w', d'$ if there is a σ-k-bisimulation $\boldsymbol{\beta}_0, \dots, \boldsymbol{\beta}_k$ with $\boldsymbol{\beta}_k \ni ((w, d), (w', d'))$ between $\mathfrak{M}$ and $\mathfrak{M}'$. We write $\mathfrak{M}, w, d \equiv_\sigma^k \mathfrak{M}', w', d'$ when $\mathfrak{M}, w, d \models \varphi$ iff $\mathfrak{M}', w', d' \models \varphi$, for every σ-formula φ with $md(\varphi) \leq k$.

We can define formulas $\tau^k_{\mathfrak{M},\sigma}$, generalising the characteristic formulas of (Goranko and Otto 2007), that describe every model $\mathfrak{M}$ up to σ-k-bisimulations in the sense that the following equivalences hold (see the full paper for details):

Lemma 24. *For any models $\mathfrak{M}$ with w, d and $\mathfrak{N}$ with v, e, and any $k < \omega$, the following conditions are equivalent*:

(i) $\mathfrak{N}, v, e \equiv^k_\sigma \mathfrak{M}, w, d$;

(ii) $\mathfrak{N}, v, e \models \tau^k_{\mathfrak{M},\sigma}(w, d)$;

(iii) $\mathfrak{N}, v, e \sim^k_\sigma \mathfrak{M}, w, d$.

Intuitively, $\tau^k_{\mathfrak{M},\sigma}(w, d)$ is the strongest formula of modal depth k that is true at w, d in $\mathfrak{M}$. For any formula φ with $md(\varphi) \leq k$, we now set

$$\exists^{\sim\sigma,k}\varphi \;=\; \bigvee_{\mathfrak{M},w,d\models\varphi} \tau^k_{\mathfrak{M},\sigma}(w, d).$$

Thus, for any $\mathfrak{N}, v, e$, we have $\mathfrak{N}, v, e \models \exists^{\sim\sigma,k}\varphi$ iff there is $\mathfrak{M}, w, d$ with $\mathfrak{M}, w, d \models \varphi$ and $\mathfrak{N}, v, e \sim^k_\sigma \mathfrak{M}, w, d$, i.e., $\exists^{\sim\sigma,k}$ is an existential depth restricted bisimulation quantifier (D'Agostino and Lenzi 2006; French 2006). Clearly, $\models_{\mathsf{Q}^1\mathsf{K}} \varphi \to \exists^{\sim\sigma,k}\varphi$.

Theorem 25. *The following conditions are equivalent, for any formula ψ with $md(\psi) = k'$ and $n = \max\{k, k'\}$:*

(a) *there is χ such that $sig(\chi) \subseteq \sigma$, $\models_{\mathsf{Q}^1\mathsf{K}} \varphi \to \chi$, and $\models_{\mathsf{Q}^1\mathsf{K}} \chi \to \psi$;*

(b) $\models_{\mathsf{Q}^1\mathsf{K}} \exists^{\sim\sigma,n}\varphi \to \psi$.

Proof. (a) $\Rightarrow$ (b) If $\not\models_{\mathsf{Q}^1\mathsf{K}} \exists^{\sim\sigma,n}\varphi \to \psi$, there is $\mathfrak{M}, w, d$ with $\mathfrak{M}, w, d \models \exists^{\sim\sigma,n}\varphi$ and $\mathfrak{M}, w, d \models \neg\psi$. By the definition of $\exists^{\sim\sigma,n}\varphi$, we then have $\mathfrak{M}, w, d \models \tau^n_{\mathfrak{M}',\sigma}(w', d')$ and $\mathfrak{M}', w', d' \models \varphi$, for some model $\mathfrak{M}'$, w', d'. By Lemma 24, $\mathfrak{M}', w', d' \sim^n_\sigma \mathfrak{M}, w, d$. Using a standard unfolding argument, we may assume that (W, R) in $\mathfrak{M}$ and (W', R') in $\mathfrak{M}'$ are tree-shaped with respective roots w, w'. As φ and ψ have modal depth $\leq n$, we may also assume that the depth of (W, R) and (W', R') is $\leq n$. But then $\mathfrak{M}', w', d' \sim_\sigma \mathfrak{M}, w, d$, contrary to (a). The implication (b) $\Rightarrow$ (a) is trivial. $\dashv$

We do not know whether $\exists^{\sim\sigma,k}\varphi$ is equivalent to a formula whose size can be bounded by an elementary function in $|\sigma|$, $|\varphi|$, k. For pure $\mathcal{ALC}$, it is indeed equivalent to an exponential-size concept (ten Cate et al. 2006).

Condition (b) in Theorem 25 gives an obvious non-elementary algorithm for checking whether given formulas have an interpolant in $\mathsf{Q}^1\mathsf{K}$. Thus, by Theorem 4, we obtain:

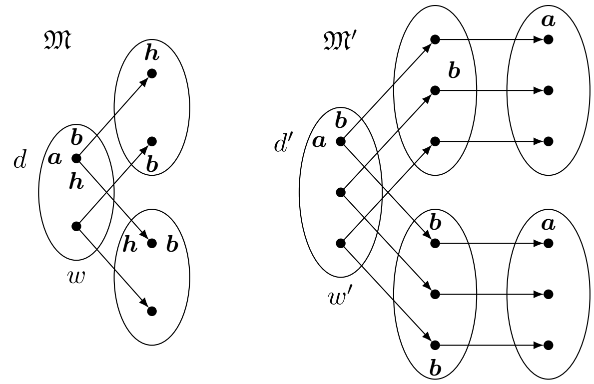
Theorem 26. *IEP and EDEP for $\mathsf{Q}^1\mathsf{K}$ are decidable in non-elementary time.*

The proof above seems to give a hint that UIEP for $\mathsf{Q}^1\mathsf{K}$ might also be decidable as (an analogue of) $\exists^{\sim\sigma,k}\varphi$ of modal depth $md(\varphi)$ is a uniform interpolant of any propositional modal formula φ in K (Visser 1996). The next example illustrates why this is not the case for 'two-dimensional' $\mathsf{Q}^1\mathsf{K}$.

Example 27. Suppose $\sigma = \{\boldsymbol{a}, \boldsymbol{b}\}$,

$$\varphi = \forall\big((\boldsymbol{a} \leftrightarrow \boldsymbol{b} \leftrightarrow \boldsymbol{h}) \wedge (\boldsymbol{h} \leftrightarrow \Box\boldsymbol{h} \leftrightarrow \Diamond\boldsymbol{h})\big) \wedge \Diamond\forall(\boldsymbol{b} \leftrightarrow \boldsymbol{h}),$$
$$\psi = \forall(\boldsymbol{a} \leftrightarrow \Box\Box\boldsymbol{a} \leftrightarrow \Diamond\Diamond\boldsymbol{a}) \wedge \Box\Diamond\top \to \Diamond\forall(\boldsymbol{b} \leftrightarrow \Diamond\boldsymbol{a}).$$

One can check (see full paper) that $\models_{\mathsf{Q}^1\mathsf{K}} \varphi \to \psi$. However, $\not\models_{\mathsf{Q}^1\mathsf{K}} \exists^{\sim\sigma,1}\varphi \to \psi$ as, for $\mathfrak{M}$ and $\mathfrak{M}'$ below, $\mathfrak{M}, w, d \models \varphi$ and $\mathfrak{M}', w', d' \not\models \psi$ but $\mathfrak{M}, w, d \sim^1_\sigma \mathfrak{M}', w', d'$. $\dashv$



In fact, by adapting the undecidability proof for $\mathsf{Q}^1\mathsf{S5}$ we prove the following:

Theorem 28. (i) *(S)CEP for $\mathsf{Q}^1\mathsf{K}$ is undecidable.*
(ii) *UIEP for $\mathsf{Q}^1\mathsf{K}$ is undecidable.*

8 Outlook

Craig interpolation and Beth definability have been studied for essentially all logical systems, let alone those applied in KR, AI, verification and databases. In fact, one of the first questions typically asked about a logic L of interest is whether L has interpolants for *all* valid implications $\varphi \to \psi$. Some L enjoy this property, while others miss it. This paper and preceding (Jung and Wolter 2021; Artale et al. 2021) open a new, *non-uniform* perspective on interpolation/definability for the latter type of L by regarding formulas φ and ψ as input (say, coming from an application) and deciding whether they have an interpolant in L.

In the context of first-order modal logics, challenging open questions that arise from this work are: What is the tight complexity of IEP for $\mathsf{Q}^1\mathsf{S5}$ and $\mathsf{S5}_{\mathcal{ALC}^u}$? Is the non-elementary upper bound for IEP in $\mathsf{Q}^1\mathsf{K}$ optimal? Is IEP decidable for $\mathsf{K}_{\mathcal{ALC}^u}$? More generally, what happens if we replace $\mathsf{S5}$ and K by other standard modal logics, e.g., $\mathsf{S4}$, multimodal $\mathsf{S5}$, or the linear temporal logic *LTL*, and/or use in place of $\mathcal{ALC}^u$ other DLs or other decidable fragments of FO such as the guarded or two-variable fragment?

A different line of research is computing interpolants. For logics with CIP, this is typically done using resolution, tableau, or sequent calculi. A more recent approach is based on type-elimination known from complexity proofs for modal and guarded logics (Benedikt, ten Cate, and Vanden Boom 2016; ten Cate 2022). While no attempt has yet been made to use traditional methods for computing interpolants in logics without CIP, type elimination has been adapted to $\mathcal{ALC}$ with role inclusions (Artale et al. 2020) that does not have CIP. Rather than eliminating types, one eliminates pairs of sets of types—i.e., mosaics in our proofs above. The question whether these proofs can be turned into an algorithm computing interpolants in, say, $\mathsf{Q}^1\mathsf{S5}$ is non-trivial and open. More generally, one can try to develop calculi for the consequence relation '$\varphi \models \psi$ iff there are no $sig(\varphi) \cap sig(\psi)$-bisimilar models satisfying φ and $\neg\psi$' and use them to compute interpolants; see (Barwise and van Benthem 1999) for a model-theoretic account of such consequence relations for infinitary logics without CIP.

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