Credible Models of Belief Update

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Abstract

In this work, we address one important problem of Katsuno and Mendelzon update operators, that is to require that any updated belief base must entail any new input in a consistent way. This assumes that any situation can be updated into one satisfying that input, which is unrealistic. To solve this problem, we must relax either the success or the consistency principle. Each case leads to a distinct family of update operators, that we semantically characterize by plausibility relations over possible worlds, considering a credibility limit that aims to forbid unrealistic changes. We discuss in which cases one family is more adequate than the other one.

1 Introduction

The aim of Belief Change Theory (Alchourrón, Gärdenfors, and Makinson 1985; Gärdenfors 1988; Hansson 1999) is to provide a formal framework for understanding how an agent's beliefs evolve in response to new evidence.

Over the past 35 years, various operators have been proposed to handle different types of situations and evidence (Fermé and Hansson 2011; Fermé and Hansson 2018). At the heart of this theory lie belief revision operators (Alchourrón, Gärdenfors, and Makinson 1985; Gärdenfors 1988; Katsuno and Mendelzon 1991; Hansson 1999), which are designed to correct an agent's beliefs based on more reliable evidence.

Another important class of operators are update operators (Katsuno and Mendelzon 1992; Winslett 1988; Herzig and Rifi 1999; Boutilier 1995; Lang 2007). The difference between revision and update operators is that revision operators aim to correct an agent's beliefs, whereas update operators aim to incorporate the results of changes in the world, without assuming that the agent's previous beliefs were incorrect. This difference is often summarized as belief revision being concerned with changing beliefs in a static world, while update is concerned with the evolution of beliefs in a dynamic world.

Another distinction between revision and update is that revision involves a selection process, where we choose the most plausible models of the new evidence based on our current beliefs, while update involves a transition process, where we consider the transitions caused by the change in the new evidence for each model of our current beliefs.

However, while there have been many extensions proposed and studied for belief revision operators, such as syntax dependence (Hansson 1999), non-prioritized revision (Schlechta 1997; Makinson 1998; Hansson 1998; Fermé and Hansson 1999; Hansson et al. 2001; Booth et al. 2012), and iteration (Darwiche and Pearl 1997; Booth and Meyer 2006; Jin and Thielscher 2007; Konieczny and Pino Pérez 2008; Schwind, Konieczny, and Pérez 2022), belief update has not received much attention since its initial characterization by Katsuno and Mendelzon (Katsuno and Mendelzon 1992). There have been some criticisms and adaptations proposed in (Herzig and Rifi 1999; Boutilier 1995; Lang 2007), but the original proposal by Katsuno and Mendelzon remains the standard one.

In this work, we aim to raise what we believe is the most significant criticism of the Katsuno and Mendelzon characterization and propose a more convincing one.

The criticism is that the Katsuno and Mendelzon characterization assumes that any situation (possible world) can be reached from any initial situation, which is not a reasonable assumption as some transitions are simply not possible in the real world. For example, if we represent the state of a cup as being either broken or unbroken, we know that a cup can go from an unbroken state to a broken state very quickly, but the reverse is not possible: it is not possible to "unbreak" a cup. Therefore, if our belief is that the cup is broken, there is no credible future where it can become unbroken. Thus, in all possible worlds where the cup is broken, it is not possible to reach a plausible possible world where the cup is unbroken.

Katsuno and Mendelzon introduced the idea of using partial pre-orders, or in the limiting case, total pre-orders[1], to represent the plausibility relation associated with each possible world. This departs from revision operators, which associate a total pre-order to each belief base as a whole, allowing one to represent the various more or less plausible transitions from each possible world. However, the assumption that all worlds are reachable from the initial one is open to criticism.

In this work, we propose a solution to this issue by

[1]Note that using partial or total pre-orders is not the most important distinction between update and revision, since in (Katsuno and Mendelzon 1992) they also define an update using total pre-orders, and in (Benferhat, Lagrue, and Papini 2005) it is shown how to define revision operators using partial pre-orders.

limiting the set of reachable worlds from each possible world. We accomplish this by using a set of credible worlds to encode this limit, similar to what has been done for credibility-limited belief revision (Hansson et al. 2001; Booth et al. 2012; Booth et al. 2014).

We provide a characterisation of these operators, where we only need to remove the consistency postulate **(U3)** from the standard Katsuno and Mendelzon characterization. In the representation theorem, we consider the credibility-limit for each world to encode the set of credible/accessible worlds from that possible world.

Let us provide an illustrative example in order to show why the Katsuno-Mendelzon characterization is not satisfactory and why our model is required.

Example 1. *In a room, there is a cup that can be on the table or on the floor (t), it can be empty or not (e), and it can be broken or not (b). Our current beliefs about the cup are that it is on the table, not empty, and not broken, or it fell off the table, in which case it will be empty and may be either broken or intact. A broken cup is necessarily empty, and once broken, it cannot be restored to an unbroken state. Therefore, our beliefs are: $\varphi \equiv \beta_1 \vee \beta_2 \vee \beta_3$, where $\beta_1 = t \wedge \neg e \wedge \neg b$, $\beta_2 = \neg t \wedge e \wedge \neg b$, and $\beta_3 = \neg t \wedge e \wedge b$.*

Suppose a reliable friend who has left the room tells us that the cup is not empty ($\alpha = \neg e$). According to the Katsuno-Mendelzon characterization, we need to find the most plausible worlds that correspond to each possibility. It is straightforward to update β_1 with the new evidence α: since $\beta_1 \vdash \alpha$, we keep β_1. However, for β_2, a possible explanation is that someone refilled the cup while it was on the floor, which is strange but possible. This leads to $\beta_4 = \neg t \wedge \neg e \wedge \neg b$.

However, there is no feasible transition from a state where the cup is broken and therefore empty (i.e., β_3) to a state where the cup is not empty (i.e., $\neg e$). The Katsuno-Mendelzon model requires us to find some worlds that are plausible (reachable) from β_3, which is counterintuitive in this situation and can even lead to impossible scenarios. For example, if our initial belief is $\psi = (\neg t \wedge e \wedge b) \vee (t \wedge e \wedge b)$, where the cup is broken, it is difficult to accept that changes in the world could go from an empty, broken cup to a not-empty, unbroken cup. In this case, it may be more appropriate to acknowledge the impossibility and conclude that solving it requires a different type of change (such as revision).

So, one has to be aware that with our model, sometimes, one can reach an inconsistent belief base, even if the previous belief base and the new evidence are consistent. This will just express the fact that the new evidence is not conceivable from the previous belief base.

If one wants to avoid this case, we propose a second model where we ensure consistency, but whose semantics is different and seems to be more related to cases where we have less confidence in the new evidence, or to cases that correspond to anticipated predictions.

However, we believe that one should not be afraid of the possible inconsistent result that expresses the fact that our model of the world was wrong. We will discuss this case in more detail in the conclusion since this situation is the starting point for future work.

Below, in Section 2, we will first provide the required formal definitions and notations for this work. Then we will recall the classical Katsuno-Mendelzon model for update. In Section 3, we will present our generalized credibility-limited update model. In Section 4, we will provide a second characterization where we ensure the consistency of the result. We will provide some examples to illustrate the behavior of the corresponding operators in Section 5. Finally, the paper will conclude with Section 6, where we will discuss these operators and future work.

2 Preliminaries

Let $\mathcal{L}_\mathcal{P}$ be a propositional language built up from a finite set of propositional variables $\mathcal{P}$ and the usual connectives. The symbol $\bot$ (resp. $\top$) is the Boolean constant always false (resp. true). An interpretation (or world) is a mapping from $\mathcal{P}$ to $\{0, 1\}$. The set of all interpretations (or worlds) is denoted by Ω. $\vdash$ denotes logical entailment, $\equiv$ logical equivalence, and $[\![\varphi]\!]$ denotes the set of models of the formula φ. Given a set of worlds $E \subseteq \Omega$, we denote by γ_S any formula such that $[\![\gamma_S]\!] = S$. When S is a singleton set $S = \{\omega\}$, γ_S is abbreviated as γ_ω. Given a set of worlds F, an ordering $\leq$ over F and a set $E \subseteq F$, we denote by $\min(E, \leq)$ the set $\min(E, \leq) = \{\omega \in E \mid \forall \omega' \in E, \omega' \not< \omega\}$.

An update operator $\diamond$ is a mapping associating two formulae φ (the agent's belief base) and α (the incoming information / piece of evidence) with a new formula $\varphi \diamond \alpha$ (the updated belief base).

Definition 1 (KM update operator (Katsuno and Mendelzon 1992))**.** *An operator $\diamond$ is a* KM update operator *if for all formulae φ, ψ, α, β, the following conditions are satisfied:*

- **(U1)** $\varphi \diamond \alpha \vdash \alpha$
- **(U2)** *If $\varphi \vdash \alpha$, then $\varphi \diamond \alpha \equiv \varphi$*
- **(U3)** *If $\varphi \nvdash \bot$ and $\alpha \nvdash \bot$, then $\varphi \diamond \alpha \nvdash \bot$*
- **(U4)** *If $\varphi \equiv \psi$ and $\alpha \equiv \beta$ then $\varphi \diamond \alpha \equiv \psi \diamond \beta$*
- **(U5)** $(\varphi \diamond \alpha) \wedge \beta \vdash \varphi \diamond (\alpha \wedge \beta)$
- **(U6)** *If $\varphi \diamond \alpha \vdash \beta$ and $\varphi \diamond \beta \vdash \alpha$, then $\varphi \diamond \alpha \equiv \varphi \diamond \beta$*
- **(U7)** *If φ is a complete[2] formula,*
  then $(\varphi \diamond \alpha) \wedge (\varphi \diamond \beta) \vdash \varphi \diamond (\alpha \vee \beta)$
- **(U8)** $(\varphi \vee \psi) \diamond \alpha \equiv (\varphi \diamond \alpha) \vee (\psi \diamond \alpha)$

KM update operators have been characterized in terms of partial orderings over worlds, more precisely, faithful assignments:

Definition 2 (Faithful assignment (Katsuno and Mendelzon 1992))**.** *A* faithful assignment *is a mapping associating each world $\omega_i \in \Omega$ with a partial order $\leq_{\omega_i}$ over Ω such that for all $\omega_i, \omega \in \Omega$, if $\omega \neq \omega_i$ then $\omega_i <_i \omega$.*

Proposition 1 ((Katsuno and Mendelzon 1992))**.** *An update operator $\diamond$ is a KM update operator if and only if there exists*

[2]A *complete* formula is a formula with a unique model.

a faithful assignment $\omega_i \mapsto \leq_{\omega_i}$ such that for all formulae φ, α,

$$[\![\varphi \diamond \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!], \leq_{\omega_i}).$$

This gives us a characterization of Katsuno and Mendelzon's update operators as an operation where we look, for each possible world ω_i, at the most plausible worlds (what we call the most plausible transitions) reachable from ω_i after the occurrence of α.

3 Credibility-Limited Update

One important technical difference between belief revision and belief update is that the relation used for finding the most plausible worlds in belief revision is a total pre-order, whereas in belief update it is a partial order. While the use of a partial relation could allow the prohibition of certain transitions between possible worlds, standard Katsuno-Mendelzon update operators do not capitalize on this capability to disallow specific transitions, as they still satisfy the following reachability property:

(reachability) $\forall \omega, \omega' \in \Omega \ \ \omega \leq_\omega \omega'$

As explained in the introduction, in real life, some transitions are not possible, and there are some possible worlds that cannot be reached from a given possible world. For example, it is impossible to reach a world where the cup is not broken from a state where the cup is broken.

This means that we aim to remove the reachability property in order to fully leverage the power of the partial relation and obtain more suitable update operators.

From a postulate perspective, removing postulate **(U3)** is sufficient to achieve this goal, resulting in a more general setting than the standard Katsuno-Mendelzon framework.

At the semantic level, this requires the creation of a partition for each possible world, classifying the possible worlds into two categories: credible and not credible. Only the credible worlds are reachable from a given possible world.

Definition 3 (Credibility-limited update operator)**.** *An operator $\diamond$ is a* credibility-limited (CL) update operator *if it satisfies* **(U1-U2)***, and* **(U4-U8)**.

We now intend to provide a representation result in terms of credible faithful assignments:

Definition 4 (Credible faithful assignment)**.** *A* credible faithful assignment *is a mapping associating each world $\omega_i \in \Omega$ with a pair $(\mathcal{C}_i, \leq_{\omega_i})$, where $\{\omega_i\} \subseteq \mathcal{C}_i \subseteq \Omega$ and $\leq_{\omega_i}$ is a partial ordering over $\mathcal{C}_i$ such that for each $\omega \in \mathcal{C}_i$, if $\omega_i \neq \omega$, then $\omega_i <_i \omega$.*

Note that in this definition the plausibility ordering is defined only on the worlds of $\mathcal{C}_i$, so worlds that do not appear in this set can not be reached from ω_i.

Theorem 1. *An update operator $\diamond$ is a CL update operator if and only if there exists a credible faithful assignment $\omega_i \mapsto (\mathcal{C}_i, \leq_{\omega_i})$ such that for all formulae φ, α,*

$$[\![\varphi \diamond \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}).$$

Credibility-limited update clearly generalizes KM update, as every KM update operator is trivially a CL update operator. However, the converse is not true in general. An interesting observation is that expansion (i.e., conjunction) is a simple example of a CL update operator that is not a KM update operator.

Definition 5 (Expansion)**.** *The* expansion operator $+$ *is defined for all formulae φ, α by $\varphi + \alpha = \varphi \wedge \alpha$.*

It is easy to see that:

Proposition 2. $+$ *satisfies* **(U1-U2)***, and* **(U4-U8)** *but not* **(U3)**.

Credibility-limited update generalizes both KM update and expansion. Additionally, expansion can be viewed as the strongest and most skeptical/change-reluctant form of credibility-limited update. This is evident from the fact that:

Proposition 3. *Let $\diamond$ be a CL update operator. Then for all formulae φ, α, we have that $\varphi \wedge \alpha \vdash \varphi \diamond \alpha$.*

Given a credibility-limited update operator, to restore consistency, one can simply associate with it a "weaker" version that satisfies postulate **(U3)**:

Definition 6. *Let $\diamond$ be an update operator. The* drastic extension *of $\diamond$, denoted by $\diamond^d$ is defined for all formulae φ, α by:*

$$\varphi \diamond^d \alpha = \begin{cases} \alpha & \text{if } \exists \psi \vdash \varphi,\, \psi \nvdash \bot,\, \psi \diamond \alpha \vdash \bot, \\ \varphi \diamond \alpha & \text{otherwise.} \end{cases}$$

Proposition 4. *If $\diamond$ is a CL update operator, then $\diamond^d$ is a KM update operator, and for all formulae φ, α, we have that $\varphi \diamond \alpha \vdash \varphi \diamond^d \alpha$.*

Summing up Propositions 3 and 4, we get for all formulae φ, α that:

$$\varphi + \alpha \vdash \varphi \diamond \alpha \vdash \varphi \diamond^d \alpha \qquad (1)$$

Note also that the drastic extension of expansion is simply the *drastic update operator* $\diamond^d$ (Marquis and Schwind 2014), defined for all formulae φ, α by $\varphi \diamond^d \alpha = \varphi$ if $\varphi \vdash \alpha$, otherwise $\varphi \diamond^d \alpha = \alpha$ (the proof is direct: if $\varphi \vdash \alpha$, then $\varphi + \alpha \equiv \varphi$ by **(U2)**; and if $\varphi \nvdash \alpha$, then one can find a formula $\psi \vdash \varphi \wedge \neg\alpha$, i.e., $\psi + \alpha = \psi \wedge \alpha \vdash \bot$, so in this case $\varphi +^d \alpha = \alpha$ by definition of the drastic extension).

Another interesting observation is that, although the consistency postulate **(U3)** is not satisfied, credibility-limited update operators satisfy a weaker form of consistency, which we call "consistency preservation" (this observation trivially follows from **(U8)**).

(CP) If $\varphi \diamond \alpha \nvdash \bot$ and $\varphi \vdash \psi$, then $\psi \diamond \alpha \nvdash \bot$

Proposition 5. *Every Credibility-Limited update operator satisfies* **(CP)**.

4 Consistent Credibility-Limited Update

In this section, we propose an alternative model for update with a credibility limit. The technical difference with respect to the CL update of the previous section is that, with CL update, the possible worlds that do not lead to any credible world of the new information lead to a contradiction, whereas here we simply reject the transition caused by the new information by keeping the original possible world. This different behavior can be motivated by two different justifications.

The first motivation is that in some applications, we want to avoid reaching a contradiction (to satisfy postulate **(U3)**). This leads to the name of these operators: *consistent* credibility-limited (CCL) update operators.

The second motivation is more intuitive and is related to the status of the new piece of information. In CL update, the new piece of information is evidence (a direct observation), so we must strictly apply the transition to all believed possible worlds. On the other hand, CCL update is more appropriate in cases where the transition implied by the new piece of information can fail in some possible worlds. This may occur if we do not completely trust the new piece of information and therefore reject it when it leads nowhere from some possible worlds. This is also the case where we use update to predict the state of the world after some change that requires certain preconditions to be true. For example, suppose we send a robot to a room with instructions to fill a cup. Then we have to update our beliefs to accommodate this change, taking into account that the cup will not be empty. However, this will only be the case if the cup is not broken. Therefore, in our possible states where the cup is not broken, we will follow the transition to new states where the cup is not empty, but in our possible states where the cup is broken, we will not change these states.

Definition 7 (Consistent credibility-limited update operator)**.** *An operator* $\diamond$ *is a* consistent credibility-limited (CCL) update operator *if it satisfies* **(U2-U8)** *and the following conditions, for all formulae* φ, α*:*

(RSC) *If* φ *is complete, then* $\varphi \diamond \alpha \vdash \alpha$ *or* $\varphi \diamond \alpha \equiv \varphi$

(SM) *If* $\alpha \vdash \beta$ *and* $\varphi \diamond \alpha \vdash \alpha$*, then* $\varphi \diamond \beta \vdash \beta$

(SM) and **(RSC)** come from postulates used for credibility-limited revision (Booth et al. 2012)[3]. **(RSC)**, which stands for *Relative Success for Complete formulae*, states that the result of the update for a complete formula should either imply the new information or leave the beliefs of the agent unchanged. Intuitively, the former case occurs when the update succeeds, while the latter happens when the new information is rejected due to insufficient credibility or an impossible transition. As for **(SM)**, *Success Monotonicity*, it is reasonable to expect that direct consequences of a succeeded formula are also succeeded formulae.

Let us state the corresponding representation theorem:

Theorem 2. *An update operator* $\diamond$ *is a CCL update operator if and only if there exists a credible faithful assignment* $\omega_i \mapsto (\leq_{\omega_i}, \mathcal{C}_i)$ *such that for all formulae* φ, α,

$$[\![\varphi \diamond \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} f(\omega_i, \alpha)$$

where for each $\omega_i \in \Omega$ *and each formula* α*,* $f(\omega_i, \alpha)$ *is defined as*

$$f(\omega_i, \alpha) = \begin{cases} \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) & \text{if } [\![\alpha]\!] \cap \mathcal{C}_i \neq \emptyset \\ \{\omega_i\} & \text{otherwise} \end{cases}$$

So one can see that, semantically, for each possible world, there is a decision to either go to the most plausible new worlds if a possible world from the models of the new piece of information is reachable (credible), or to remain in the current possible world.

5 Example and Discusion

Let us come back to the illustrative example from the introduction and formalize it in order to illustrate the behavior of our operators.

In a room, there is a cup. The cup can be on the table (t) or on the floor, it can be empty (e) or not, and it can be broken (b) or not. We will take the order teb for the interpretations, so 101 means that t and b hold and that e does not hold.

Our current beliefs about the cup are that it is on the table, not empty, and not broken, or it may have fallen off the table onto the floor, in which case it will be empty and can be either intact or broken. So our belief base is a formula φ such that $[\![\varphi]\!] = \{\omega_1, \omega_2, \omega_3\}$, with $\omega_1 = 100$, $\omega_2 = 010$, and $\omega_3 = 011$.

Broken is an irreversible state, i.e., we cannot go from a broken state to an unbroken state. Moreover, a broken cup is clearly empty. We encode these physical constraints in the credible worlds associated with each possible world. Note that, as credible worlds are a set of possible worlds, they can be encoded by a propositional formula. So we have that $C(\omega_1) = C(\omega_2) = [\![b \to e]\!]$ and $C(\omega_3) = [\![b \wedge e]\!]$.

CL update. Suppose a reliable friend who left the room tells us that he saw that the cup is not empty ($\alpha = \neg e$).

What should our new beliefs $\varphi \diamond \alpha$ be? According to the representation theorem, using CL update, we have to update each model of our current beliefs individually.

For ω_1, since it is consistent with α, we don't need to do anything. So, $[\![\varphi_{\omega_1} \diamond \alpha]\!] = \{\omega_1\} = \{100\}$.

As for ω_2, since it is not a model of α, we need to look for the most plausible credible possible worlds for ω_2. Suppose that the plausibility order associated with ω_2 is $\{010\} <_{\omega_2} \{110, 000\} <_{\omega_2} \{100\} <_{\omega_2} \{011, 111\}$. Then, the most plausible states reachable from ω_2 are the ones where the cup is now on the table and empty or on the floor and not empty, which are more plausible than the cup being empty and on the table. The least plausible worlds are the ones where the cup is now broken. So, with this order, we obtain $[\![\varphi_{\omega_2} \diamond \alpha]\!] = \{000\}$.

Lastly, ω_3 is the most interesting case here since it is a "broken" state, and none of the models of the new piece of information $\alpha = \neg e$ is credible for ω_3. It is not credible to go from a "broken" to an "unbroken" state, so $[\![\varphi_{\omega_3} \diamond \alpha]\!] = \bot$.

The result is the disjunction of the results obtained by each model. Therefore, after the CL update, we believe that the cup is not empty and not broken: $[\![\varphi \diamond \alpha]\!] = \{100, 000\}$.

[3]More precisely, **(RSC)** is a weaker version of the original Relative Success postulate introduced in (Booth et al. 2012), which is discussed at the end of this section.

CCL update. Let us consider another scenario now. We want to update our beliefs based on the same formula as before ($\alpha = \neg e$), but the new piece of information is of different nature. We send a robot to the room with instructions to fill the cup[4]. We need to update our beliefs to reflect what we expect to happen once the robot has had enough time to perform the action. We know that in some cases, such as if the cup is broken, the action will not be possible. In order to represent the new state of the world, we need to use CCL update.

For the first two models, there are reachable (credible) possible worlds in the models of α, so we obtain the same results as before, i.e., $[\![\varphi_{\omega_1} \diamond \alpha]\!] = \{100\}$ and $[\![\varphi_{\omega_2} \diamond \alpha]\!] = \{000\}$.

However, for ω_3, there are no credible worlds in α. This means that if the true world is ω_3, then the robot will not be able to perform α (fill the broken cup), and the world will remain in this state, i.e., $[\![\varphi_{\omega_3} \diamond \alpha]\!] = \{011\}$.

The result is the disjunction of the results obtained by each model, so $[\![\varphi \diamond \alpha]\!] = \{100, 000, 011\}$. Therefore, we believe that if the cup is not broken, then it is not empty, and if the cup is broken, it is empty and on the floor.

Before concluding this section, let us use our example to discuss how the standard postulate of Relative Success satisfied by credibility-limited *revision* operators (Booth et al. 2012), interacts with our CL and CCL update operators. This postulate, denoted by **(RS)**, is stronger than our postulate **(RSC)**, since it applies to every belief base, not only complete ones:

(RS) $\varphi \diamond \alpha \vdash \alpha$ or $\varphi \diamond \alpha \equiv \varphi$

First, since CL operators satisfy **(U1)**, they clearly satisfy **(RS)**. However, this is generally not the case for CCL operators: in our example, the CCL operator $\diamond$ satisfies **(RSC)** but violates **(RS)** since $\varphi \diamond \alpha \nvdash \alpha$ and $\varphi \diamond \alpha \not\equiv \varphi$. This is due to the fact that the model 010 of φ is effectively "updated" into the model 000 of α since 000 is reachable from 010, whereas the model 011 of φ remains in the updated beliefs since no world from α is reachable from 011.

This example also shows why **(RS)** is not desirable for CCL update. When the robot is asked to fill the cup, **(RS)** present us with a dilemma, since it would require our updated beliefs either to entail that the cup is filled, and thus disregarding the initial plausible situation where the cup was initially broken (011); or to remain entirely unchanged, ignoring a reasonable update of the two initial plausible situations where the cup was not broken (100 and 010). Hence a more reasonable result in this case is to perform a change when it is deemed credible according to the each specific initial situation one may be in φ. Accordingly, this example shows that the notion of relative success is only satisfied "locally", i.e., how new information with relative confidence can be integrated into an agent's beliefs while adhering to an update spirit.

[4]Note that the example used by (Katsuno and Mendelzon 1992) to motivate the definition of their update operators was also to send a robot in a room with instructions.

φ	$(t \land \neg e \land \neg b) \lor (\neg t \land e)$	$\{100, 010, 011\}$
α	$\neg e$	$\{000, 001, 100, 101\}$
$\varphi \diamond \alpha$	$\neg e \land \neg b$	$\{100, 000\}$
$\varphi \diamond \alpha$	$(\neg e \land \neg b) \lor (\neg t \land b \land e)$	$\{100, 000, 011\}$

Table 1: Summary of the illustrative example.

6 Conclusion

In this work, we proposed a belief update model that addresses an issue with Katsuno and Mendelzon's standard model, which assumes that any situation can be reached from any other situation. This assumption is often not realistic in practical scenarios where some transitions are physically or legally impossible.

To overcome this issue, we introduced the notion of credible worlds, which represent the possible worlds that are reachable from a given possible world. The update operators take these credible worlds into account to appropriately handle updates.

However, we found that there are two ways to handle cases where the new piece of information is not credible with respect to a given possible world, and the choice of the appropriate way depends on the status of the new piece of information.

If we have very high confidence in this new piece of information, for instance for direct observations, then we must conclude that all possible worlds that reject the new information were wrong (they do not comply with reality, and therefore were incorrect models of the world). This definition leads us to the class of CL (credibility-limited) update operators. However, one must be aware that with these operators, we can sometimes reach an inconsistent belief base when updating with a consistent new piece of information in the case where we detect a credibility issue for each possible world of our current beliefs. We will discuss a possible treatment of this case in the future work paragraph below.

Before that, let us recall the second family of operators we defined, CCL (consistent credibility-limited) update operators, which always ensure a consistent result by removing the success postulate. With these operators, if the new piece of information is not credible from a possible world, then this piece of information is rejected, and we keep the old possible world in our beliefs. This behavior corresponds to new pieces of information in which we have some confidence, but less than with CL operators. This can be the case if the new piece of information is not an observation but a message that we received from another agent (who can be wrong), or if the new piece of information is a foreseen consequence of an action that may fail (for example, we know that the robot is asked to fill the cup, and this action can only occur successfully when the cup is not broken).

As future work, we aim to solve the inconsistent case with CL update operators. When we reach an inconsistent belief base after a CL update by a consistent formula that is not credible for each possible world of our current beliefs, it means that our current possible worlds were wrong, as they do not comply with reality, and cannot incorporate the result of the evolution of the world into our current model of the world. Therefore, our current view of the world is wrong and needs to be corrected. And the solution to adjust an agent's beliefs when trustworthy evidence contradicts them is belief revision, an operation that modifies an agent's beliefs based on reliable evidence that contradicts them. Our next goal is to characterize operators that can perform updates when possible, as small, foreseen evolutions of the world. Still, when faced with an inconceivable situation from the current beliefs' point of view, the operators will perform a revision.

Appendix: Proofs

Proof of Theorem 1. *(Only if part)* Let $\omega_i \mapsto (\mathcal{C}_i, \leq_{\omega_i})$ be a credible faithful assignment, and let $\diamond$ be an update operator such that for all formulae φ, α, $[\![\varphi \diamond \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Let us show that it is a CL update operator, i.e., that is satisfies **(U1-U2)** and **(U4-U8)**. The proofs for **(U1)**, **(U4)** and **(U8)** are direct by definition of $\diamond$, so let us prove the remaining postulates are satisfied.

(U2) Assume that $\varphi \vdash \alpha$, and let us show that $\varphi \diamond \alpha \equiv \varphi$. Let us first show that for each world $\omega_i \in [\![\varphi]\!]$, $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = \{\omega_i\}$. So let $\omega_i \in [\![\varphi]\!]$. Since $\varphi \vdash \alpha$, we know that $\omega \in [\![\alpha]\!]$. And since $\omega_i \in \mathcal{C}_i$ by definition of $\mathcal{C}_i$, we get that $\omega_i \in [\![\alpha]\!] \cap \mathcal{C}_i$. Yet $\leq_{\omega_i}$ is faithful, so for each world $\omega \in \mathcal{C}_i$, we have that $\omega_i <_i \omega$. This means that $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = \{\omega_i\}$, for each world $\omega_i \in [\![\varphi]\!]$. Then, $\bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = \bigcup_{\omega_i \in [\![\varphi]\!]} \{\omega_i\}$, from which we get that $[\![\varphi \diamond \alpha]\!] = [\![\varphi]\!]$. Hence, $\varphi \diamond \alpha \equiv \varphi$.

(U5) Let us first show that for each world $\omega_i \in [\![\varphi]\!]$, $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \cap [\![\beta]\!] \subseteq \min([\![\alpha \wedge \beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Assume toward a contradiction that there is a world $\omega \in \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \cap [\![\beta]\!]$ such that $\omega \notin \min([\![\alpha \wedge \beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Since $\omega \in [\![\alpha \wedge \beta]\!] \cap \mathcal{C}_i$, this means that there is a world $\omega' \in [\![\alpha \wedge \beta]\!] \cap \mathcal{C}_i$ such that $\omega' <_i \omega$. Yet $\omega' \in [\![\alpha]\!] \cap \mathcal{C}_i \cap [\![\beta]\!]$, so $\omega \notin \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \cap [\![\beta]\!]$, which leads to a contradiction. We got that $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \cap [\![\beta]\!] \subseteq \min([\![\alpha \wedge \beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, which means that $\bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \cap [\![\beta]\!] \subseteq \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha \wedge \beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, thus $[\![(\varphi \diamond \alpha) \wedge \beta]\!] \subseteq [\![\varphi \diamond (\alpha \wedge \beta)]\!]$. Hence, $(\varphi \diamond \alpha) \wedge \beta \vdash \varphi \diamond (\alpha \wedge \beta)$.

(U6) Assume that $\varphi \diamond \alpha \vdash \beta$ and $\varphi \diamond \beta \vdash \alpha$. We only need to prove that $\varphi \diamond \alpha \vdash \varphi \diamond \beta$: the proof for $\varphi \diamond \beta \vdash \varphi \diamond \alpha$ is similar since α and β play symmetrical roles in the statement of the postulate. Toward a contradiction, assume that $\varphi \diamond \alpha \nvdash \varphi \diamond \beta$. This means that there exists a world $\omega \in [\![\varphi \diamond \alpha]\!]$ such that $\omega \notin [\![\varphi \diamond \beta]\!]$. By definition of $\diamond$, this implies that there exists a world $\omega_i \in [\![\varphi]\!]$ such that $\omega \in \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$ and $\omega \notin \min([\![\beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Yet $\varphi \diamond \alpha \vdash \beta$, so since $\omega \in [\![\varphi \diamond \alpha]\!]$, we get that $\omega \in [\![\beta]\!]$. And since $\omega \in \mathcal{C}_i$, we get that $\omega \in [\![\beta]\!] \cap \mathcal{C}_i$. So $[\![\beta]\!] \cap \mathcal{C}_i \neq \emptyset$. But $\omega \notin \min([\![\beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, so there exists a world $\omega' \in [\![\beta]\!] \cap \mathcal{C}_i$, i.e., $\omega' \in \min([\![\beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, such that $\omega' <_i \omega$. Yet $\varphi \diamond \beta \vdash \alpha$, so we get that $\omega' \in [\![\alpha]\!]$. We found a world $\omega' \in [\![\alpha]\!] \cap \mathcal{C}_i$ such that $\omega' <_i \omega$, which contradicts the fact that $\omega \in \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Hence, $\varphi \diamond \alpha \vdash \varphi \diamond \beta$. This proves that $\varphi \diamond \alpha \equiv \varphi \diamond \beta$.

(U7) Let φ be a complete formula, i.e., $\varphi = \gamma_{\omega_i}$ for some world $\omega_i \in \Omega$. We need to show that $(\gamma_{\omega_i} \diamond \alpha) \wedge (\gamma_{\omega_i} \diamond \beta) \vdash \gamma_{\omega_i} \diamond (\alpha \vee \beta)$. Toward a contradiction, assume that there exists a world $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!] \cap [\![\gamma_{\omega_i} \diamond \beta]\!]$ such that $\omega \notin [\![\gamma_{\omega_i} \diamond (\alpha \vee \beta)]\!]$. By definition of $\diamond$, we know that $\omega \in \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \cap \min([\![\beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. This means that $[\![\alpha]\!] \cap [\![\beta]\!] \cap \mathcal{C}_i \neq \emptyset$, and in particular that $[\![\alpha \vee \beta]\!] \cap \mathcal{C}_i \neq \emptyset$. So let $\omega' \in \min([\![\alpha \vee \beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Since $\omega \notin \min([\![\alpha \vee \beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, we get that $\omega' <_i \omega$. If $\omega' \in [\![\alpha]\!]$, then $\omega' \in [\![\alpha]\!] \cap \mathcal{C}_i$, yet $\omega' <_i \omega$, which contradicts the fact that $\omega \in \min([\![\alpha]\!] \cap \mathcal{C}_i)$. Similarly, if $\omega' \in [\![\beta]\!]$, then $\omega' \in [\![\beta]\!] \cap \mathcal{C}_i$, so $\omega' <_i \omega$, contradicts the fact that $\omega \in \min([\![\beta]\!] \cap \mathcal{C}_i)$. Both cases lead to a contradiction, so for every world $\omega \in \Omega$, if $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!] \cap [\![\gamma_{\omega_i} \diamond \beta]\!]$ then $\omega \in [\![\gamma_{\omega_i} \diamond (\alpha \vee \beta)]\!]$. Stated equivalently, we got that $(\gamma_{\omega_i} \diamond \alpha) \wedge (\gamma_{\omega_i} \diamond \beta) \vdash \gamma_{\omega_i} \diamond (\alpha \vee \beta)$.

This concludes the (only if) part of the proof.

(If part) Let $\diamond$ be a CL update operator, and let us consider the assignment associating every world ω_i with the pair $(\mathcal{C}_i, \leq_{\omega_i})$ where $\mathcal{C}_i$ is defined as the set $\mathcal{C}_i = \{\omega \in \Omega \mid [\![\gamma_{\omega_i} \diamond \gamma_\omega]\!] = \{\omega\}\}$, and $\leq_{\omega_i}$ is the relation $\leq_{\omega_i} \subseteq \mathcal{C}_i \times \mathcal{C}_i$ defined for all worlds ω, $\omega' \in \mathcal{C}_i$ as $\omega \leq_{\omega_i} \omega'$ if and only if ($\omega = \omega_i$ or $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega\}$). Let $\omega_i \in \Omega$. From **(U2)** we know that $[\![\gamma_{\omega_i} \diamond \gamma_{\omega_i}]\!] = \{\omega_i\}$, so we directly get that $\{\omega_i\} \subseteq \mathcal{C}_i$ by definition of $\mathcal{C}_i$. Let us show that $\leq_{\omega_i}$ is a (partial) order.

(Reflexivity) Let $\omega \in \mathcal{C}_i$, we must show that $\omega \leq_{\omega_i} \omega$. Since $\omega \in \mathcal{C}_i$, by definition of $\mathcal{C}_i$ we get that $[\![\gamma_{\omega_i} \diamond \gamma_\omega]\!] = \{\omega\}$. Thus $\omega \leq_{\omega_i} \omega$ by definition of $\leq_{\omega_i}$.

(Antisymmetry) Let $\omega \leq_{\omega_i} \omega'$, $\omega' \leq_{\omega_i} \omega$, and let us show that $\omega = \omega'$. Assume toward a contradiction that $\omega \neq \omega'$.

Let us first show that $\omega \neq \omega_i$ and $\omega' \neq \omega_i$. Toward a contradiction, assume that $\omega = \omega_i$. Then by **(U2)** we get that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = [\![\gamma_\omega \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega\} \neq \{\omega'\}$. Since $\omega = \omega_i$, we know that $\omega' \neq \omega_i$. Yet $\omega' \leq_{\omega_i} \omega$, so $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega'\}$, which leads to a contradiction. Hence, $\omega \neq \omega_i$. We can use a similar argument to prove that $\omega' \neq \omega_i$. Now, by definition of $\leq_{\omega_i}$, since $\omega \leq_{\omega_i} \omega'$ and $\omega' \leq_{\omega_i} \omega$ and since we proved that $\omega \neq \omega_i$ and $\omega' \neq \omega_i$, we get that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega\}$ and $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega'\}$. This contradicts our initial assumption that $\omega \neq \omega'$. Hence, $\omega = \omega'$. This concludes the proof that $\leq_{\omega_i}$ satisfies (antisymmetry).

(Transitivity) Let $\omega, \omega', \omega'' \in \mathcal{C}_i$, and assume that $\omega \leq_{\omega_i} \omega'$ and $\omega' \leq_{\omega_i} \omega''$. We must show that $\omega \leq_{\omega_i} \omega''$. The proof is trivial by definition of $\leq_{\omega_i}$ in the case when $\omega = \omega_i$, so assume that $\omega \neq \omega_i$. Let us first prove that $\omega' \neq \omega_i$ and $\omega'' \neq \omega_i$. Assume toward a contradiction that $\omega' = \omega_i$. Since $\omega \neq \omega_i$ and $\omega \leq_{\omega_i} \omega'$, we get that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega\}$ by definition of $\leq_{\omega_i}$. Yet from **(U2)** and since $\omega' = \omega_i$, we have that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = [\![\gamma_{\omega'} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega'\} = \{\omega_i\}$. We got that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega\}$ and $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega_i\}$, which contradicts $\omega \neq \omega_i$. Hence, $\omega' \neq \omega_i$. The same arguments can be used to prove that $\omega'' \neq \omega_i$.

At this point, we know that $\{\omega, \omega', \omega''\} \cap \{\omega_i\} = \emptyset$. When $\omega = \omega'$ or $\omega' = \omega''$, we trivially get that $\omega \leq_{\omega_i} \omega''$; and when $\omega = \omega''$, we also get that $\omega \leq_{\omega_i} \omega''$ since $\leq_{\omega_i}$ satisfies (symmetry). Then, assume that ω, ω' and ω'' are pairwise different, i.e., $\omega \neq \omega'$, $\omega' \neq \omega''$ and $\omega \neq \omega''$. By definition of $\leq_{\omega_i}$ and since $\omega \leq_{\omega_i} \omega'$ and $\omega' \leq_{\omega_i} \omega''$, we know that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega\}$ and $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega',\omega''\}}]\!] = \{\omega'\}$. By **(U5)**, $(\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}) \wedge \gamma_{\{\omega,\omega'\}} \vdash \gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}$. Yet $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega\}$, which means that $\omega' \notin [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}]\!]$. Similarly, by **(U5)**, $(\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}) \wedge \gamma_{\{\omega',\omega''\}} \vdash \gamma_{\omega_i} \diamond \gamma_{\{\omega',\omega''\}}$. Yet $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega',\omega''\}}]\!] = \{\omega'\}$, which means that $\omega'' \notin [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}]\!]$. Hence, by **(U1)**, we get that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}]\!] \subseteq \{\omega\}$. Yet $[\![\gamma_{\omega_i} \diamond \gamma_{\omega}]\!] = \{\omega\}$ by definition of $\mathcal{C}_i$ and since $\omega \in \mathcal{C}_i$. By **(U6)**, since $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}]\!] \subseteq \{\omega\}$ and $[\![\gamma_{\omega_i} \diamond \gamma_{\omega}]\!] \subseteq \{\omega,\omega',\omega''\}$, we get that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}]\!] = [\![\gamma_{\omega_i} \diamond \gamma_{\omega}]\!] = \{\omega\}$. By **(U6)** again, since $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega''\}}]\!] \subseteq \{\omega,\omega',\omega''\}$ (by **(U1)**) and $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}]\!] \subseteq \{\omega,\omega''\}$, we get that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega''\}}]\!] = [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}]\!] = \{\omega\}$. This shows that $\omega \leq_{\omega_i} \omega''$ and concludes the proof that $\leq_{\omega_i}$ satisfies (transitivity).

To show that the assignment $\omega_i \mapsto (\mathcal{C}_i, \leq_{\omega_i})$ is faithful, we only need to prove that for each $\omega \in \Omega$, if $\omega \neq \omega_i$ then $\omega_i <_i \omega$, but this is direct from the fact that $\omega_i \leq_{\omega_i} \omega$ by definition of $\leq_{\omega_i}$ and since $\leq_{\omega_i}$ satisfies (antisymmetry). This concludes the proof that $\omega_i \mapsto (\mathcal{C}_i, \leq_{\omega_i})$ is a credible faithful assignment.

We now intend to show that for each formula α and each world $\omega_i \in \Omega$, $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = [\![\gamma_{\omega_i} \diamond \alpha]\!]$. Assume first that $\omega_i \in [\![\alpha]\!]$. Since $\omega_i \in \mathcal{C}_i$ and $\omega_i <_i \omega^j$ for each $\omega^j \in [\![\alpha]\!]$, we get that $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = \{\omega_i\}$. And by **(U2)**, $[\![\gamma_{\omega_i} \diamond \alpha]\!] = \{\omega_i\}$. So, $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = [\![\gamma_{\omega_i} \diamond \alpha]\!]$. So assume now in the rest of the proof that $\omega_i \notin [\![\alpha]\!]$.

Let us first show that $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \subseteq [\![\gamma_{\omega_i} \diamond \alpha]\!]$. So let $\omega \in \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, we must show that $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!]$. Let us write the set of models of α as $[\![\alpha]\!] = \{\omega^1, \ldots, \omega^k\}$ $(k \geq 1)$. Since $\omega \in [\![\alpha]\!]$, we can also write $[\![\alpha]\!] = \{\omega, \omega^1\} \cup \ldots \cup \{\omega, \omega^k\}$. Let $\omega^j \in [\![\alpha]\!]$ and let us prove that $\omega \in [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!]$.
Case 1: $\omega^j \in \mathcal{C}_i$. Assume toward a contradiction that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] = \emptyset$. We know that $\omega \neq \omega_i$ (since $\omega \in [\![\alpha]\!]$ and $\omega_i \notin [\![\alpha]\!]$), so $\omega \not\leq_{\omega_i} \omega^j$ by definition of $\leq_{\omega_i}$. This contradicts the fact that $\omega \in \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, since $\omega^j \in [\![\alpha]\!] \cap \mathcal{C}_i$. Hence, $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] \neq \emptyset$. And in the case when $\omega^j \neq \omega$, since $\omega \in [\![\alpha]\!] \cap \mathcal{C}_i$, we know that $\omega^j \not\leq_{\omega_i} \omega$ (since $\leq_{\omega_i}$ is antisymmetric), thus $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] \neq \{\omega^j\}$ by definition of $\leq_{\omega_i}$. We have that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] \subseteq \{\omega, \omega^j\}$ (by **(U1)**), that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] \neq \emptyset$, and if $\omega^j \neq \omega$ that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] \neq \{\omega^j\}$. Hence, $\omega \in [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!]$.
Case 2: $\omega^j \notin \mathcal{C}_i$. Assume toward a contradiction that $\omega \notin [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!]$. By **(U1)**, $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] \subseteq \{\omega^j\}$. Yet $[\![\gamma_{\omega_i} \diamond \gamma_{\omega^j}]\!] = \emptyset$ since $\omega^j \notin \mathcal{C}_i$, so since $[\![\gamma_{\omega_i} \diamond \gamma_{\omega^j}]\!] \subseteq \{\omega, \omega^j\}$, by **(U6)** we get that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] = [\![\gamma_{\omega_i} \diamond \gamma_{\omega^j}]\!] = \emptyset$. Yet $[\![\gamma_{\omega_i} \diamond \gamma_{\omega}]\!] = \{\omega\}$ since $\omega \in \mathcal{C}_i$. So $[\![\gamma_{\omega_i} \diamond \gamma_{\omega}]\!] \subseteq \{\omega, \omega^j\}$ and $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] \subseteq \{\omega\}$, and by **(U6)** again we get that $[\![\gamma_{\omega_i} \diamond \gamma_{\omega}]\!] = [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] = \emptyset$, which leads to a contradiction. Hence, $\omega \in [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!]$.
We have proved that $\omega \in [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!]$ for each $\omega^j \in [\![\alpha]\!]$. Thus $\omega \in [\![(\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^1\}}) \wedge \ldots \wedge (\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^k\}})]\!]$. Using **(U7)** multiple times, we get that $\omega \in [\![(\gamma_{\omega_i} \diamond (\gamma_{\{\omega,\omega^1\}} \vee \ldots \vee \gamma_{\{\omega,\omega^k\}}))]\!]$, that is, $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!]$. This shows that $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \subseteq [\![\gamma_{\omega_i} \diamond \alpha]\!]$.

Let us now show the other inclusion, i.e., $[\![\gamma_{\omega_i} \diamond \alpha]\!] \subseteq \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Let $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!]$, and assume toward a contradiction that $\omega \notin \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$.
Case 1: $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = \emptyset$. In this case, we have that $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$. This means that for each world $\omega^j \in [\![\alpha]\!]$, $[\![\gamma_{\omega_i} \diamond \gamma_{\omega^j}]\!] = \emptyset$ (by definition of $\mathcal{C}_i$ and from **(U1)**). Yet $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!]$, so by **(U1)** we know that $\omega \in [\![\alpha]\!]$, thus $[\![\gamma_{\omega_i} \diamond \gamma_{\omega}]\!] = \emptyset$. Now, since $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!]$, we can write that $[\![\gamma_{\omega_i} \diamond \alpha]\!] \cap \{\omega\} = \{\omega\}$. Yet by **(U5)**, we know that $(\gamma_{\omega_i} \diamond \alpha) \wedge \gamma_{\omega} \vdash \gamma_{\omega_i} \diamond \gamma_{\omega}$. That is, $\omega \in [\![\gamma_{\omega_i} \diamond \gamma_{\omega}]\!]$. This contradicts the fact that $[\![\gamma_{\omega_i} \diamond \gamma_{\omega}]\!] = \emptyset$.
Case 2: $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \neq \emptyset$. In this case, we have that $[\![\alpha]\!] \cap \mathcal{C}_i \neq \emptyset$. Since $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!]$, we can write that $[\![\gamma_{\omega_i} \diamond \alpha]\!] \cap \{\omega\} = \{\omega\}$. And similarly to case 1 above, we get by **(U5)** that $\omega \in [\![\gamma_{\omega_i} \diamond \gamma_{\omega}]\!]$. So by definition of $\mathcal{C}_i$ and from **(U1)**, we get that $\omega \in [\![\alpha]\!] \cap \mathcal{C}_i$. Yet we initially assumed (toward a contradiction) that $\omega \notin \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, which means that there exists a world ω' (i.e., $\omega' \in \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$) such that $\omega' <_{\omega_i} \omega$. By **(U5)**, $(\gamma_{\omega_i} \diamond \alpha) \wedge \gamma_{\{\omega,\omega'\}} \vdash \gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}$. Yet $\omega \not\leq_{\omega_i} \omega'$, so $\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}} = \{\omega'\}$ (by definition of $\leq_{\omega_i}$). Hence, $\omega \notin [\![\gamma_{\omega_i} \diamond \alpha]\!]$, which leads to a contradiction.

We got for each world $\omega_i \in \Omega$ and for each formula α that $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = [\![\gamma_{\omega_i} \diamond \alpha]\!]$. Then by **(U8)**, we get for all formulae φ, α that $[\![\varphi \diamond \alpha]\!] = \bigcup\{\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \mid \omega_i \in [\![\varphi]\!]\}$, which concludes the proof. ■

Proof of Proposition 2. The fact that $+$ satisfies **(U1-U2)** and **(U4-U8)** can be verified trivially. Likewise, a counter-example for **(U3)** can trivially be found by taking two consistent formulae φ, α such that $\varphi \wedge \alpha \vdash \bot$. ■

Proof of Proposition 3. Let $\diamond$ be a CL update operator, and φ, α be two formulae. The case where $\varphi \wedge \alpha \vdash \bot$ is trivial, so assume that $\varphi \wedge \alpha \nvdash \bot$. Let $\omega \in [\![\varphi \wedge \alpha]\!]$. We need to show that $\omega \in [\![\varphi \diamond \alpha]\!]$. Since $\gamma_\omega \vdash \alpha$, from **(U2)** we know that $\gamma_\omega \diamond \alpha \equiv \gamma_\omega$. And since $\gamma_\omega \vdash \varphi$, from **(U8)** we get that $\varphi \diamond \alpha \equiv (\varphi \vee \gamma_\omega) \diamond \alpha \equiv (\varphi \diamond \alpha) \vee (\gamma_\omega \diamond \alpha) \equiv (\varphi \diamond \alpha) \vee \gamma_\omega$, which means that $\gamma_\omega \vdash \varphi \diamond \alpha$, i.e., that $\omega \in [\![\varphi \diamond \alpha]\!]$. This concludes the proof. ■

Proof of Proposition 4 (proof sketch). Let $\diamond$ be a Credibility-Limited update operator, and let us show that $\diamond^d$ is a KM update operator. By Theorem 1, there exists a credible faithful assignment $\omega_i \mapsto (\leq_{\omega_i}, \mathcal{C}_i)$ such that for all formulae φ, α, $[\![\varphi \diamond \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. For each world ω_i, let $\leq^{\Omega}_{\omega_i}$ be the binary relation over all worlds from Ω defined as $\leq^{\Omega}_{\omega_i} = \leq_{\omega_i} \cup E_i$ where $E_i = \bigcup\{\{(\omega,\omega'), (\omega',\omega')\} \mid \omega \in \mathcal{C}_i, \omega' \in \Omega \setminus \mathcal{C}_i\}$. It is not difficult to verify that the assignment $\omega_i \mapsto \leq^{\Omega}_{\omega_i}$ is faithful (the full proof can be found at (Fermé et al. 2023)).

Now, we intend to show that for all formulae φ, α, $[\![\varphi \diamond^d \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!], \leq^{\Omega}_{\omega_i})$. Let φ, α be two formulae. We consider two cases:
Case 1: assume first that there exists a formula $\psi \vdash$

φ, $\psi \nvdash \bot$, $\psi \diamond \alpha \vdash \bot$. We need to show that $[\![\varphi \diamond^d \alpha]\!] = [\![\alpha]\!]$. We know from Theorem 1 that $[\![\psi \diamond \alpha]\!] = \bigcup_{\omega_i \in [\![\psi]\!]} \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Yet $[\![\psi \diamond \alpha]\!] = \emptyset$, which means that for each world $\omega_i \in [\![\psi]\!]$, we have that $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$, or stated equivalently, that for each world $\omega \in [\![\alpha]\!]$, $\omega \notin \mathcal{C}_i$. Let $\omega_i \in [\![\psi]\!]$. Since $\omega_i \mapsto \leq^{\Omega}_{\omega_i}$ is a faithful assignment, $\omega_i <^{\Omega}_i \omega$. And we can see by definition of $\leq^{\Omega}_{\omega_i}$ that for all ω, $\omega' \notin \mathcal{C}_i$, if $\omega \neq \omega'$ then $\omega \not\leq^{\Omega}_{\omega_i} \omega'$, i.e., all non-credible worlds w.r.t. ω_i are pairwise incomparable. Yet $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$, so $\min([\![\alpha]\!], \leq^{\Omega}_{\omega_i}) = [\![\alpha]\!]$. And since $\omega_i \in [\![\psi]\!]$ and $\psi \models \varphi$, we get that $\bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!], \leq^{\Omega}_{\omega_i}) = [\![\alpha]\!]$, and so $[\![\varphi \diamond^d \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!], \leq^{\Omega}_{\omega_i})$.

Case 2: assume that case 1 does not hold. In particular, this means that for each *complete* formula $\psi = \gamma_{\omega_i}$ such that $\gamma_{\omega_i} \vdash \varphi$, we have that $\gamma_{\omega_i} \diamond \varphi \nvdash \bot$. Yet we know from Theorem 1 that $[\![\gamma_{\omega_i} \diamond \alpha]\!] = \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, so this means that (i) for each world $\omega_i \in [\![\varphi]\!]$, $[\![\alpha]\!] \cap \mathcal{C}_i \neq \emptyset$. Now for each world $\omega_i \in [\![\varphi]\!]$, by definition of $\leq^{\Omega}_{\omega_i}$, for all worlds $\omega \in \mathcal{C}_i$, $\omega' \in \Omega$, we can easily see that:

$$\begin{array}{lll} \text{(ii)} & \omega \leq^{\Omega}_{\omega_i} \omega' \text{ iff } \omega \leq_{\omega_i} \omega' & \text{if } \omega' \in \mathcal{C}_i, \\ \text{(iii)} & \omega <^{\Omega}_i \omega', & \text{otherwise.} \end{array}$$

For each world $\omega_i \in [\![\varphi]\!]$, we got that $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq^{\Omega}_{\omega_i})$ (from (ii)), and $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq^{\Omega}_{\omega_i}) = \min([\![\alpha]\!], \leq^{\Omega}_{\omega_i})$ (from (i) and (iii)), so $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = \min([\![\alpha]\!], \leq^{\Omega}_{\omega_i})$. Hence, $[\![\varphi \diamond^d \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!], \leq^{\Omega}_{\omega_i})$.

We have shown that $\omega_i \mapsto \leq^{\Omega}_{\omega_i}$ is a faithful assignment and that for all formulae φ, α, $[\![\varphi \diamond^d \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} \min([\![\alpha]\!], \leq^{\Omega}_{\omega_i})$. From Proposition 1, this means that $\diamond^d$ is a KM update operator.

The fact that $\varphi \diamond \alpha \vdash \varphi \diamond^d \alpha$ for all formulae φ, α, is direct by definition of $\diamond^d$ and since $\diamond^d$ satisfies **(U1)**. This concludes the proof. ■

Proof of Theorem 2. *(Only if part)* Let $\omega_i \mapsto (\mathcal{C}_i, \leq_{\omega_i})$ be a credible faithful assignment, and $\diamond$ be an update operator such that for all formulae φ, α, $[\![\varphi \diamond \alpha]\!] = \bigcup_{\omega_i \in [\![\varphi]\!]} f(\omega_i, \alpha)$, where for each $\omega_i \in \Omega$ and each formula α, $f(\omega_i, \alpha)$ is defined as

$$f(\omega_i, \alpha) = \begin{cases} \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}), & \text{if } [\![\alpha]\!] \cap \mathcal{C}_i \neq \emptyset, \\ \{\omega_i\}, & \text{otherwise.} \end{cases}$$

Let us show that $\diamond$ is a CCL update operator, i.e., that is satisfies **(RSC)**, **(SM)** and **(U2-U8)**. The proofs for **(RSC)**, **(U3)**, **(U4)** and **(U8)** are direct by definition of $\diamond$, so let us prove that the remaining postulates are satisfied.

(SM) Assume that $\varphi \diamond \alpha \vdash \alpha$ and $\alpha \vdash \beta$, and let us show that $\varphi \diamond \beta \vdash \beta$. So let $\omega \in [\![\varphi \diamond \beta]\!]$ and let us prove that $\omega \in [\![\beta]\!]$. By definition of $\diamond$, it is enough to prove for each world $\omega_i \in [\![\varphi]\!]$ that $f(\omega_i, \beta) \subseteq [\![\beta]\!]$. So let $\omega_i \in [\![\varphi]\!]$. We fall into one of the following two cases:

Case 1: $[\![\beta]\!] \cap \mathcal{C}_i \neq \emptyset$. Then $f(\omega_i, \beta) = \min([\![\beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, so we directly get that $f(\omega_i, \beta) \subseteq [\![\beta]\!]$.

Case 2: $[\![\beta]\!] \cap \mathcal{C}_i = \emptyset$. Since $[\![\alpha]\!] \subseteq [\![\beta]\!]$, we also get that $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$. So $f(\omega_i, \alpha) = f(\omega_i, \beta)$ $(= \{\omega_i\})$. Yet $\varphi \diamond \alpha \vdash \alpha$, so $f(\omega_i, \alpha) \subseteq [\![\alpha]\!] \subseteq [\![\beta]\!]$. Hence, $f(\omega_i, \beta) \subseteq [\![\beta]\!]$.

(U2) Assume that $\varphi \vdash \alpha$, and let us show that $\varphi \diamond \alpha \equiv \varphi$. We can prove that for each world $\omega_i \in [\![\varphi]\!]$, $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = \{\omega_i\}$ identically to the proof of Theorem 1. Then by definition of $\diamond$, we get that $f(\omega_i, \alpha) = \{\omega_i\}$. Then, $\bigcup_{\omega_i \in [\![\varphi]\!]} f(\omega_i, \alpha) = \bigcup_{\omega_i \in [\![\varphi]\!]} \{\omega_i\}$, from which we get that $[\![\varphi \diamond \alpha]\!] = [\![\varphi]\!]$. Hence, $\varphi \diamond \alpha \equiv \varphi$.

(U5) Let us first show that for each world $\omega_i \in [\![\varphi]\!]$, $f(\omega_i, \alpha) \cap [\![\beta]\!] \subseteq f(\omega_i, \alpha \wedge \beta)$. When $[\![\alpha]\!] \cap \mathcal{C}_i \neq \emptyset$ and $[\![\beta]\!] \cap \mathcal{C}_i \neq \emptyset$, we must prove that $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \cap [\![\beta]\!] \subseteq \min([\![\alpha \wedge \beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, but this is done identically to the proof of Theorem 1. When $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$ then $[\![\alpha \wedge \beta]\!] \cap \mathcal{C}_i = \emptyset$, so $f(\omega_i, \alpha) \cap [\![\beta]\!] \subseteq f(\omega_i, \alpha \wedge \beta)$. The remaining case is when $[\![\alpha]\!] \cap \mathcal{C}_i \neq \emptyset$ and $[\![\alpha \wedge \beta]\!] \cap \mathcal{C}_i = \emptyset$. Then, $[\![\alpha]\!] \cap \mathcal{C}_i \cap [\![\beta]\!] = \emptyset$, thus $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \cap [\![\beta]\!] = \emptyset$, so $f(\omega_i, \alpha) \cap [\![\beta]\!] = \emptyset$, and we get $f(\omega_i, \alpha) \cap [\![\beta]\!] \subseteq f(\omega_i, \alpha \wedge \beta)$. We got $f(\omega_i, \alpha) \cap [\![\beta]\!] \subseteq f(\omega_i, \alpha \wedge \beta)$ for each $\omega_i \in [\![\varphi]\!]$. So, $\bigcup_{\omega_i \in [\![\varphi]\!]} f(\omega_i, \alpha) \cap [\![\beta]\!] \subseteq \bigcup_{\omega_i \in [\![\varphi]\!]} f(\omega_i, \alpha \wedge \beta)$, thus $[\![(\varphi \diamond \alpha) \wedge \beta]\!] \subseteq [\![\varphi \diamond (\alpha \wedge \beta)]\!]$. Hence, $(\varphi \diamond \alpha) \wedge \beta \vdash \varphi \diamond (\alpha \wedge \beta)$.

(U6) Assume that $\varphi \diamond \alpha \vdash \beta$ and $\varphi \diamond \beta \vdash \alpha$. Similarly to the proof of Theorem 1, we only need to prove that $\varphi \diamond \alpha \vdash \varphi \diamond \beta$. Toward a contradiction, assume that $\varphi \diamond \alpha \nvdash \varphi \diamond \beta$. This means that there exists a world $\omega \in [\![\varphi \diamond \alpha]\!]$ such that $\omega \notin [\![\varphi \diamond \beta]\!]$. By definition of $\diamond$, this implies that there exists a world $\omega_i \in [\![\varphi]\!]$ such that $\omega \in f(\omega_i, \alpha)$ and $\omega \notin f(\omega_i, \beta)$.

Case 1: $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$. Then $f(\omega_i, \alpha) = \{\omega_i\}$. But since $\varphi \diamond \alpha \vdash \beta$, $f(\omega_i, \alpha) \subseteq [\![\beta]\!]$, so $\omega_i \in [\![\beta]\!]$. Yet $\omega_i \in \mathcal{C}_i$ by definition of $\mathcal{C}_i$, so $[\![\beta]\!] \cap \mathcal{C}_i \neq \emptyset$. In that case, $f(\omega_i, \beta) = \min([\![\beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \neq \emptyset$. Since $\varphi \diamond \beta \vdash \alpha$, $f(\omega_i, \beta) \subseteq [\![\alpha]\!]$, so $\min([\![\beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \subseteq [\![\alpha]\!]$, which means that there is a world $\omega \in [\![\beta]\!] \cap \mathcal{C}_i \cap [\![\alpha]\!]$ and contradicts the fact that $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$.

Case 2: $[\![\alpha]\!] \cap \mathcal{C}_i \neq \emptyset$. In this case, since $f(\omega_i, \alpha) = \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \neq \emptyset$, and since $\varphi \diamond \alpha \vdash \beta$, $f(\omega_i, \alpha) \subseteq [\![\beta]\!]$, so $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \subseteq [\![\beta]\!]$, thus $[\![\beta]\!] \cap \mathcal{C}_i \neq \emptyset$, so $f(\omega_i, \beta) = \min([\![\beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Then the proof that a contradiction is raised in this case is done identically to the proof of Theorem 1.

Since both cases lead to a contradiction, we get that $\varphi \diamond \alpha \vdash \varphi \diamond \beta$. Hence, $\varphi \diamond \alpha \equiv \varphi \diamond \beta$.

(U7) Let φ be a complete formula, i.e., $\varphi = \gamma_{\omega_i}$ for some world $\omega_i \in \Omega$. We need to show that $(\gamma_{\omega_i} \diamond \alpha) \wedge (\gamma_{\omega_i} \diamond \beta) \vdash \gamma_{\omega_i} \diamond (\alpha \vee \beta)$, i.e., that $[\![\gamma_{\omega_i} \diamond \alpha]\!] \cap [\![\gamma_{\omega_i} \diamond \beta]\!] \subseteq [\![\gamma_{\omega_i} \diamond (\alpha \vee \beta)]\!]$. So let $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!] \cap [\![\gamma_{\omega_i} \diamond \beta]\!]$. We have $\omega \in f(\omega_i, \alpha) \cap f(\omega_i, \beta)$. Let us show that $\omega \in [\![\gamma_{\omega_i} \diamond (\alpha \vee \beta)]\!]$. We fall into one of the following cases:

Case 1: $[\![\alpha]\!] \cap \mathcal{C}_i = [\![\beta]\!] \cap \mathcal{C}_i = \emptyset$. In this case, $f(\omega_i, \alpha) = f(\omega_i, \beta) = \{\omega_i\}$, and so $\omega = \omega_i$. Yet $([\![\alpha]\!] \cup [\![\beta]\!]) \cap \mathcal{C}_i = \emptyset$, so $[\![\alpha \vee \beta]\!] \cap \mathcal{C}_i = \emptyset$, thus $f(\omega_i, \alpha \vee \beta) = \{\omega_i\}$. This shows that $\omega \in f(\omega_i, \alpha \vee \beta)$, so $\omega \in [\![\gamma_{\omega_i} \diamond (\alpha \vee \beta)]\!]$.

Case 2: $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$ iff $[\![\beta]\!] \cap \mathcal{C}_i \neq \emptyset$. We can assume that $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$ and $[\![\beta]\!] \cap \mathcal{C}_i \neq \emptyset$: the proof in the dual case when $[\![\alpha]\!] \cap \mathcal{C}_i \neq \emptyset$ and $[\![\beta]\!] \cap \mathcal{C}_i = \emptyset$ is similar since α and β play symmetrical roles. Since $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$, we get that $f(\omega_i, \alpha) = \{\omega_i\}$. Then, $\omega = \omega_i$.

Yet $([\![\alpha]\!] \cup [\![\beta]\!]) \cap \mathcal{C}_i \neq \emptyset$, so $[\![\alpha \vee \beta]\!] \cap \mathcal{C}_i \neq \emptyset$, thus $f(\omega_i, \alpha \vee \beta) = \min([\![\alpha \vee \beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Yet $\omega_i \in [\![\alpha \vee \beta]\!]$, $\omega_i \in \mathcal{C}_i$ by definition of $\mathcal{C}_i$, and $\omega_i <_{\omega_i} \omega'$ for each world $\omega' \in [\![\alpha \vee \beta]\!] \cap \mathcal{C}_i$ by definition of $\leq_{\omega_i}$. This means that $\omega_i \in \min([\![\alpha \vee \beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. And since $\omega = \omega_i$, we get that $\omega \in \min([\![\alpha \vee \beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, so $\omega \in f(\omega_i, \alpha \vee \beta)$, thus $\omega \in [\![\gamma_{\omega_i} \diamond (\alpha \vee \beta)]\!]$.
Case 3: $[\![\alpha]\!] \cap \mathcal{C}_i \neq \emptyset$ and $[\![\beta]\!] \cap \mathcal{C}_i \neq \emptyset$. In this case, $f(\omega_i, \alpha) = \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$ and $f(\omega_i, \beta) = \min([\![\beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. And since $([\![\alpha]\!] \cup [\![\beta]\!]) \cap \mathcal{C}_i \neq \emptyset$, we also have that $f(\omega_i, \alpha \vee \beta) = \min([\![\alpha \vee \beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Then the proof that $\omega \in \min([\![\alpha \vee \beta]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$ for this case is done identically to the proof of Theorem 1.
In all cases, we showed $[\![\gamma_{\omega_i} \diamond \alpha]\!] \cap [\![\gamma_{\omega_i} \diamond \beta]\!] \subseteq [\![\gamma_{\omega_i} \diamond (\alpha \vee \beta)]\!]$. So, $(\gamma_{\omega_i} \diamond \alpha) \wedge (\gamma_{\omega_i} \diamond \beta) \vdash \gamma_{\omega_i} \diamond (\alpha \vee \beta)$.
This concludes the (only if) part of the proof.

(If part) Let $\diamond$ be a CCL update operator, and let us consider the assignment associating every world ω_i with the pair $(\mathcal{C}_i, \leq_{\omega_i})$ where $\mathcal{C}_i$ is defined as the set $\mathcal{C}_i = \{\omega \in \Omega \mid [\![\gamma_{\omega_i} \diamond \gamma_\omega]\!] = \{\omega\}\}$, and $\leq_{\omega_i}$ is the relation $\leq_{\omega_i} \subseteq \mathcal{C}_i \times \mathcal{C}_i$ defined for all worlds $\omega, \omega' \in \mathcal{C}_i$ as $\omega \leq_{\omega_i} \omega'$ if and only if ($\omega = \omega_i$ or $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega\}$). Let $\omega_i \in \Omega$. Note that the construction of the assignment above is identical to the one from a CL operator $\diamond$ in the (if) part of the proof of Theorem 1. As a consequence, some parts of this proof are identical to some of the (if) parts of the proof of Theorem 1, namely those when the postulate **(U1)** is not used. So we can already state that $\{\omega_i\} \subseteq \mathcal{C}_i$, that $\leq_{\omega_i}$ satisfies (reflexivity) and (antisymmetry). We still need to show that $\leq_{\omega_i}$ satisfies (transitivity), which will be enough to conclude that the assignment $\omega \mapsto (\leq_{\omega_i}, \mathcal{C}_i)$ is a credible faithful assignment.

(Transitivity) Let $\omega, \omega', \omega'' \in \mathcal{C}_i$, and assume that $\omega \leq_{\omega_i} \omega'$ and $\omega' \leq_{\omega_i} \omega''$. We must show that $\omega \leq_{\omega_i} \omega''$. The proof that $\{\omega, \omega', \omega''\} \cap \{\omega_i\} = \emptyset$ is covered in the proof of Theorem 1, as well as the case when the three worlds ω, ω' and ω'' are not pairwise different. So assume that $\omega \neq \omega'$, $\omega' \neq \omega''$ and $\omega \neq \omega''$. By definition of $\leq_{\omega_i}$ and since $\omega \leq_{\omega_i} \omega'$ and $\omega' \leq_{\omega_i} \omega''$, we know that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega\}$ and $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega',\omega''\}}]\!] = \{\omega'\}$. By **(U5)**, $(\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}) \wedge \gamma_{\{\omega,\omega'\}} \vdash \gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}$. Yet $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega\}$, which means that $\omega' \notin [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}]\!]$. Using a similar reasoning, we get that $\omega'' \notin [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}]\!]$. Hence, by **(RSC)** and **(U3)**, we get that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}]\!] \in \{\{\omega\}, \{\omega_i\}\}$. But $[\![\gamma_{\omega_i} \diamond \gamma_\omega]\!] = \{\omega\}$ by definition of $\mathcal{C}_i$ and since $\omega \in \mathcal{C}_i$, so $\gamma_{\omega_i} \diamond \gamma_\omega \vdash \gamma_\omega$. And since $\gamma_\omega \vdash \gamma_{\{\omega,\omega',\omega''\}}$, by **(SM)** we get that $\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}} \vdash \gamma_{\{\omega,\omega',\omega''\}}$, that is, $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}}]\!] = \{\omega\}$ (recall that $\{\omega, \omega', \omega''\} \cap \{\omega_i\} = \emptyset$). Using **(SM)** again and a similar argument, we get $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega''\}})]\!] \subseteq \{\omega, \omega''\}$. Then by **(U6)**, since $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega''\}})]\!] \subseteq \{\omega, \omega', \omega''\}$ and $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}})]\!] \subseteq \{\omega, \omega''\}$, we get that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega''\}})]\!] = [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega',\omega''\}})]\!] = \{\omega\}$. This shows that $\omega \leq_{\omega_i} \omega''$, so $\leq_{\omega_i}$ satisfies (transitivity).

The rest of the proof showing that $\omega_i \mapsto (\mathcal{C}_i, \leq_{\omega_i})$ is faithful is identical to the one of Theorem 1.

We now intend to show that for each formula α, we have that $f(\omega_i, \alpha) = [\![\gamma_{\omega_i} \diamond \alpha]\!]$. Let us write the set of models of α as $[\![\alpha]\!] = \{\omega^1, \ldots, \omega^k\}$ ($k \geq 1$). Assume first that $\omega_i \in [\![\alpha]\!]$. Since $\omega_i \in \mathcal{C}_i$ and $\omega_i <_i \omega^j$ for each $\omega^j \in [\![\alpha]\!]$, we get that $f(\omega_i, \alpha) = \{\omega_i\}$. And by **(U2)**, $[\![\gamma_{\omega_i} \diamond \alpha]\!] = \{\omega_i\}$. Hence, $f(\omega_i, \alpha) = [\![\gamma_{\omega_i} \diamond \alpha]\!]$. So assume now in the rest of the proof that $\omega_i \notin [\![\alpha]\!]$. We consider two cases separately:

Case 1: $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$. Then $f(\omega_i, \alpha) = \{\omega_i\}$. We want to show that $[\![\gamma_{\omega_i} \diamond \alpha]\!] = \{\omega_i\}$. Using **(RSC)** we can see that it is enough to show that $\gamma_{\omega_i} \diamond \alpha \nvdash \alpha$. Let us first show that for each world $\omega^j \in [\![\alpha]\!]$, $\gamma_{\omega_i} \diamond \gamma_{\omega^j} \equiv \gamma_{\omega_i}$. So let $\omega^j \in [\![\alpha]\!]$. Since $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$, we get that $\omega^j \notin \mathcal{C}_i$, and then by definition of $\mathcal{C}_i$ we know that $[\![\gamma_{\omega_i} \diamond \gamma_{\omega^j}]\!] \neq \{\omega^j\}$. And by **(U3)**, $[\![\gamma_{\omega_i} \diamond \gamma_{\omega^j}]\!] \neq \emptyset$. So by **(RSC)**, we get that $[\![\gamma_{\omega_i} \diamond \gamma_{\omega^j}]\!] = \{\omega_i\}$. We got that $\gamma_{\omega_i} \diamond \gamma_{\omega^j} \equiv \gamma_{\omega_i}$ for each world $\omega^j \in [\![\alpha]\!]$. Thus $\omega_i \in [\![(\gamma_{\omega_i} \diamond \gamma_{\omega^1}) \wedge \ldots \wedge (\gamma_{\omega_i} \diamond \gamma_{\omega^k})]\!]$. Using **(U7)** multiple times, we get that $\omega_i \in [\![(\gamma_{\omega_i} \diamond (\gamma_{\omega^1} \vee \ldots \vee \gamma_{\omega^k}))]\!]$, that is, $\omega_i \in [\![\gamma_{\omega_i} \diamond \alpha]\!]$. But we know that $\omega_i \in \mathcal{C}_i$, and since $[\![\alpha]\!] \cap \mathcal{C}_i = \emptyset$, $\omega_i \notin [\![\alpha]\!]$. We got that $\omega_i \in [\![\gamma_{\omega_i} \diamond \alpha]\!] \setminus [\![\alpha]\!]$, which shows that $\gamma_{\omega_i} \diamond \alpha \nvdash \alpha$. Then from **(RSC)**, $[\![\gamma_{\omega_i} \diamond \alpha]\!] = \{\omega_i\}$, which concludes the proof for case 1.

Case 2: $[\![\alpha]\!] \cap \mathcal{C}_i \neq \emptyset$. So $f(\omega_i, \alpha) = \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, so we want to show that $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) = [\![\gamma_{\omega_i} \diamond \alpha]\!]$.

Let us first show that $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \subseteq [\![\gamma_{\omega_i} \diamond \alpha]\!]$. So let $\omega \in \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, and let us show that $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!]$. Since $\omega \in [\![\alpha]\!]$, we can write $[\![\alpha]\!] = \{\omega, \omega^1\} \cup \ldots \cup \{\omega, \omega^k\}$. Let $\omega^j \in [\![\alpha]\!]$ and let us prove that $\omega \in [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!]$. Since $\omega \in \mathcal{C}_i$, $\omega \in [\![\gamma_{\omega_i} \diamond \gamma_\omega]\!]$, so by **(SM)** and **(RSC)** and since $\omega_i \neq \omega$, we get that $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] \subseteq \{\omega, \omega^j\}$.
Case 2-i: $\omega^j \in \mathcal{C}_i$. If $\omega^j \neq \omega$, since $\omega \in [\![\alpha]\!] \cap \mathcal{C}_i$, we know that $\omega^j \not\leq_{\omega_i} \omega$ (since $\leq_{\omega_i}$ is antisymmetric), thus $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] \neq \{\omega^j\}$ by definition of $\leq_{\omega_i}$. So from **(U3)**, we get that $\omega \in [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!]$.
Case 2-ii: $\omega^j \notin \mathcal{C}_i$. Assume toward a contradiction that $\omega \notin [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!]$. Then by **(U3)** $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!] = \{\omega^j\}$ (and $\omega \neq \omega^j$). But by **(U5)**, $(\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}) \wedge \gamma_{\omega^j} \vdash \gamma_{\omega_i} \diamond \gamma_{\omega^j}$, which means that $\omega^j \in [\![\gamma_{\omega_i} \diamond \gamma_{\omega^j}]\!]$ and contradicts $\omega^j \notin \mathcal{C}_i$.
We proved that $\omega \in [\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^j\}}]\!]$ for each $\omega^j \in [\![\alpha]\!]$. Thus $\omega \in [\![(\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^1\}}) \wedge \ldots \wedge (\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega^k\}})]\!]$. Using **(U7)** multiple times, we get that $\omega \in [\![(\gamma_{\omega_i} \diamond (\gamma_{\{\omega,\omega^1\}} \vee \ldots \vee \gamma_{\{\omega,\omega^k\}}))]\!]$, i.e., $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!]$. This shows that $\min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i}) \subseteq [\![\gamma_{\omega_i} \diamond \alpha]\!]$ in case 2.

Let us now show the other inclusion in case 2, i.e., $[\![\gamma_{\omega_i} \diamond \alpha]\!] \subseteq \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Let $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!]$, and assume toward a contradiction that $\omega \notin \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$. Since $\omega \in [\![\gamma_{\omega_i} \diamond \alpha]\!]$, we get that $[\![\gamma_{\omega_i} \diamond \alpha]\!] \cap \{\omega\} = \{\omega\}$. By **(U5)**, $(\gamma_{\omega_i} \diamond \alpha) \wedge \gamma_\omega \vdash \varphi \diamond \gamma_\omega$, thus $\omega \in [\![\gamma_{\omega_i} \diamond \gamma_\omega]\!]$. So by definition of $\mathcal{C}_i$, $\omega \in [\![\alpha]\!] \cap \mathcal{C}_i$. Yet $\omega \notin \min([\![\alpha]\!] \cap \mathcal{C}_i, \leq_{\omega_i})$, so there is a world $\omega' \in [\![\alpha]\!] \cap \mathcal{C}_i$ such that $\omega' <_i \omega$. By **(U5)**, $(\gamma_{\omega_i} \diamond \alpha) \wedge \gamma_{\{\omega,\omega'\}} \vdash \gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}$. Yet $[\![\gamma_{\omega_i} \diamond \gamma_{\{\omega,\omega'\}}]\!] = \{\omega'\}$ by definition of $\leq_{\omega_i}$. Since $\omega \not\leq_{\omega_i} \omega'$, we get $\omega \notin [\![\gamma_{\omega_i} \diamond \alpha]\!]$, which leads to a contradiction.

We got for each world $\omega_i \in \Omega$ and for each formula α that $f(\omega_i, \alpha) = [\![\gamma_{\omega_i} \diamond \alpha]\!]$. Then from **(U8)**, we get for all formulae φ, α that $[\![\varphi \diamond \alpha]\!] = \bigcup\{f(\omega_i, \alpha) \mid \omega_i \in [\![\varphi]\!]\}$, which concludes the proof. ■

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