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Streamlining Input/Output Logics with Sequent Calculi

Agata Ciabattoni, **Dmitry Rozplokhas**
Institute of Logic and Computation, TU Wien, Vienna, Austria
{agata, dmitry}@logic.at

Abstract

Input/Output (I/O) logic is a general framework for reasoning about conditional norms and/or causal relations. We streamline Bochman's causal I/O logics via proof-search-oriented sequent calculi. Our calculi establish a natural syntactic link between the derivability in these logics and in the original I/O logics. As a consequence of our results, we obtain new, simple semantics for all these logics, complexity bounds, embeddings into normal modal logics, and efficient deduction methods. Our work encompasses many scattered results and provides uniform solutions to various unresolved problems.

1 Introduction

Input/Output (I/O) logic is a general framework proposed by (Makinson and van der Torre 2000) to reason about conditional norms. I/O logic is not a single logic but rather a family of logics, each viewed as a "transformation engine", which converts an input (condition under which the obligation holds) into an output (what is obligatory under these conditions). Many different I/O logics have been defined, e.g., (Makinson and van der Torre 2001; van der Torre and Parent 2013; Parent and van der Torre 2014; Stolpe 2015), and also used as building blocks for causal reasoning (Bochman 2003; Bochman 2004; Bochman and Lifschitz 2015; Bochman 2021), laying down the logical foundations for the causal calculus (McCain and Turner 1997), and for legal reasoning (Ciabattoni, Parent, and Sartor 2021). I/O logics manipulate Input-Output pairs[1] (A, B), which consists of boolean formulae representing either conditional obligations (in the case of the original I/O logics) or causal relations (A causes B, in the case of their causal counterparts). Different I/O logics are defined by varying the mechanisms of obtaining new pairs from a set of pairs (entailment problem). Each I/O logic is characterized by its own semantics. The original I/O logics use a procedural approach, while their causal counterparts adopt bimodels, which in general consist of pairs of arbitrary deductively-closed sets of formulae. Additionally, each I/O logic is equipped with a proof calculus, consisting of axioms and rules but not suitable for proof search.

This paper deals with the four original I/O logics OUT_1-OUT_4 in (Makinson and van der Torre 2000) and their causal counterpart $\mathrm{OUT}_1^{\perp}$-$\mathrm{OUT}_4^{\perp}$ in (Bochman 2004). We introduce proof-search-oriented sequent calculi and use them to bring together scattered results and to provide uniform solutions to various unresolved problems. Indeed (van Berkel and Straßer 2022) characterized many I/O logics through an argumentative approach using sequent-style calculi. Their calculi are not proof search-oriented. First sequent calculi of this kind for some I/O logics, including OUT_1 and OUT_3, have been proposed in (Lellmann 2021). Their implementation provides an alternative decidability proof, although not optimal (entailment is shown to be in Π_3^P). Moreover, the problem of finding proof-search-oriented calculi for OUT_2 and OUT_4 was left open there. A prover for these two logics was introduced in (Benzmüller et al. 2019). The prover encodes in classical Higher Order Logic their embeddings from (Makinson and van der Torre 2000) into the normal modal logics **K** and **KT**. Finding an embedding of OUT_1 and OUT_3 into normal modal logics was left as an open problem, that (van der Torre and Parent 2013) indicates as difficult, if possible at all. An encoding of various I/O logics into more complicated logics (adaptive modal logics) is in (Straßer, Beirlaen, and Putte 2016). Using their procedural semantics, (Steen 2021) defined goal-directed decision procedures for the original I/O logics, without mentioning the complexity of the task. (Sun and Robaldo 2017) showed that the entailment problem for OUT_1, OUT_2, and OUT_4 is co-NP-complete, while for OUT_3 the complexity was determined to lie within the first and second levels of the polynomial hierarchy, without exact resolution.

In this paper we follow a new path that streamlines the considered logics. Inspired by the modal embedding of $\mathrm{OUT}_2^{\perp}$ and $\mathrm{OUT}_4^{\perp}$ in (Bochman 2003), we design well-behaving sequent calculi for Bochman's causal I/O logics. The normal form of derivations in these calculi allows a simple syntactic link between derivability in the original I/O logics and in their causal versions to be established, making it possible to utilize our calculi for the original I/O logics as well. As a by-product:

- We introduce a simple possible worlds semantics.
- We prove co-NP-completeness and provide efficient automated procedures for the entailment problem; the latter are obtained via reduction to unsatisfiability of a classical logic formula of polynomial size.
- We provide embeddings into the shallow fragment of the

[1]Production rules $A \Rightarrow B$, in Bochman's terminology.

Logic	(TOP)	(BOT)	(WO)	(SI)	(AND)	(OR)	(CT)
OUT_1	✓		✓	✓	✓		
OUT_2	✓		✓	✓	✓	✓	
OUT_3	✓		✓	✓	✓		✓
OUT_4	✓		✓	✓	✓	✓	✓
$\mathrm{OUT}_1^\perp$	✓	✓	✓	✓	✓		
$\mathrm{OUT}_2^\perp$	✓	✓	✓	✓	✓	✓	
$\mathrm{OUT}_3^\perp$	✓	✓	✓	✓	✓		✓
$\mathrm{OUT}_4^\perp$	✓	✓	✓	✓	✓	✓	✓

Table 1: Defining rules for the considered I/O logics

modal logics **K**, **KD** (i.e., standard deontic logic (von Wright 1951)), and their extension with axiom **F**.

These results are uniformly obtained *for all* four original I/O logics and their causal versions.

2 Preliminaries

In the I/O logic framework, conditional norms (or causal relations) are expressed as pairs (B, Y) of propositional boolean formulae. The semantics is operational, rather than truth-functional. The meaning of the deontic/causal concepts in these logics is given in terms of a set of procedures yielding outputs for inputs. The basic mechanism underpinning these procedures is detachment (modus ponens).

On the syntactic side, different I/O logics are obtained by varying the mechanisms of obtaining new input-output pairs from a given set of these pairs. The mechanisms introduced in the original paper (Makinson and van der Torre 2000) are based on the following (axioms and) rules ($\models$ denotes semantic entailment in classical propositional logic):

(TOP) $(\top, \top)$ is derivable from no premises

(BOT) $(\bot, \bot)$ is derivable from no premises

(WO) (A, X) derives (A, Y) whenever $X \models Y$

(SI) (A, X) derives (B, X) whenever $B \models A$

(AND) (A, X_1) and (A, X_2) derive $(A, X_1 \wedge X_2)$

(OR) (A_1, X) and (A_2, X) derive $(A_1 \vee A_2, X)$

(CT) (A, X) and $(A \wedge X, Y)$ derive (A, Y)

Different I/O logics are given by different subsets R of these rules, see Fig. 1. The basic system, called *simple-minded output* OUT_1, consists of the rules {(TOP), (WO), (SI), (AND)}. Its extension with (OR) (for reasoning by cases) leads to *basic output* logic OUT_2, with (CT) (for reusability of outputs as inputs in derivations) to *simple-minded reusable output* logic OUT_3, and with both (OR) and (CT) to *basic reusable output* logic OUT_4. Their causal counterpart (Bochman 2004), that we denote by $\mathrm{OUT}_i^\perp$ for $i = 1, \ldots, 4$, extends the corresponding logics with (BOT).

Definition 1. *Given a set of pairs G and a set R of rules, a derivation in an I/O logic of a pair (B, Y) from G is a tree with (B, Y) at the root, each non-leaf node derivable from its immediate parents by one of the rules in R, and each leaf node is an element of G or an axiom from R.*

We indicate by $G \vdash_{OUT*} (B, Y)$ that the pair (B, Y) is derivable in the I/O logic $OUT*$ from the set of pairs in G

$$\frac{B \Rightarrow}{G \vdash (B, Y)}\ (\mathrm{IN}) \qquad \frac{\Rightarrow Y}{G \vdash (B, Y)}\ (\mathrm{OUT})$$

Figure 1: Concluding rules (same for all causal I/O logic)

(*entailment problem*). We will refer to (B, Y) as the *goal pair*, to the formulae B and Y as the *goal input* and *goal output* respectively, and to the pairs in G as *deriving pairs*.

Remark 1. *A derivation in I/O logic is a sort of natural deduction proof, acting on pairs, rather than formulae. This proof theory is however not helpful to decide whether $G \vdash_{OUT*} (B, Y)$ holds, or to prove metalogical results (e.g., complexity bounds). The main reason is that derivations have no well-behaved normal forms, and in general are difficult to find. (Lellmann 2021) introduced the first proof-search oriented calculi, which operate effectively only in the absence of (OR) (hence not for $OUT_2^\perp$ and $OUT_4^\perp$). The calculi use sequents that manipulate pairs expressed using the conditional logic connective $>$.*

3 Sequent Calculi for Causal I/O Logics

We present sequent-style calculi for the causal I/O logic $\mathrm{OUT}_1^\perp$-$\mathrm{OUT}_4^\perp$. Their basic objects are

$$\textit{I/O sequents} \quad (A_1, X_1), \ldots, (A_n, X_n) \vdash (B, Y)$$

dealing with pairs, as well as

$$\textit{Genzen's LK sequents} \quad A_1, \ldots A_n \Rightarrow B_1, \ldots, B_m$$

dealing with boolean formulae (meaning that $\{A_1, \ldots, A_n\} \models (B_1 \vee \cdots \vee B_m)$). Our calculi are defined by extending the sequent calculus LK[2] for classical logic with three rules manipulating I/O sequents: one *elimination rule* —different for each logic— that removes one of the deriving pair while modifying the goal pair, and two *concluding rules* that transform the derivation of the goal pair into an LK derivation of either the goal input or the goal output. The latter rules, which are the same for all the considered logics, are in Fig. 1.

Definition 2. *A* derivation *in our calculi is a finite labeled tree whose internal nodes are I/O or LK sequents s.t. the label of each node follows from the labels of its children using the calculus rules. We say that an I/O sequent $(A_1, X_1), \ldots, (A_n, X_n) \vdash (B, Y)$ is derivable if all the leaves of its derivation are LK axioms.*

A derivation of an I/O sequent consists of two phases. Looking at it bottom up, we first encounter rules dealing with pairs (pair elimination and concluding rules) followed by LK rules. The calculi, in a sense, uphold the ideological principles guiding I/O logics: pairs (i.e. conditional norms) are treated separately from the boolean statements.

It is easy to see that using (IN) and (OUT) we can derive (TOP) and (BOT); their soundness in the weakest causal I/O logic $\mathrm{OUT}_1^\perp$ is proven below.

Lemma 1. *(IN) and (OUT) are derivable in $OUT_1^\perp$.*

Proof. If $B \Rightarrow$ we have the following derivation in $\mathrm{OUT}_1^\perp$: from $(\bot, \bot)$ and $B \models \bot$ (i.e., $B \Rightarrow$) we get $(B, \bot)$ by (SI); the required pair (B, Y) follows by (WO).

[2] We assume the readers to be familiar with LK.

Assume $\Rightarrow Y$. From $(\top, \top)$ and $\Rightarrow Y$ (i.e., $\top \models Y$) by (WO) we get $(\top, Y)$, from which (B, Y) follows by (SI). $\square$

Henceforth, when presenting derivations in our calculi, we will omit the LK sub-derivations.

3.1 Basic Production Inference $\text{OUT}_2^\perp$

The calculus $\mathcal{SC}_2^\perp$ for the causal[3] basic output logic $\text{OUT}_2^\perp$ is obtained by adding to the core calculus (consisting of LK with the rules (IN) and (OUT)) the pair elimination rule (E_2) in Fig. 2.

Remark 2. *The rule (E_2) is inspired by the embedding in (Bochman 2003) of $OUT_2^\perp$ into the modal logic* $\mathbf{K}$*:* $(A_1, X_1), \ldots, (A_n, X_n) \vdash_{OUT_2^\perp} (B, Y)$ *iff* $(*)$ $(A_1 \to \Box X_1), \ldots, (A_n \to \Box X_n), B \vdash \Box Y$ *is derivable in* $\mathbf{K}$*. To provide the rule's intuition we make use of the sequent calculus GK for* $\mathbf{K}$ *in (Ono 1998). GK extends LK with the following rule for introducing boxes (or eliminating them, looking at the rule bottom up):*

$$\frac{A_1, \ldots, A_n \vdash B}{\Box A_1, \ldots, \Box A_n \vdash \Box B}\ (\Box R)$$

To prove the sequent $(*)$ in GK we can apply the LK rule for $\to$ to one of the implications $(A_i \to \Box X_i)$ on the left. This creates two premises: (a) $G', B \vdash_K \Box Y, A_i$ and (b) $G', B, \Box X_i \vdash_K \Box Y$ (where G' is the set of all implications on the left-hand side but $(A_i \to \Box X_i)$). Now (a) $G', B \vdash_K \Box Y, A_i$ is equivalent in $\mathbf{K}$ *to (derivable in GK if and only if so is) the sequent $G', (B \land \neg A_i) \vdash_K \Box Y$, that using the embedding again leads to the first premise of (E_2); (b) $G', B, \Box X_i \vdash_K \Box Y$ is equivalent (for suitable G') to $G', B \vdash_K \Box(Y \lor \neg X_i)$, which leads to the second premise of (E_2).*

We prove below the soundness and completeness of the calculus $\mathcal{SC}_2^\perp$ for $\text{OUT}_2^\perp$. We start by describing a useful characterization of derivability in $\mathcal{SC}_2^\perp$ of an I/O sequent $(A_1, X_1), \ldots, (A_n, X_n) \vdash (B, Y)$ via derivability of certain sequents in LK.

Notation 1. *$\mathcal{P}(X)$ will denote the set of all partitions of the set X, i.e. $\mathcal{P}(X) = \{(I, J) \mid I \cup J = X, I \cap J = \emptyset\}$*

Notice that if a concluding rule (IN) or (OUT) can be applied to the conclusion of (E_2), it can also be applied to its premises. This observation implies that if $(A_1, X_1), \ldots, (A_n, X_n) \vdash (B, Y)$ is derivable in $\mathcal{SC}_2^\perp$ there is a derivation in which the concluding rules are applied only when all deriving pairs are eliminated. We use this *I/O normal form* of derivations in the proof of the following lemma.

Lemma 2 (Characterization lemma for $\mathcal{SC}_2^\perp$)**.** *$(A_1, X_1), \ldots, (A_n, X_n) \vdash (B, Y)$ is derivable in $\mathcal{SC}_2^\perp$ iff for all partitions $(I, J) \in \mathcal{P}(\{1, \ldots, n\})$, either $B \Rightarrow \{A_i\}_{i \in I}$ or $\{X_j\}_{j \in J} \Rightarrow Y$ is derivable in LK.*

Proof. By induction on n. Base case: $n = 0$. The only partition is $(\emptyset, \emptyset)$. From derivability of $B \Rightarrow$ or $\Rightarrow Y$ follows $\vdash (B, Y)$ by either (IN) or (OUT); the converse also holds.

Inductive case: from n to $n+1$. Let $G = \{(A_1, X_1), \ldots, (A_{n+1}, X_{n+1})\}$ and consider only derivations in $\mathcal{SC}_2^\perp$ in I/O normal form (so the last applied rule can only be (E_2)). We characterize the condition when there exists a derivation of $G \vdash (B, Y)$ whose last rule applied is (E_2) eliminating a pair (A_k, X_k), for some $k \in \{1, \ldots, n+1\}$. This application leads to two premises: $G' \vdash (B \land \neg A_k, Y)$ and $G' \vdash (B, Y \lor \neg X_k)$, where $G' = G \setminus \{(A_k, X_k)\}$. By the inductive hypothesis, the derivability of these premises is equivalent to the derivability of the following sequents: for each $(I', J') \in \mathcal{P}(\{1, \ldots, n+1\} \setminus \{k\})$:

- (a1) $B \land \neg A_k \Rightarrow \{A_i\}_{i \in I'}$ or (a2) $\{X_j\}_{j \in J'} \Rightarrow Y$, and
- (b1) $B \Rightarrow \{A_i\}_{i \in I'}$ or (b2) $\{X_j\}_{j \in J'} \Rightarrow Y \lor \neg X_k$.

(a1) is equivalent in LK to (a1)' $B \Rightarrow \{A_i\}_{i \in I' \cup \{k\}}$, and (b2) to (b2)' $\{X_j\}_{j \in J' \cup \{k\}} \Rightarrow Y$. Hence (a1)', (a2) give the required condition for the partition $(I' \cup \{k\}, J')$, while (b1), (b2)' for the partition $(I', J' \cup \{k\})$. $\square$

The soundness and completeness proof of $\mathcal{SC}_2^\perp$ makes use of the admissibility in the calculus of the structural rules for I/O sequents (weakening, contraction, and cut) in Fig. 3. Recall that a rule is *admissible* if its addition does not change the set of sequents that can be derived.

Lemma 3. *The rules (IO-Wk), (IO-Ctr) and (IO-Cut) in Fig. 3 are admissible in $\mathcal{SC}_2^\perp$.*

Proof. By Lem. 2 we can reduce the admissibility of these structural rules to the admissibility of weakening, contraction, and cut in LK. Consider the case (IO-Cut). Let $G = \{(D_1, W_1) \ldots (D_m, W_m)\}$, $G' = \{(A_1, X_1), \ldots (A_n, X_n)\}$ and $(A_{n+1}, X_{n+1}) = (C, Z)$. Now $G, G' \vdash (B, Y)$ is derivable in $\mathcal{SC}_2^\perp$ iff for any $(I_1, J_1) \in \mathcal{P}(\{1, \ldots, n\})$ and $(I_2, J_2) \in \mathcal{P}(\{1, \ldots, m\})$ either $B \Rightarrow \{A_i\}_{i \in I_1}, \{D_i\}_{i \in I_2}$ or $\{X_j\}_{j \in J_1}, \{W_j\}_{j \in J_2} \Rightarrow Y$ is derivable in LK. It is tedious but easy to see that this holds by applying Lem. 2 to the hypotheses $(C, Z), G \vdash (B, Y)$ and $G' \vdash (C, Z)$, and using the structural rules of LK. $\square$

Theorem 1 (Soundness and completeness of $\mathcal{SC}_2^\perp$)**.** *$G \vdash (B, Y)$ is derivable in $\mathcal{SC}_2^\perp$ iff (B, Y) is derivable from the pairs in G in $OUT_2^\perp$.*

Proof. (Completeness) Assume that (B, Y) is derivable in $\text{OUT}_2^\perp$. We prove by induction on the derivation tree that for each pair (A, X) occurring in it, the I/O sequent $G \vdash (A, X)$ is derivable in $\mathcal{SC}_2^\perp$. The case $(A, X) \in G$ is:

$$\frac{\dfrac{A \land \neg A \Rightarrow}{G' \vdash (A \land \neg A, X)}\ (\text{IN}) \qquad \dfrac{\Rightarrow X \lor \neg X}{G' \vdash (A, X \lor \neg X)}\ (\text{OUT})}{(A, X), G' \vdash (A, X)}\ (\text{E}_2)$$

We show the case of (SI) ($B \models A$ iff $B \land \neg A \Rightarrow$):

$$\frac{\dfrac{\overset{\text{by I.H.}}{\vdots}}{G \vdash (A, Y)} \qquad \dfrac{\dfrac{B \land \neg A \Rightarrow}{\vdash (B \land \neg A, Y)}\ (\text{IN}) \qquad \dfrac{\Rightarrow Y \lor \neg Y}{\vdash (B, Y \lor \neg Y)}\ (\text{OUT})}{(A, Y) \vdash (B, Y)}\ (\text{E}_2)}{G \vdash (B, Y)}\ (\text{IO-Cut})$$

[3]Called basic production inference in (Bochman 2004).

$$\frac{G \vdash (B \wedge \neg A, Y) \qquad G \vdash (B, Y \vee \neg X)}{(A, X),\ G \vdash (B, Y)}\ (\mathrm{E}_2) \qquad \frac{B \Rightarrow A \qquad G \vdash (B, Y \vee \neg X)}{(A, X),\ G \vdash (B, Y)}\ (\mathrm{E}_1)$$

$$\frac{G \vdash (B \wedge \neg A, Y) \qquad G \vdash (B \wedge X, Y \vee \neg X)}{(A, X),\ G \vdash (B, Y)}\ (\mathrm{E}_4) \qquad \frac{B \Rightarrow A \qquad G \vdash (B \wedge X, Y \vee \neg X)}{(A, X),\ G \vdash (B, Y)}\ (\mathrm{E}_3)$$

Figure 2: Sequent rules for pair elimination (one for each considered causal I/O logic)

$$\frac{G \vdash (B, Y)}{(A, X), G \vdash (B, Y)}\ (\text{IO-Wk}) \qquad \frac{(A, X), (A, X), G \vdash (B, Y)}{(A, X), G \vdash (B, Y)}\ (\text{IO-Ctr}) \qquad \frac{G \vdash (C, Z) \qquad (C, Z), G' \vdash (B, Y)}{G, G' \vdash (B, Y)}\ (\text{IO-Cut})$$

Figure 3: Structural I/O rules (admissible in all our calculi)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A,X)}{(A, Y \vee X)}\ (\mathrm{WO})}{(B \wedge A, Y \vee X)}\ (\mathrm{SI}) \qquad \dfrac{(B, Y \vee \neg X)}{(B \wedge A, Y \vee \neg X)}\ (\mathrm{SI})}{(B \wedge A, (Y \vee X) \wedge (Y \vee \neg X))}\ (\mathrm{AND})}{(B \wedge A, Y \vee (X \wedge \neg X))}\ (\mathrm{WO})}{(B \wedge A, Y)}\ (\mathrm{WO}) \qquad (B \wedge \neg A, Y)}{((B \wedge A) \vee (B \wedge \neg A), Y)}\ (\mathrm{OR})}{(B \wedge (A \vee \neg A), Y)}\ (\mathrm{SI})}{(B, Y)}\ (\mathrm{SI})$$

Figure 4: Derivation of the rule (E_2) in $\mathrm{OUT}_2^{\perp}$

For (AND): we derive $(B, X_1), (B, X_2) \vdash (B, X_1 \wedge X_2)$, and apply (IO-Cut) twice followed by many applications of contraction to the resulting derivation as follows

$$\dfrac{\dfrac{\dfrac{\overset{\textit{by I.H.}}{\vdots}}{G \vdash (B, X_1)} \qquad \dfrac{\dfrac{\overset{\textit{by I.H.}}{\vdots}}{G \vdash (B, X_2)} \qquad (B, X_1),\ (B, X_2) \vdash (B, X_1 \wedge X_2)}{G,\ (B, X_1) \vdash (B, X_1 \wedge X_2)}}{G, G \vdash (B, X_1 \wedge X_2)}}{G \vdash (B, X_1 \wedge X_2)}\ (\text{IO-Ctr}) \times n$$

The claim follows by Lem 3.

Soundness See Lem. 1 and Fig. 4 for the rule (E_2). ☐

3.2 Causal Production Inference $\mathrm{OUT}_4^{\perp}$

The calculus $\mathcal{SC}_4^{\perp}$ for the causal version of reusable output logic $\mathrm{OUT}_4^{\perp}$ extends the core calculus (consisting of LK with the the rules (IN) and (OUT)) with the pair elimination rule (E_4) in Fig. 2.

Inspired by the normal modal logic embedding of $\mathrm{OUT}_4^{\perp}$ in (Bochman 2003), the shape of the rule (E_4) requires to amend the statement of the characterization lemma. The proof of this lemma is similar to the one for $\mathcal{SC}_2^{\perp}$

Lemma 4 (Characterization lemma for $\mathcal{SC}_4^{\perp}$). *$(A_1, X_1), \ldots, (A_n, X_n) \vdash (B, Y)$ is derivable in $\mathcal{SC}_4^{\perp}$ iff for all $(I, J) \in \mathcal{P}(\{1, \ldots, n\})$, either $B, \{X_j\}_{j \in J} \Rightarrow \{A_i\}_{i \in I}$ or $\{X_j\}_{j \in J} \Rightarrow Y$ is derivable in LK.*

Theorem 2 (Soundness and completeness of $\mathcal{SC}_4^{\perp}$). *$G \vdash (B, Y)$ is derivable in $\mathcal{SC}_4^{\perp}$ iff the pair (B, Y) is derivable from the pairs in G in $\mathrm{OUT}_4^{\perp}$.*

Proof. (Completeness) To derive (CT) in $\mathcal{SC}_4^{\perp}$ we first derive $(A, X), (A \wedge X, Y) \vdash (A, Y)$, and then apply (IO-Cut) and (IO-Ctr) (Fig. 3). The claim follows by the admissibility of these structural rules in $\mathcal{SC}_4^{\perp}$, which can be reduced to the admissibility of the structural rules in LK as for $\mathcal{SC}_2^{\perp}$.

(Soundness) Replace in Fig. 4 the subtree that derives $(B \wedge A, Y \vee \neg X)$ from the pair $(B, Y \vee \neg X)$ by the following derivation, which uses the rule (CT)

$$\dfrac{\dfrac{(A, X)}{(B \wedge A, X)}\ (\mathrm{SI}) \qquad \dfrac{(B \wedge X, Y \vee \neg X)}{(B \wedge A \wedge X, Y \vee \neg X)}\ (\mathrm{SI})}{(B \wedge A, Y \vee \neg X)}\ (\mathrm{CT})$$

☐

3.3 Production Inference $\mathrm{OUT}_1^{\perp}$ and Regular Production Inference $\mathrm{OUT}_3^{\perp}$

The calculi $\mathcal{SC}_1^{\perp}$ and $\mathcal{SC}_3^{\perp}$ for the causal simple-minded output $\mathrm{OUT}_1^{\perp}$ and simple-minded reusable output $\mathrm{OUT}_3^{\perp}$ consist of LK with (IN) and (OUT) extended with the pair elimination rules (E_1) and (E_3) in Fig. 2, respectively.

Remark 3. *Unlike $\mathrm{OUT}_2^{\perp}$ and $\mathrm{OUT}_4^{\perp}$, there is no modal embedding in the literature to provide guidance for the development of the pair elimination rules for $\mathrm{OUT}_1^{\perp}$ and $\mathrm{OUT}_3^{\perp}$. These rules are instead designed by appropriately modifying (E_2) and (E_4). Indeed $\mathrm{OUT}_1^{\perp}$ and $\mathrm{OUT}_3^{\perp}$ impose restrictions on $\mathrm{OUT}_2^{\perp}$ and $\mathrm{OUT}_4^{\perp}$, respectively, by prohibiting the combination of inputs. The rules (E_1) and (E_3) are defined by reflecting this limitation.*

Due to the LK premise in their peculiar rules, derivations in $\mathcal{SC}_1^{\perp}$ and $\mathcal{SC}_3^{\perp}$ have a simpler form w.r.t. derivations in $\mathcal{SC}_2^{\perp}$ and $\mathcal{SC}_4^{\perp}$; this form could be exploited for the soundness and completeness proof. We proceed instead as for the latter calculi by proving the characterization lemma. This lemma will be key to solve the open problems about computational bounds and modal embeddings for $\mathrm{OUT}_1^{\perp}$ and $\mathrm{OUT}_3^{\perp}$. The proof of the lemma for $\mathcal{SC}_1^{\perp}$ and $\mathcal{SC}_3^{\perp}$ is less straightforward than for the other logics. The intuition here is that the characterization considers all possible ways to apply the rule (E_3) (or (E_1)), by partitioning the premises $(A_1, X_1), \ldots, (A_n, X_n)$ into two disjoint sets (I of remaining deriving pairs and J of eliminated pairs). We will focus on the lemma for $\mathcal{SC}_3^{\perp}$, the one for $\mathcal{SC}_1^{\perp}$ being a simplified case (with a very similar proof). Its proof relies on the following result:

Lemma 5. *If $(A,X), G \vdash (B,Y)$ is derivable in $\mathcal{SC}_3^\perp$, then so is $G \vdash (B \wedge X, Y \vee \neg X)$.*

Proof. Easy induction on the length of the derivation. We proceed by case distinction on the last applied rule. ☐

Lemma 6 (Characterization lemma for $\mathcal{SC}_3^\perp$)**.** *$(A_1,X_1),\dots,(A_n,X_n) \vdash (B,Y)$ is derivable in $\mathcal{SC}_3^\perp$ iff for all $(I,J) \in \mathcal{P}(\{1,\dots,n\})$, one of the following holds:*

- *$B, \{X_j\}_{j\in J} \Rightarrow A_i$ is derivable in LK for some $i \in I$,*
- *$B, \{X_j\}_{j\in J} \Rightarrow$ is derivable in LK,*
- *$\{X_j\}_{j\in J} \Rightarrow Y$ is derivable in LK.*

Proof. ($\Rightarrow$): Let $(I,J) \in \mathcal{P}(\{1,\dots,n\})$ be any partition. By (several application of) Lem. 5 to each (A_j,X_j) with $j \in J$, we get that the I/O sequent $(*)$ $\{(A_i,X_i) \mid i \in I\} \vdash (B \wedge \bigwedge_{j\in J} X_j, Y \vee \bigvee_{j\in J} \neg X_j)$ is derivable in $\mathcal{SC}_3^\perp$. We consider the last rule (r) applied in the derivation of $(*)$. Three cases can arise:

- (r) = (E$_3$) eliminating the deriving pair (A_k, X_k) with $k \in I$; hence $B \wedge \bigwedge_{j\in J} X_j \Rightarrow A_k$ (and therefore $B, \{X_j\}_{j\in J} \Rightarrow A_k$) is derivable in LK,
- (r) = (IN) then $B, \{X_j\}_{j\in J} \Rightarrow$ is derivable in LK.
- (r) = (OUT) then $\{X_j\}_{j\in J} \Rightarrow Y$ is derivable in LK.

($\Leftarrow$): We stepwise construct a derivation in $\mathcal{SC}_3^\perp$ of $(A_1,X_1),\dots,(A_n,X_n) \vdash (B,Y)$. We start with $(I,J) = (\{1,\dots,n\},\emptyset)$, and we distinguish the three cases from the assumption of the lemma: (1) $B, \{X_j\}_{j\in J} \Rightarrow A_i$ for some $i \in I$, (2) $B, \{X_j\}_{j\in J} \Rightarrow$, and (3) $\{X_j\}_{j\in J} \Rightarrow Y$. If either (2) or (3) holds the derivation follows by applying a concluding rule ($J = \emptyset$). If (1) holds, we apply (E$_3$) bottom up getting $\{(A_t,X_t)\}_{t\in\{1,\dots,n\}\setminus\{i\}} \vdash (B \wedge X_i, Y \vee \neg X_i)$ as the second premise. We now apply the same reasoning to this latter sequent (considering the partition $(I,J) = (\{1,\dots,n\}\setminus\{i\},\{i\})$), and keep applying it for the second premise of (E$_3$), until a concluding rule is applied. This will eventually happen since I loses the index of the eliminated pair at each step. ☐

The characterization lemma for $\mathcal{SC}_1^\perp$ has the following formulation (the proof is similar to the proof for $\mathcal{SC}_3^\perp$).

Lemma 7 (Characterization lemma for $\mathcal{SC}_1^\perp$)**.** *$(A_1,X_1),\dots,(A_n,X_n) \vdash (B,Y)$ is derivable in $\mathcal{SC}_1^\perp$ iff for all partitions $(I,J) \in \mathcal{P}(\{1,\dots,n\})$, at least one of the following holds:*

- *$B \Rightarrow A_i$ is derivable in LK for some $i \in I$,*
- *$B \Rightarrow$ is derivable in LK,*
- *$\{X_j\}_{j\in J} \Rightarrow Y$ is derivable in LK.*

Theorem 3 (Soundness and completeness of $\mathcal{SC}_1^\perp$ and $\mathcal{SC}_3^\perp$)**.** *$G \vdash (B,Y)$ is derivable in $\mathcal{SC}_1^\perp$ ($\mathcal{SC}_3^\perp$) iff (B,Y) is derivable from the pairs in G in $OUT_1^\perp$ (OUT_3).*

Proof. (*Completeness*) uses the admissibility proof of the structural rules in Fig. 3 The case $(A,X) \vdash (A,X)$ is now handled as follows:

$$\dfrac{A \Rightarrow A \qquad \dfrac{\Rightarrow X \vee \neg X}{\vdash (A, X \vee \neg X)}\,(\text{OUT})}{(A,X) \vdash (A,X)}\,(\text{E}_1)$$

The derivation of the rule (SI) directly uses the premise $B \Rightarrow A$, instead of the rule (IN).

(Soundness): For $\mathcal{SC}_1^\perp$, the derivation of (E$_1$) is obtained by juxtaposing to the subderivation of $(B \wedge A, Y)$ from (A,X) and $(B, Y \vee \neg X)$ in Fig. 4 the following

$$\dfrac{\dfrac{\vdots}{(B \wedge A, Y)}}{(B,Y)}\,(\text{SI})$$

noticing that the side condition $B \Rightarrow A$ of the rule (E$_1$) implies that B and $B \wedge A$ are classically equivalent. The soundness of the rule (E$_3$) is very similar since the translation of the rule (E$_4$) from the proof of Theorem 2 also contains a sub-derivation of the pair $(B \wedge A, Y)$ from (A,X) and $(B \wedge X, Y \vee \neg X)$ which uses only the rules of OUT$_3$ ((WO), (SI), (AND) and (CT)). ☐

4 Causal I/O Logics vs. Original I/O Logics

We establish a syntactic correspondence between derivability in the original I/O logics and in their causal version. This correspondence obtained utilizing the sequent calculi $\mathcal{SC}_1^\perp$-$\mathcal{SC}_4^\perp$, will enable to use them for OUT$_1$-OUT$_4$, and to transfer all results arising from the calculi for the causal I/O logics to the original I/O logics.

Note that $\mathcal{SC}_1^\perp$-$\mathcal{SC}_4^\perp$ rely on the axiom (BOT), which is absent in the original I/O logics. An inspection of the soundness proofs for our calculi shows that (BOT) is solely employed in the translation of the rule (IN) (Lemma 1). Can we simply remove this rule and hence get rid of axiom (BOT)? Yes, but only for $\mathcal{SC}_1^\perp$ and $\mathcal{SC}_3^\perp$, where, as evidenced by the completeness proof, (IN) is used to derive (BOT) and not utilized elsewhere. Hence, by removing the rule (IN) from $\mathcal{SC}_1^\perp$ and $\mathcal{SC}_3^\perp$ we get sequent calculi for OUT$_1$ and OUT$_3$. These calculi are close to the sequent calculi inspired by conditional logics introduced in (Lellmann 2021). The same does not hold for $\mathcal{SC}_2^\perp$ and $\mathcal{SC}_4^\perp$, where (IN) is needed, e.g., to derive (SI). Instead of developing ad hoc calculi to handle the original I/O logics, we leverage $\mathcal{SC}_1^\perp$-$\mathcal{SC}_4^\perp$ using the following result:

Theorem 4. *$(A_1,X_1),\dots,(A_n,X_n) \vdash_{OUT_k} (B,Y)$ iff $(A_1,X_1),\dots,(A_n,X_n) \vdash_{OUT_k^\perp} (B,Y)$ and $X_1,\dots,X_n \models Y$ in classical logic, for each $k = 1,\dots,4$.*

Proof. ($\Rightarrow$) Derivability in OUT$_k$ implies derivability in the stronger logic OUT$_k^\perp$. The additional condition $X_1,\dots,X_n \models Y$ can be proved by an easy induction on the length of the derivation in the original I/O logics: it is enough to check that for every rule if the outputs of all (pairs-)premises follow from $X_1 \wedge \cdots \wedge X_n$, then so is the output of the (pair-)conclusion.

($\Leftarrow$) Consider our sequent calculi $\mathcal{SC}_1^\perp$ - $\mathcal{SC}_4^\perp$. The translation constructed in their soundness theorem shows how to map a derivation of the I/O sequent $(A_1, X_1), \ldots, (A_n, X_n) \vdash (B, Y)$ in $\mathcal{SC}_k^\perp$ into a I/O derivation of (B, Y) from the pairs $\{(A_1, X_1), \ldots, (A_n, X_n)\}$ using the rules of the logic $\mathrm{OUT}_k^\perp$. As observed before, (BOT) appears in this transformed derivation only inside sub-derivations of the pairs (B', Y') derived in $\mathcal{SC}_k^\perp$ by the rule (IN). We can replace every such sub-derivation with a derivation that does not contain the axiom (BOT) and uses only the rules of the weakest logic OUT_1. This latter derivation relies on the premise $B' \Rightarrow$ of the rule (IN) (which implies $B' \models \perp$), on the condition $X_1 \wedge \cdots \wedge X_n \models Y$ from the statement of the lemma and the fact that $Y \models Y'$ since in all our calculi the goal output in the premises of the elimination rules is the same or weaker than the goal output in the conclusion. The required derivation is the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(A_1, X_1)}{(\perp, X_1)}\text{(SI)} \quad \cdots \quad \dfrac{(A_n, X_n)}{(\perp, X_n)}\text{(SI)}}{(\perp, X_1 \wedge \cdots \wedge X_n)}\text{(AND)} \times (n-1)}{(\perp, Y')}\text{(WO)}}{(B', Y')}\text{(SI)}}{}$$

After the replacement of all indicated sub-derivation of (B', Y') with the ones obove, we will get a derivation of (B, Y) that does not use the axiom (BOT) and thus (B, Y) is derivable in the original I/O calculus in (Makinson and van der Torre 2000) for OUT_k. $\square$

Remark 4. *The constructive proof above heavily relies on the restricted form of the I/O derivations resulting from translating our sequent derivations. If at all possible, finding ways to eliminate the use of the (BOT) axiom in arbitrary I/O derivations within $\mathrm{OUT}_k^\perp$ would be a challenging task. The power of structural proof theory lies in its capacity to solely examine well-behaved derivations.*

5 Applications

Our proof-theoretic investigation is used here to establish the following results for the original and the causal I/O logics: uniform possible worlds semantics (Sec. 5.1), co-NP-completeness and automated deduction methods (Sec. 5.2), and new embeddings into normal modal logics (Sec. 5.3).

5.1 Possible Worlds Semantics

We provide possible worlds semantics for both the original and the causal I/O logics. Our semantics is a generalization of the bimodels semantics in (Bochman 2004) for $\mathrm{OUT}_2^\perp$; it turns out to be simpler than them for the remaining causal logics, and than the procedural semantics for the original I/O logics. As we will see, this semantics facilitates clean and uniform solutions to various unresolved inquiries regarding I/O logics that were only partially addressable.

First, notice that a contrapositive reading of the characterization lemmas leads to countermodels for non-derivable statements in all considered causal I/O logics. These countermodels consist of (a partition and) several boolean interpretations (two for $\mathrm{OUT}_2^\perp$, $\mathrm{OUT}_4^\perp$ and their causal versions, and $(n+2)$ for $\mathrm{OUT}_1^\perp$, $\mathrm{OUT}_3^\perp$ and their causal versions) that falsify the LK sequents from the respective lemma statement. We show below that a suitable generalization of these countermodels provides alternative semantic characterizations for both the original and the causal I/O logics.

A possible worlds semantics for the causal I/O logics was introduced by (Bochman 2004) using *bimodels*. For the simplest case of $\mathrm{OUT}_2^\perp$, bimodel is a pair of worlds (here 'world' can be seen as a synonym for boolean interpretation) corresponding to input and output states.

Definition 3. *(Bochman 2004) A pair (A, X) is said to be* valid *in a bimodel (in, out) if $in \vDash A$ implies $out \vDash X$.*

The adequacy of this semantics implies, in particular, that $G \vdash_{\mathrm{OUT}_2^\perp} (B, Y)$ if and only if the validity of all pairs from G implies validity of (B, Y) for all bimodels. The notion of bimodels for $\mathrm{OUT}_1^\perp$, $\mathrm{OUT}_3^\perp$ and $\mathrm{OUT}_4^\perp$ is more complex, with input and output states consisting of arbitrary deductively closed sets of formulae, instead of worlds.

To construct our semantics, we look at the countermodels provided by the characterization lemma from the point of view of the simplest bimodels of $\mathrm{OUT}_2^\perp$. Lemma 2 says indeed that if $(A_1, X_1), \ldots, (A_n, X_n) \vdash (B, Y)$ is not derivable in $\mathcal{SC}_2^\perp$ there is a partition (I, J), and two boolean interpretations *in* and *out* such that: *in* falsifies the LK-sequent $B \Rightarrow \{A_i\}_{i \in I}$ (meaning that $in \vDash B$ and $in \nvDash A_i$ for all $i \in I$) and *out* falsifies the LK-sequent $\{X_j\}_{j \in J} \Rightarrow Y$ (meaning that $out \vDash X_j$ for all $j \in J$ and $out \nvDash Y$). These two interpretations lead to a bimodel that falsifies $(A_1, X_1), \ldots, (A_n, X_n) \vdash (B, Y)$; indeed all pairs (A_i, X_i) for $i \in I$ are valid in (in, out) as $in \nvDash A_i$, all pairs (A_j, X_j) for $j \in J$ are valid in (in, out) as $out \vDash X_j$, but (B, Y) is not valid in (in, out) because $in \vDash B$ and $out \nvDash Y$.

Reasoning in a similar way about the countermodels for $\mathrm{OUT}_1^\perp$ given by Lem. 7, we observe there are now multiple input worlds, each falsifying in $B \Rightarrow A_i$ the input A_i (plus one additional input world that arises from the sequent $B \Rightarrow$). This leads to the following generalization of bimodels with multiple input worlds.

Definition 4. *An* I/O model *is a pair (In, out) where out is the output world, and In is a set of input worlds.*

The definition of validity in an I/O model will be modified to require that the input formula is true in all input worlds, rather than just in the unique input world. This update ensures that the existence of a single input world falsifying A is enough to establish the validity of the pair (A, X). Moreover, the additional ability to reuse outputs as inputs in the logics $\mathrm{OUT}_3^\perp$ and $\mathrm{OUT}_4^\perp$ can be expressed in these models by the requirement that a triggered output X should hold in the input worlds too. This leads to the following two definitions of validity in I/O models – one for the logics $\mathrm{OUT}_1^\perp$ and $\mathrm{OUT}_2^\perp$, the other for $\mathrm{OUT}_3^\perp$ and $\mathrm{OUT}_4^\perp$.

Definition 5.

- *An I/O pair (A, X) is* 1-2-valid *in an I/O model (In, out) if $(\forall in \in In.\ in \vDash A)$ implies $out \vDash X$.*

Logic	Frame condition	Notion of validity
$\text{OUT}_1^{\perp}$	$\|In\| \geq 1$	1-2-validity
$\text{OUT}_2^{\perp}$	$\|In\| = 1$	1-2-validity
$\text{OUT}_3^{\perp}$	$\|In\| \geq 1$	3-4-validity
$\text{OUT}_4^{\perp}$	$\|In\| = 1$	3-4-validity
OUT_1	no conditions	1-2-validity
OUT_2	$\|In\| \leq 1$	1-2-validity
OUT_3	no conditions	3-4-validity
OUT_4	$\|In\| \leq 1$	3-4-validity

Table 2: Conditions on I/O models (size of the set *In* of input worlds) and corresponding notions of validity for I/O models.

- *An I/O pair* (A, X) *is* 3-4-valid *in an I/O model* (In, out) *if* $(\forall in \in In.\ in \models A)$ *implies* $(\forall w \in \{out\} \cup In.\ w \models X)$.

When clear from the context, henceforth we will use the term *validity* to mean either 1-2-validity (hence referring to the logics OUT_1, OUT_2, $\text{OUT}_1^{\perp}$, and $\text{OUT}_2^{\perp}$) or 3-4-validity (hence referring to OUT_3, OUT_4, $\text{OUT}_3^{\perp}$, and $\text{OUT}_4^{\perp}$).

Proposition 1 (Semantics for $\text{OUT}_k^{\perp}$)**.** $G \vdash_{OUT_k^{\perp}} (B, Y)$ *iff for all I/O models (satisfying the conditions in Tab. 2) validity of all pairs from* G *implies validity of* (B, Y).

Proof. The proofs rely on the characterization lemmas for each of the four logics. We detail the proof for $\text{OUT}_3^{\perp}$, the others being similar. The equivalence is proved by negating both statements.

($\Leftarrow$) Suppose $(A_1, X_1), \ldots, (A_n, X_n) \vdash (B, Y)$ is not derivable in $\mathcal{SC}_3^{\perp}$ (and hence in $\text{OUT}_3^{\perp}$). We show the existence of a countermodel. By Lem. 6 there exists a partition (I, J) such that the sequents $\{X_j\}_{j\in J} \Rightarrow Y$, $B, \{X_j\}_{j\in J} \Rightarrow$ and $B, \{X_j\}_{j\in J} \Rightarrow A_i$ for all $i \in I$ are not derivable in LK. By the soundness and completeness of LK w.r.t. the classical truth tables semantics there exist:

- an interpretation *out*, s.t. $out \models X_j \;\; \forall j \in J$ and $out \not\models Y$,
- an interpretation in_0, s.t. $in_0 \models B$ and $in_0 \models X_j \;\; \forall j \in J$,
- for every $i \in I$ an interpretation in_i, s.t. $in_i \models B$, $in_i \models X_j$ for all $j \in J$ and $in_i \not\models A_i$.

Then the I/O model $M = (\{in_0\} \cup \{in_i \mid i \in I\}, out)$ (which has at least one input world) is a countermodel for the derivability of the I/O sequent $(A_1, X_1), \ldots, (A_n, X_n) \vdash (B, Y)$. Indeed, every pair (A_i, X_i) for $i \in I$ is 3-4-valid in M since A_i is not true in the input world in_i; every pair (A_j, X_j) for $j \in J$ is 3-4-valid in M since X_j is true in all worlds of M; but the pair (B, Y) is not 3-4-valid since B is true in all input worlds, while Y is not true in the output world *out*.

($\Rightarrow$) Let $M = (In, out)$ (with $In \neq \emptyset$) be a countermodel. I.e. M 3-4-validates all pairs $\{(A_1, X_1), \ldots, (A_n, X_n)\}$, but does not 3-4-validate the pair (B, Y). We prove that the sequent $(A_1, X_1) \ldots, (A_n, X_n) \vdash (B, Y)$ is not derivable in $\text{OUT}_3^{\perp}$. We show the existence of a partition (I, J), such that none of the LK sequents $\{X_j\}_{j\in J} \Rightarrow Y$, $\{X_j\}_{j\in J} \Rightarrow Y$, and $B, \{X_j\}_{j\in J} \Rightarrow$ is derivable. The claim follows then by Lem. 6. For such partition, we chose J as $\{j \colon \forall in \in In.\ in \models A_j\}$, and I as the rest of the indices. Notice that X_j is true in all worlds of M for every $j \in J$ by definition of 3-4-validity of (A_j, X_j). Also, the fact that (B, Y) is not 3-4-valid in M means that B holds in all input worlds, but there exists a world $w^* \in \{out\} \cup In$, s.t. $w^* \not\models Y$. Then:

- $\{X_j\}_{j\in J} \Rightarrow Y$ is not derivable in LK, because this sequent does not hold in the world w^*.
- $B, \{X_j\}_{j\in J} \Rightarrow$ is not derivable in LK, because this sequent does not hold in any input world of M (and there is at least one by the condition $In \neq \emptyset$).
- For any $i \in I$ there exists an input world in_i, s.t. $in_i \not\models A_i$ (by the choice of I). Hence $B, \{X_j\}_{j\in J} \Rightarrow A_i$ is not derivable in LK, as this sequent does not hold in in_i. $\square$

Dropping the condition of having at least one input world leads to models for the original I/O logics.

Proposition 2 (Semantics for OUT_k)**.** $G \vdash_{OUT_k} (B, Y)$ *iff for all I/O models (satisfying the conditions in Tab. 2) validity of all pairs from* G *implies validity of* (B, Y).

Proof. By Th. 4, (B, Y) is derivable from $(A_1, X_1), \ldots,$ (A_n, X_n) in OUT_k iff it is derivable in $\text{OUT}_k^{\perp}$ together with the additional condition $X_1, \ldots, X_n \models Y$. We prove that this additional condition is equivalent to the fact that every model with zero input worlds that validates all pairs from G also validates (B, Y). Notice that this will prove the proposition, as the only difference between the proposed semantics for a causal I/O logic and the corresponding original one is that the latter additionally considers models with zero input worlds (see Tab. 2).

For both notions of validity, the validity of a pair (A, X) in $(\emptyset, out)$ is equivalent to $out \models X$. Now, $X_1, \ldots, X_n \models Y$ means that every interpretation that satisfies every X_i also satisfies Y, which is equivalent to the fact that every model $(\emptyset, out)$ (with arbitrary interpretation *out*) that validates every (A_i, X_i) also validates (B, Y). $\square$

Remark 5. *A natural interpretation for the I/O models in the deontic context regards input world(s) as (different possible instances of) the real world, and the output world as the ideal world, where all triggered obligations are fulfilled.*

5.2 Complexity and Automated Deduction

We investigate the computational properties of the four original I/O logics and their causal versions. One corollary of our previous results is co-NP-completeness for all of them. Moreover, we can explicitly reduce the entailment problem in all these logics to the (un-)satisfiability of one classical propositional formula of polynomial size, a thoroughly studied problem with a huge variety of efficient tools available.

Corollary 1. *The entailment problem is a co-NP-complete problem for all eight considered I/O logics.*

Proof. The characterization lemmas for the logics $\text{OUT}_1^{\perp}$-$\text{OUT}_4^{\perp}$ imply that the non-derivability of a pair from n pairs can be non-deterministically verified in polynomial time by guessing the non-fulfilling partition (consisting of n bits) and then non-deterministically checking the non-derivability of all sequents (at most $(n + 2)$) for this partition; the latter task can be done in linear time. For the original I/O logics, by Th. 4 we also need to verify that the additional condition does not hold (guessing a falsifying boolean assignment). Thus, the entailment problem belongs to co-NP for all considered I/O logics. The co-NP-completeness follows by the fact that any arbitrary propositional formula Y is classically valid iff $(\top, Y)$ can be derived from no pairs in any calculus for the considered logics (notice that the additional condition of Th. 4 also boils down to the classical validity of Y). $\square$

We provide an explicit reduction the derivability in I/O logics to the classical validity. For OUT_2 and OUT_4 this is already contained in (Makinson and van der Torre 2000). Using the semantics introduced in Sec. 5.1, we obtain this result for Bochmann's causal I/O logics and their original version in a uniform way.

Prop. 1 shows that the underivability of $G \vdash (B, Y)$ in the causal I/O logics is equivalent to the existence of an I/O model that validates all pairs in G, but does not validate (B, Y). For $\text{OUT}_2^{\perp}$ and $\text{OUT}_4^{\perp}$ a countermodel should have exactly one input world, while for $\text{OUT}_1^{\perp}$ and $\text{OUT}_3^{\perp}$ there is always one with at most $(|G| + 1)$ input worlds.

We will encode existence of a countermodel to $G \vdash (B, Y)$ with exactly $\mathcal{N}_k$ input worlds (with $\mathcal{N}_k = 1$ for $k = 2, 4$, and $\mathcal{N}_k = |G| + 1$ for $k = 1, 3$) in classical logic. For the encoding, we assign to the input worlds the numbers from 1 to $\mathcal{N}_k$, and 0 to the output world. Let $\mathcal{V}$ be the finite set of all propositional variables that occur in the formulae of G or (B, Y). For every variable $x \in \mathcal{V}$, our encoding will use $(\mathcal{N}_k + 1)$ copies of this variable $\{x^0, \dots, x^{\mathcal{N}_k}\}$ with the intuitive interpretation that x^l is true iff x is true in the world number l. For an arbitrary formula A with variables from $\mathcal{V}$, let us denote by A^l the copy of A in which every variable $x \in \mathcal{V}$ is replaced by its labeled version x^l. We read the formula A^l as "A is true in the world number l". The exact connection with $G \vdash (B, Y)$ is stated below.

Lemma 8. $(A_1, X_1), \dots, (A_n, X_n) \vdash_{OUT_k^{\perp}} (B, Y)$ *iff the classical propositional formula* $\neg\mathcal{P}_n^k((B, Y)) \wedge \bigwedge_{(A,X)\in G} \mathcal{P}_n^k((A, X))$ *is unsatisfiable, where*

- $\mathcal{P}_n^k((A, X)) = (\bigwedge_{l=1}^{\mathcal{N}_k} A^l) \rightarrow X^0$ *for* $k = 1, 2$
- $\mathcal{P}_n^k((A, X)) = (\bigwedge_{l=1}^{\mathcal{N}_k} A^l) \rightarrow (\bigwedge_{l=0}^{\mathcal{N}_k} X^l)$ *for* $k = 3, 4$

Proof. We prove the contrapositive version.

($\Rightarrow$) Let $\mathcal{V}^L$ the set of all labeled copies of variables in $\mathcal{V}$ ($\mathcal{V}^L = \{x^l \mid x \in \mathcal{V}, l \in \{0, \dots, \mathcal{N}_k\}\}$). Suppose there is a valuation $v \colon \mathcal{V}^L \rightarrow \{0, 1\}$, that satisfies the formula in the statement (i.e. $v \vDash \mathcal{P}_n^k((A, X))$ for all $(A, X) \in G$ and $v \nvDash \mathcal{P}_n^k((B, Y))$). v can be decomposed into $(\mathcal{N}_k + 1)$ valuations $v_l \colon \mathcal{V} \rightarrow \{0, 1\}$, one for each label ($v_l(x) = v(x^l)$). It is easy to see that $(*)$: For every formula A with variables in $\mathcal{V}$, $v \vDash A^l$ iff $v_l \vDash A$ (it can be proven by trivial induction). The valuations $\{v_l\}$ can then be turned into an I/O model $M = (\{v_1, \dots, v_{\mathcal{N}_k}\}, v_0)$. Then using the reading of $v \vDash A^l$ given by $(*)$ we can see that $v \vDash \mathcal{P}_n^k((A, X))$ (for $k = 1, 2$) iff (A, X) is 1-2-valid in M, and $v \vDash \mathcal{P}_n^k((A, X))$ (for $k = 3, 4$) iff (A, X) is 3-4-valid in M. Therefore, since v satisfies the formula in the statement, M validates all pairs from G and does not validate (B, Y), which implies that $G \vdash (B, Y)$ is not derivable in $\text{OUT}_k^{\perp}$.

($\Leftarrow$) Here instead of decomposing a valuation of labeled variables into $(\mathcal{N}_k + 1)$ worlds, we use a countermodel $(\{in_1, \dots, in_{\mathcal{N}_k}\}, out)$ to define a valuation $v \colon \mathcal{V}^L \rightarrow \{0, 1\}$ of labeled variables (with $v(x^0) = out(x)$ and $v(x^l) = in_l(x)$). The proof proceeds as in the other direction. $\square$

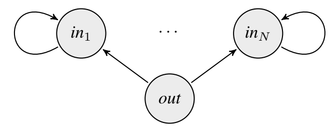

Figure 5: Kripke countermodel construction for I/O model $(\{in_1, \dots, in_N\}, out)$. Arrows represent the accessibility relation.

The result is extended to the original I/O logics via Th. 4.

Lemma 9. $(A_1, X_1), \dots, (A_n, X_n) \vdash_{OUT_k} (B, Y)$ *iff the classical propositional formula* $\mathcal{F}_n^k \vee (\neg Y \wedge \bigwedge_{i=1}^{n} X_i)$ *is unsatisfiable, where* $\mathcal{F}_n^k$ *is the formula encoding derivability of* $(A_1, X_1), \dots, (A_n, X_n) \vdash (B, Y)$ *in* $OUT_k^{\perp}$ *from Lem. 8.*

Proof. The disjunct $(\neg Y \wedge (X_1 \wedge \cdots \wedge X_n))$ arises from Th. 4 (derivability in OUT_k is equivalent to derivability in $\text{OUT}_k^{\perp}$ *and* the classical entailment of Y from $\{X_i\}_{i=1}^n$). The claim follows by Lem. 8. $\square$

5.3 Embeddings into Normal Modal Logics

We provide uniform embeddings into normal modal logics. The embeddings are a corollary of soundness and completeness of I/O logics w.r.t. I/O models.

More precisely we show that $G \vdash (B, Y)$ in I/O logics iff a certain sequent consisting of shallow formulae only (meaning that the formulae do not contain nested modalities) is valid in suitable normal modal logics. To do that we establish a correspondence between pairs and shallow formulae.

The I/O models already use the terminology of Kripke semantics that define normal modal logic. To establish a precise link between the two semantics we need only to define the accessibility relation on worlds. We will treat the set of input worlds *In* as the set of worlds accessible from the output world *out* (see Fig. 5). Under this view on input worlds, 1-2-validity (resp. 3-4-validity) of the pair (A, X) is equivalent to the truth of the modal formula $\Box A \rightarrow X$ (resp. $\Box A \rightarrow X \wedge \Box X$) in the world *out*.

Also, the conditions on the number of input worlds that are used in Prop. 1 and Prop 2 to distinguish different I/O logics can be expressed in normal modal logics by standard Hilbert axioms. Specifically, axiom $\mathbf{D}\colon \Box A \rightarrow \Diamond A$ forces Kripke models to have at least one accessible world, while $\mathbf{F}\colon \Diamond A \rightarrow \Box A$ forces them to have at most one accessible world. As proved below, the embedding works for the basic modal logic $\mathbf{K}$ extended with $\mathbf{D}$ (which results in the well-known standard deontic logic (von Wright 1951) $\mathbf{KD}$), with $\mathbf{F}$, or both axioms.

Henceforth we abbreviate, e.g., validity in the logics $\mathbf{K}$ (respectively $\mathbf{K}+\mathbf{F}$) with $\models_{\mathbf{K}/\mathbf{K}+\mathbf{F}}$.

Theorem 5. *(B, Y) is derivable from pairs G in*

- *OUT_1 and OUT_2 iff $G^{\Box}_{1/2} \models_{\mathbf{K}/\mathbf{K}+\mathbf{F}} \Box B \to Y$*
- *OUT_3 and OUT_4 iff $G^{\Box}_{3/4} \models_{\mathbf{K}/\mathbf{K}+\mathbf{F}} \Box B \to Y \wedge \Box Y$*
- *$OUT_1^{\perp}$ and $OUT_2^{\perp}$ iff $G^{\Box}_{1/2} \models_{\mathbf{KD}/\mathbf{KD}+\mathbf{F}} \Box B \to Y$*
- *$OUT_3^{\perp}$ and $OUT_4^{\perp}$ iff $G^{\Box}_{3/4} \models_{\mathbf{KD}/\mathbf{KD}+\mathbf{F}} \Box B \to Y \wedge \Box Y$*

*where $G^{\Box}_{1/2} = \{\Box A_i \to X_i \mid (A_i, X_i) \in G\}$,
and $G^{\Box}_{3/4} = \{\Box A_i \to X_i \wedge \Box X_i \mid (A_i, X_i) \in G\}$.*

Proof. We show these equivalences by transforming the I/O countermodels given by Prop. 1 and Prop. 2 into Kripke countermodels for the corresponding modal logic and vice versa. The transformations will be the same for all the considered logics. We detail the case of $\mathrm{OUT}_3^{\perp}$.

($\Leftarrow$) Assume $G^{\Box}_3 \models_{\mathbf{KD}} \Box B \to Y \wedge \Box Y$ does not hold. Then there should exist a Kripke model M in which every world has at least one world accessible from it, and a world w in M, such that $w \models \Box A \to X \wedge \Box X$ for every $(A, X) \in G$ and $w \not\models \Box B \to Y \wedge \Box Y$. Let $N(w)$ be the set of all worlds accessible from w in M. Then the I/O model $(N(w), w)$ will be a countermodel for $G \vdash (B, Y)$; notice indeed that $w \models \Box A \to X \wedge \Box X$ means exactly 3-4-validity of (A, X) in $(N(w), w)$, so all pairs in G are 3-4-valid in $(N(w), w)$, but (B, Y) is not 3-4-valid, and $|N(w)| \geq 1$. Hence $G \vdash (B, Y)$ is not derivable in $\mathrm{OUT}_3^{\perp}$.

($\Rightarrow$) Assume $G \vdash (B, Y)$ is not derivable in $\mathrm{OUT}_3^{\perp}$. Then there is some I/O model $(\mathit{In}, \mathit{out})$ with $|\mathit{In}| \geq 1$, s.t. all pairs from G are 3-4-valid in $(\mathit{In}, \mathit{out})$ and (B, Y) is not 3-4-valid in $(\mathit{In}, \mathit{out})$. Consider the Kripke model M that consists of worlds $\mathit{In} \cup \{\mathit{out}\}$ with accessibility relation defined as shown in Fig. 5 (all input worlds are accessible from the output world and every input world is accessible from itself). M satisfies the frame condition for $\mathbf{KD}$ as there is at least one accessible world from *out* because of $|\mathit{In}| \geq 1$, and exactly one accessible world for every input world (itself). *out* satisfies the modal formula $A \to X \wedge \Box X$ (with A and X being propositional formulae) in the Kripke model M iff the I/O pair (A, X) is 3-4-valid in $(\mathit{in}, \mathit{out})$. So in M, $\mathit{out} \models A \to X \wedge \Box X$ for every pair $(A, X) \in G$ and $\mathit{out} \not\models B \to Y \wedge \Box Y$. Therefore $G^{\Box}_3 \models_{\mathbf{KD}} \Box B \to Y \wedge \Box Y$ does not hold. $\Box$

Remark 6. *Modal embeddings were already known for the causal logics $OUT_2^{\perp}$ and $OUT_4^{\perp}$. The embedding for $OUT_2^{\perp}$ was translating the pair (A, X) into the $\mathbf{K}$ formula $A \to \Box X$. In (Makinson and van der Torre 2000) this embedding was stated for OUT_2 and OUT_4 together with the additional condition $X_1, \ldots, X_n \models Y$ (appearing in our Th. 4). Note that moving the modality to inputs allows for a more refined embedding. The validity of the statement $(A \to \Box X), \ldots, (A \to \Box X) \models (B \to \Box Y)$ is indeed the same in all four target logics we use (**K**, **KD**, **K + F** and **KD + F**), while the validity of $(\Box A \to X), \ldots, (\Box A \to X) \models (\Box B \to Y)$ is different.*

6 Conclusions

We have introduced sequent calculi for I/O logics. Our calculi provide a natural syntactic connection between derivability in the four original I/O logic (Makinson and van der Torre 2000) and in their causal version (Bochman 2004). Moreover, the calculi yield natural possible worlds semantics, complexity bounds, embeddings into normal modal logics, as well as efficient deduction methods. It is worth noticing that our methods for the entailment problem offer derivability certificates (i.e. derivations) or counter-models as solutions. The efficient discovery of the latter can be accomplished using SAT solvers, along the line of (Lahav and Zohar 2014). The newly introduced possible worlds semantics might be used to import in I/O logics tools and results from standard modal theory.

Our work encompasses many scattered results and presents uniform solutions to various unresolved problems; among them, it contains first proof-search oriented calculi for $\mathrm{OUT}_2^{\perp}$ and $\mathrm{OUT}_4^{\perp}$; it provides a missing[4] direct formal connection between the semantics of the original and the causal I/O logics; it introduces a uniform embedding into normal modal logics, that also applies to OUT_1 and OUT_3, despite the absence in these logics of the (OR) rule[5]; moreover, it settles the complexity of the logics OUT_3 and $\mathrm{OUT}_3^{\perp}$. The latter logic has been used in (Bochman 2018) as the base for actual causality and in (Bochman 2004), together with $\mathrm{OUT}_4^{\perp}$, to characterize strong equivalence of causal theories w.r.t. two different semantics: general and causal non-monotonic semantics. Strong equivalence is an important notion as theories satisfying it are 'equivalent forever', that is, they are interchangeable in any larger causal theory without changing the general/causal non-monotonic semantics. Furthermore OUT_4 has been used as a base for formalizing legal concepts (Ciabattoni, Parent, and Sartor 2021). The automated deduction tools we have provided might be used also in these contexts.

In this paper, we have focused on *monotonic* I/O logics. However, due to their limitations in addressing different aspects of causal reasoning (Bochman 2021) and of normative reasoning, several non-monotonic extensions have been introduced. For example (Makinson and van der Torre 2001; Parent and van der Torre 2014) have proposed non-monotonic extensions that have also been applied to represent and reason about legal knowledge bases, as demonstrated in the work by Robaldo et al. (Robaldo et al. 2020). Our new perspective on the monotonic I/O logics contributes to increase their understanding and can provide a solid foundation for exploring non-monotonic extensions.

[4]From (van der Torre and Parent 2013): "*As a matter of facts, there is no direct (formal) connection between the semantics Bochman proposes and the operational semantics for I/O logic. The linkage between the two is established through the axiomatic characterization: both the possible-worlds semantics and the operational semantics give rise to almost the same axiom system*"

[5]*From (Makinson and van der Torre 2000): "As far as the authors are aware, it is not possible to characterise the system of simple-minded output (with or without reusability) by relabeling or modal logic in a straightforward way. The (OR) rule appears to be needed, so that we can work with complete sets.*"

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