Proceedings of the 20th International Conference on Principles of Knowledge Representation and Reasoning
Main Track

Approximating Weakly Preferred Semantics in Abstract Argumentation through Vacuous Reduct Semantics

Lydia Blümel , Matthias Thimm
Artificial Intelligence Group, FernUniversität in Hagen
lydia.bluemel@fernuni-hagen.de, matthias.thimm@fernuni-hagen.de

Abstract

We consider the recently introduced vacuous reduct semantics in abstract argumentation that allows the composition of arbitrary argumentation semantics through the notion of the reduct. We show that by recursively applying the vacuous reduct scheme we are able to cover a broad range of semantical approaches. Our main result shows that we can recover the weakly preferred semantics as the unique solution of a fixed point equation involving an infinite application of the vacuous reduct semantics based only on the very simple property of conflict-freeness. We also conduct an extensive study of the computational complexity of the recursive application of vacuous reduct semantics, which shows that it completely covers each level of the polynomial hierarchy, depending on the recursion depth.

1 Introduction

Abstract argumentation frameworks (Dung 1995) are a simple but expressive formalism for representing and reasoning with argumentative scenarios. An abstract argumentation framework is represented as a directed graph, where arguments are represented by vertices and an attack from one argument on another is represented by a directed edge. This formalism allows to concisely represent central issues in non-monotonic reasoning and many extensions to abstract argumentation frameworks have been developed to deal with advanced topics such as quantitative uncertainty (Hunter et al. 2021; Potyka 2019), strategic issues (Black, Maudet, and Parsons 2021; Thimm 2014; Governatori, Maher, and Olivieri 2021), and dynamic aspects (Doutre and Mailly 2018; Ferretti et al. 2017; Niskanen et al. 2020). Abstract argumentation frameworks are interpreted through extensions, which are sets of arguments that form a plausible outcome of the underlying argumentation and thus specify the acceptability of arguments. Many extension-based semantics (Baroni, Caminada, and Giacomin 2011; Baumann, Brewka, and Ulbricht 2020a; Cramer and van der Torre 2019) have been developed that describe a specific formal meaning of "acceptability". The two central concepts in this context are *conflict-freeness* and *admissibility*. Conflict-freeness of an extension requires that there are no attacks between acceptable arguments, while admissibility requires that acceptable arguments attacked by other arguments must in turn be defended by acceptable arguments. Most semantics for abstract argumentation rely on these two concepts.

Some previous works (Bodanza and Tohmé 2009; Baumann, Brewka, and Ulbricht 2020a; Dondio and Longo 2021; Dvorák et al. 2022; Thimm 2023) justify the need to develop semantical approaches that do *not* rely on the property of admissibility. In particular, in a setting where an argument b is attacked only by an argument a and a is attacking itself as well, a defense of b from a seems unnecessary. In fact, a can be deemed "non-sensical" due to it attacking itself, so the set $\{b\}$ itself may be regarded as acceptable, although not being admissible. A specific general framework for discussing such semantics is the vacuous reduct semantics (Thimm 2023), which allows the combination of any two semantics σ and τ. More precisely, it refines the notion of a σ-extension by further requiring from a σ-extension E that no argument is accepted by τ in the *reduct* of E, i. e., the argumentation framework obtained by removing all arguments in E and attacked by E. It can be shown (Thimm 2023) that, e. g., preferred semantics (Dung 1995)—which only considers subset-maximal admissible sets—is an instance of such a semantics, by setting σ and τ both to *admissibility*.

In this work, we further investigate the framework of vacuous reduct semantics by, in particular, focusing on the possibility of recursive application of the vacuous reduct scheme. We first show that many classical as well as non-classical semantics can be phrased as particular instances of vacuous reduct semantics. Our main result, however, concerns the weakly preferred semantics (Baumann, Brewka, and Ulbricht 2020a), which is a variation of preferred semantics relying on the notion of *weak admissibility*. We show that weakly preferred semantics can be characterised by a fixed point equation involving recursive application of the vacuous reduct scheme that only relies on the notion of conflict-freeness. We also show that we can approximate weakly preferred extensions by a bounded application of this scheme. In addition, we conduct an extensive computational complexity investigation and show that, depending on the bound of the recursive application of the vacuous reduct scheme, we completely cover each level of the polynomial hierarchy. This result is consistent with the previous result (Dvorák, Ulbricht, and Woltran 2021) on the complexity of reasoning with weakly preferred semantics, which is

PSPACE-complete, and our recursive scheme approximating weakly preferred semantics with increasing bounds on the recursive application. To summarise, the contributions of this paper are as follows:

1. We investigate several instances of the vacuous reduct semantics and show that they correspond to other semantics previously considered in the literature (Section 3).
2. We analyse the recursive application of the vacuous reduct scheme, focusing on the case of conflict-freeness as the base semantics (Section 4).
3. We characterise weakly preferred semantics using a fixed point equation involving infinite application of the vacuous reduct scheme with conflict-freeness as base semantics and show how it can be approximated by bounding the recursion depth (Section 5).
4. We analyse the computational complexity of various tasks involving vacuous reduct semantics with conflict-freeness as base semantics (Section 6)

Section 2 gives necessary preliminaries and Section 7 concludes the paper with a discussion. Due to space limitations, we omit most of the proofs of technical results. These can be found in an online appendix.[1]

2 Preliminaries

2.1 Abstract Argumentation Frameworks by Dung

This section recapitulates Dung's concept of an *abstract argumentation framework*, i.e. a tuple $F = (A, R)$(Dung 1995). Therein, A is a finite set of arguments and $R \subseteq A \times A$ is a binary relation on A that represents attacks between these arguments. $a \to b$ is a shorthand for $(a, b) \in R$ and is read as "argument a *attacks* argument b". For two sets of arguments $E, D \subseteq A$ we say E attacks D and write $E \to D$ iff there exist arguments $e \in E$, $d \in D$ such that $e \to d$. We simply write $E \to a$ instead of $E \to \{a\}$ and $a \to E$ instead of $\{a\} \to E$. We denote by $E^- = \{a \in A \mid a \to E\}$ the set of all arguments attacking E and by $E^+ = \{a \in A \mid E \to a\}$ the set of all arguments attacked by E, respectively. The *restriction* of an argumentation framework F to a subset of its arguments $E \subseteq A$ is $F_E = (E, R \cap (E \times E))$.

For formal clarity, we fix an infinite countable background set U_{Arg} as the universe of arguments and assume $A \subseteq U_{Arg}$ for any AF $F = (A, R)$. The set of all finite AFs over U_{Arg} is denoted by U_{AF}.

Definition 1. A mapping $\sigma : U_{AF} \to 2^{2^{U_{Arg}}}$ is called an *extension-based argumentation semantics* if it maps each AF $F = (A, R)$ to a set $\sigma(F) \subseteq 2^A$ of subsets of A. We call such a subset $E \in \sigma(F)$ a σ-extension of F.

An extension-based semantics for AFs is based on the idea of an extension representing a set of collectively acceptable arguments. Dung introduces several extension-based semantics for his framework. We will refer to them as the classical argumentation semantics. They are build on two principles, *conflict-freeness* and *classic defense*. A set of arguments E is said to be conflictfree if $(E \times E) \cap R = \emptyset$, i.e., there are no attacks among the members of E. An argument set E is (classically) defended by another set D if all attackers of E are in turn attacked by D, i.e., $E^- \subseteq D^+$. E is *admissible* if it is conflictfree and defends itself. The following semantics derived from these two principles are relevant for the following sections.

Definition 2. Let $F = (A, R)$ be an AF. We define the following conflictfree- and admissibility-based semantics on F:[2]

$$
\begin{aligned}
cf(F) &= \{E \subseteq A \mid (E \times E) \cap R = \emptyset\} \\
na(F) &= \{E \in cf(F) \mid \forall D \in cf(F) : E \subseteq D \Rightarrow D \subseteq E\} \\
ad(F) &= \{E \subseteq A \mid E \in cf(F) \wedge E^- \subseteq E^+\} \\
pr(F) &= \{E \in ad(F) \mid \forall D \in ad(F) : E \subseteq D \Rightarrow D \subseteq E\} \\
co(F) &= \{E \in ad(F) \mid \forall a \in A : E \text{ defends } a \Rightarrow a \in E\} \\
grd(F) &= \{E \in co(F) \mid \forall D \in co(F) : D \subseteq E \Rightarrow E \subseteq D\} \\
stb(F) &= \{E \in ad(F) \mid E \cup E^+ = A\}
\end{aligned}
$$

We say an argument a is *skeptically accepted* wrt. to a semantics σ if $a \in E$ holds for all extensions $E \in \sigma(F)$. It is *credulously accepted* if there exists at least one σ-extension E containing a otherwise the argument a is *rejected*.

2.2 Weak Admissibility and Vacuous Reduct Semantics

Given an AF $F = (A, R)$ and a set E the *reduct* of F wrt. E is $F^E = F_{A \setminus (E \cup E^+)}$ the restriction of F to all arguments which are neither in nor attacked by E. (Baumann, Brewka, and Ulbricht 2020a) use this notion to define a weakened, recursive variation of the admissibility and preferred semantics (and of some other semantics which we will not discuss here).

Definition 3. Let $F = (A, R)$ be an AF. The *weakly admissible semantics* on F is defined as the set

$$
\begin{aligned}
ad^w(F) = \{E \in cf(F) \mid \forall a \in E^- : a \in E^+ \vee \\
\forall D \in ad^w(F^E) : a \notin D\}
\end{aligned}
$$

A weakly preferred extension is a $\subseteq$-maximal weakly admissible extension.

The motivation behind weak admissibility is to capture cases where acceptable arguments cannot be classically defended, like in the following example.

Example 4. For the AF in Fig. 1, the empty set is the only admissible and preferred extension. Under weak admissibility things are different. The argument set $\{b\}$ is conflictfree and if we compute the reduct $F^{\{b\}}$ we are left with the AF consisting of only the self-attacker a and argument d, now unattacked, i.e., $F^{\{b\}} = (\{a, d\}, \{(a, a)\})$, since b attacks c and therefore $A \setminus (\{b\} \cup \{b\}^+) = \{a, d\}$. Now since $\{a\}$ is not conflictfree, the empty set and $\{d\}$

[1] http://mthimm.de/misc/lbmt_kr23_appendix.pdf

[2] Throughout this paper we treat the conflictfree and admissible semantics as argumentation semantics in their own right which define a set of conflictfree resp. admissible extensions

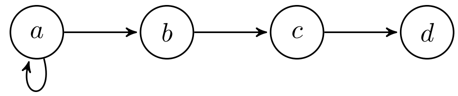

Figure 1: Motivating Problem

are the only weakly admissible extension of $F^{\{b\}}$ and the only attacker of $\{b\}$ namely a is not a member of these extensions, so $\{b\} \in ad^w(F)$. Furthermore, d being in $F^{\{b\}}$ means $\{b,d\}$ is conflictfree and has no threats in $F^{\{b,d\}}$, so $\{b,d\} \in ad^w(F)$, too. On the other hand, $\{c\} \notin ad^w(F)$ as it has an attacker b for which $\{b\} \in ad^w(F^{\{c\}})$ holds, where $F^{\{c\}} = (\{a,b\},\{(a,a),(a,b)\})$. $\{a\}$ is not weakly admissible because it is not conflictfree. We end up with $ad^w(F) = \{\emptyset, \{b\}, \{d\}, \{b,d\}\}$ and $\{b,d\}$ being the only weakly preferred extension.

(Baumann, Brewka, and Ulbricht 2020b) use the principle of *meaningless reduct* for an alternative characterization of the classic and weakly preferred semantics.

Definition 5. An argumentation semantics σ satisfies the principle of *meaningless reduct* iff for any AF $F = (A,R)$ and any $E \in \sigma(F)$ it holds that $\sigma(F^E) = \{\emptyset\}$.

The authors observe the principle is satisfied by both classic and weakly preferred semantics. Furthermore, the following holds:

Proposition 6. *Let $F = (A,R)$ be an AF. A set $E \subseteq A$ is (weakly) preferred if and only if E is (weakly) admissible and $ad(F^E) = \{\emptyset\}$ (resp. $ad^w(F^E) = \{\emptyset\}$).*

In (Thimm 2023), the general family of vacuous reduct semantics and some of its instantations are considered. We first recall the former definition.

Definition 7. Let σ, τ be two extension-based argumentation semantics. The *τ-vacuous reduct semantics* to the base of σ is the argumentation semantics $vac_\sigma(\tau) : U_{AF} \to 2^{2^{U_{Arg}}}$ that maps each AF $F = (A,R)$ to the set of extensions

$$vac_\sigma(\tau)\,(F) = \{E \in \sigma(F) \mid \tau(F^E) \subseteq \{\emptyset\}\}$$

As per this definition, a vacuous reduct semantics refines its base semantics σ by accepting only those extensions which satisfy the additional condition of having no nonempty τ-extension in their reduct. In simple words $E \in \sigma(F)$ becomes an extension if either $\tau(F^E) = \emptyset$ or $\tau(F^E) = \{\emptyset\}$. To emphasize the operative nature of combining two semantics in a universally defined way, we chose a different notation from the original work for the definition itself. Indeed, one way to read the above definition is that $vac : Sem \times Sem \to Sem$ is a binary operation on the set of all argumentation semantics Sem. A small remark on the side: The empty semantics ($empty(F) = \emptyset$) is a neutral element as a right hand argument and "nullifies" everything as a left hand argument of vac. It is easy to observe there is no symmetry in general. A future work direction might be to examine the algebraic properties of this and other operations on argumentation semantics.

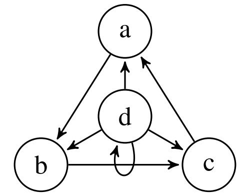

Figure 2: What happens between ad and ad^w

For an example of a vacuous reduct semantics see Prop. 6, obviously $pr = vac_{ad}(ad)$ and $pr^w = vac_{ad^w}(ad^w)$. The more interesting candidates are the $vac_{cf}(ad)$- and $vac_{cf}(vac_{cf}(ad))$-semantics which are referred to in (Thimm 2023) as the undisputed and strongly undisputed semantics, respectively. To get an intuition how adding a vacuity condition to a base semantics plays out, we will apply them to our motivating example.

Example 8. For the AF depicted in Fig. 1 we have $cf(F) = \{\emptyset, \{b\}, \{c\}, \{d\}, \{b,d\}\}$, so there are five argument sets which satisfy the base condition, i. e. conflictfree-ness. Since $F = F^\emptyset$ has the empty set as its only admissible extension, $\emptyset \in vac_{cf}(ad)(F)$. $F^{\{c\}}$ contains the self-attacker a and b, which is attacked by it, therefore has no non-empty admissible extension either, so $\{c\} \in vac_{cf}(ad)(F)$, too. For $\{b\}$ on the other hand $\{d\} \in ad(F^{\{b\}})$ so $\{b\}$ is not undisputed. To sum up, $vac_{cf}(ad)(F) = \{\emptyset, \{c\}, \{d\}, \{b,d\}\}$. For strongly undisputed semantics note first that F has non-empty undisputed extensions, so $\emptyset \notin vac_{cf}(vac_{cf}(ad))(F)$, same goes for $\{c\}$,$\{d\}$ and $\{b\}$. In fact, $\{b,d\}$ is the only strongly undisputed extension, because $F^{\{b,d\}}$ contains no non-empty conflictfree set and therefore no undisputed extension.

Let us now illustrate the difference between (strongly) undisputed and weakly admissible semantics with Ex. 8 from (Thimm 2023).

Example 9. The reader may convince herself that for the AF in Fig. 2 we get $ad(F) = ad^w(F) = \{\emptyset\}$. Things are slightly different under the $vac_{cf}(ad)$-semantics. Since the conflictfree semantics is our base, we have $\{a\}$, $\{b\}$, $\{c\}$, and $\emptyset$ as candidates for $vac_{cf}(ad)$-extensions. For $\emptyset$, we can see right away that the vacuity condition is satisfied, i. e., $ad(F^\emptyset) = ad(F) \subseteq \{\emptyset\}$. Therefore $\emptyset \in vac_{cf}(ad)(F)$. In case of, e. g., $\{a\}$ we check whether $ad(F^{\{a\}}) \subseteq \emptyset$. Since $F^{\{a\}} = (\{c,d\},\{(d,d),(d,c)\})$ this is indeed the case and by taking symmetry into account we arrive at $vac_{cf}(ad)(F) = \{\emptyset, \{a\}, \{b\}, \{c\}\}$ i. e. all conflict-free extensions end up being accepted in this case. This seems far to liberal. What happens if we take this to the next level? For the $vac_{cf}(vac_{cf}(ad))$-semantics we still have the four conflictfree sets as candidates but now we look for $vac_{cf}(ad)$-extensions in the reduct. This immediately rules out the empty set and a close inspection of $F^{\{a\}} = (\{c,d\},\{(d,d),(d,c)\})$ tells us $\{c\}$ is a $vac_{cf}(ad)$-extension of the reduct, so in fact $vac_{cf}(vac_{cf}(ad))(F) = \emptyset$.

Note that adding a new unattacked argument e would change the result in Ex. 9, none of the singletons nor the empty set would be an undisputed extension in that case. Instead the union of e with any of the previous extensions would become a $vac_{cf}(ad)$-extension. One way to think of the vacuity condition is as a sort of completeness - we are forbidden to leave certain arguments out. The strongly undisputed semantics demonstrates this does not mean we *can* always include what we left out, though. Observe that a vacuous reduct semantics can be used as the base part or the vacuous part of another vacuous reduct semantics. For instance, the undisputed semantics is used as the vacuity condition of strongly undisputed semantics. The investigations of the following sections revolve mainly around this particular feature.

3 Vacuous Reduct Semantics not yet Considered

The investigations in (Thimm 2023) put the main focus on three specific vacuous reduct semantics, $vac_{ad}(ad)$ which is shown to be the preferred semantics and $vac_{cf}(ad)$ as well as its next level $vac_{cf}(vac_{cf}(ad))$. The general nature of Def. 7 invites us to enrich this list with other combinations of semantics. Our objective in doing this is not so much to find new semantics but to better understand the interplay between semantics as base parts and/or vacuity conditions. What happens if the semantics for the vacuousness part is weakened? Are there any limits to the expressiveness of the vacuous reduct notion?

The first observation we can make regarding the second question is that the vacuity condition is only distinct up to credulous acceptance. Take for example $vac_{ad}(ad)$, the preferred semantics. The vacuity condition for a $vac_{ad}(ad)$-extension E requires that $ad(F^E) \subseteq \{\emptyset\}$. This is equivalent to asking that no arguments in F^E are credulously accepted. Since every admissible extension has a preferred extension as a superset, the set of credulously accepted arguments is the same for ad and pr. Thus we can exchange the admissible semantics for the preferred semantics in the vacuousness part and still get the preferred semantics, $ad^{pr} = pr$. Because the vacuity condition is only concerned with the reduct F^E of a potential extension E, this interchangeability is independent of the base semantics.

Proposition 10. *Let σ, τ_1 and τ_2 be argumentation semantics. If $\bigcup \tau_1(F) = \bigcup \tau_2(F)$ for any AF $F = (A, R)$ then $vac_\sigma(\tau_1) = vac_\sigma(\tau_2)$.*

Proof. Suppose some $E \in vac_\sigma(\tau_1)(F)$ exists, then $E \in \sigma(F)$ and $\tau_1(F^E) \subseteq \{\emptyset\}$. Then $\bigcup \tau_1(F) = \emptyset = \bigcup \tau_2(F)$, so $\tau_2(F^E) \subseteq \{\emptyset\}$. Since $E \in \sigma(F)$ we have $E \in vac_\sigma(\tau_2)(F)$. The other direction follows from the symmetry of the statement. □

We can narrow down our search space for interesting vacuous reduct semantics with this statement, for instance the semantics $vac_{cf}(pr)$ is just the undisputed semantics again. From a computational perspective Prop. 10 has an important implication. It suffices entirely to describe a semantics up to credulous acceptance to compute the corresponding vacuity condition. In fact, a criterion for the existence of non-empty extensions suffices. For example, when checking whether some $E \in vac_{cf}(grd)(F)$, instead of looking for a grounded extension in F^E, it is enough to check whether the reduct contains an unattacked argument. The following corollary sums up which cases are already covered by (Thimm 2023) due to Prop. 10 and which of the simpler vacuous reduct semantics not considered there coincide.

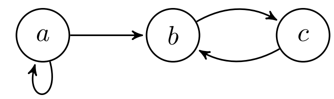

Figure 3: An admissible variation of stable semantics

Corollary 11. *Let $\sigma \in \{cf, ad\}$. Then $vac_\sigma(ad) = vac_\sigma(pr) = vac_\sigma(co)$ and $vac_\sigma(cf) = vac_\sigma(na)$.*

According to the corollary the remaining interesting vacuity conditions would be the grounded and the conflictfree semantics. For $vac_{ad}(grd)$ it was already pointed out in (Thimm 2023) that this is a characterization of the complete semantics. What can be said about $vac_{cf}(grd)$? It describes a semantics which contains everything it can defend and which only accepts extensions where this does not cause a conflict. A reader familiar with recent works on weak argumentation semantics might recognize the *ub-complete semantics* here.[3]

Proposition 12. $vac_{cf}(grd) = co^{ub}$.

Proof. $E \in co^{ub}$ iff $E \in cf(F)$ and $\chi(E) \subseteq E$ following (Blümel and Ulbricht 2022). The base condition of $vac_{cf}(grd)$ equals $E \in cf(F)$. For the vacuity condition note that $\chi(E) \subseteq E$ iff no unattacked argument exists in F^E which is equivalent to $grd(F^E) = \{\emptyset\}$. □

Let us provide an example how ub-completeness is achieved by grd-vacuity.

Example 13. The AF in Fig. 1 has four ub-complete extensions, i. e. $\emptyset$, $\{b, d\}$, $\{c\}$ and $\{d\}$. In case of $\emptyset$, $\{c\}$ and $\{d\}$ a chain of arguments starting with a self-attacker remains in the reduct, which means the grounded extension is empty. Therefore the vacuity condition is satisfied and all arguments in the reduct can be labelled undecided. For $\{b\}$ on the other hand $F^{\{b\}} = (\{a, d\}, \{(a, a)\})$ where d is unattacked and therefore has to be included. Since $grd(F^{\{b\}}) = \{\{d\}\}$ the vacuity condition is violated by $\{b\}$, so $\{b\}$ is not ub-complete.

We steer our attention towards cf-vacuity for the remainder of this section. First, let us take a look at the behaviour of the $vac_{ad}(cf)$ semantics.

[3]orig. a label-based semantics (Dondio and Longo 2021), denote by $co^{ub}(F)$ the set of all ub-complete extension, where E is an ub-complete extension iff E conflictfree and $\chi(E) \subseteq E$, where $\chi(E) = \{a \in A \mid E\ defends\ a\}$ (Blümel and Ulbricht 2022)

Example 14. The AF depicted in Fig. 3 has one admissible extension, $\{c\}$. Since $F^{\{c\}}$ only contains the self-attacker a, $\{c\}$ also satisfies the vacuity condition of having no nonempty conflictfree extension in the reduct, so $\{c\} \in vac_{ad}(cf)(F)$. Although $\{b\}$ satisfies the vacuity condition as well, it is not a $vac_{ad}(cf)$-extension, because it is not admissible in the first place.

Ex. 14 shows that $vac_{ad}(cf)$ is not the stable semantics, since $\{c\}$ is not stable, it does not attack a. Because of the base condition being admissibility we can also exclude naive semantics from the list of candidates. It is not preferred semantics either, a single odd cycle can serve as a counterexample here. To the best of our knowledge $vac_{ad}(cf)$ is a genuine new semantics, one based on admissibility and with a simple but interesting maximality condition. Since $ad(F) \subseteq cf(F)$ always, the vacuity condition tells us we are looking at a subset of the preferred semantics, one that is slightly more liberal than stable semantics in that the $vac_{ad}(cf)$-semantics does not require self-attackers to be attacked by its extensions. One can think of it as a semantics where everything has to be attacked except for those contradictions which are no direct threat.

Proposition 15. *For any AF $F = (A, R)$ it holds that*

$$vac_{ad}(cf) = \{E \in pr(F) | E \cup E^+ \cup \{a \in A | (a, a) \in R\} = A\}$$

Things get even more puzzling with the $vac_{cf}(cf)$-semantics. While its extensions are certainly maximal conflictfree i. e. $E \in vac_{cf}(cf)(F)$ implies $E \in na(F)$, maximal conflict-freeness is not sufficient. For example, in Fig. 2 the singleton $\{b\}$ is maximal conflictfree but $F^{\{b\}}$ has a conflictfree subset, the singleton $\{a\}$. The observations hint at a close relationship with stable semantics again. It turns out $vac_{cf}(cf)$ is an alternative characterization of cogent stable semantics.[4]

Proposition 16. $vac_{cf}(cf) = stb_{\delta_{self}}$

Here we witness the effect of having a very weak semantics as the vacuity condition. Even with conflictfree semantics as a base part the resulting semantics suddenly becomes one of the strictest non-classical semantics. In the next section we will demonstrate how to derive more fine-grained weak semantics by alternating between weak and strict vacuity conditions in a natural way.

4 Higher-Order-Vacuousness

The objective of this section is to answer the question: What happens if we use a vacuous reduct semantics as the vacuity condition of another vacuos reduct semantics or more generally, what happens if we apply vacuousness a fixed number of times? Say we have two semantics σ and τ given, then we can roughly distinguish two cases: Base repetition and vacuousness repetition. Before we delve deeper into the first case we will make a quick remark on why repeating the vacuity condition is not very fascinating.

[4]A set E is a cogent stable extension iff $E \cup \delta_{self}(E) = A$ and $E \cap \delta_{self}(E) = \emptyset$, where $\delta_{self}(E) = E^+ \cup \{a \in A | (a, a) \in R\}$. Denote by $stb_{\delta_{self}}(F)$ the set of all cogent stable extensions.(Blümel and Ulbricht 2022)

Proposition 17. *For any σ, τ argumentation semantics it holds that*

$$vac_\sigma(\tau) = vac_{vac_\sigma(\tau)}(\tau)$$

To introduce a general notation for semantics with a fixed number of repetition in the base part we enhance Def. 7 a little.

Definition 18. Let σ, τ be argumentation semantics. We define the τ-vacuous reduct semantics with *n-th order* base σ recursively for $n \in \mathbb{N}$ by

$$vac^n_\sigma(\tau) = \begin{cases} \tau & for\ n = 0 \\ vac_\sigma(vac^{n-1}_\sigma(\tau)) & for\ n > 0 \end{cases}$$

A specific example of base repetition are the undisputed and the strongly undisputed semantics, under the new notation $vac^1_{cf}(ad)$ and $vac^2_{cf}(ad)$, respectively. From there one can easily add another step of applying conflictfree semantics as σ in the base of $vac_\sigma(\tau)$, pushing strongly undisputed semantics itself in the position of the vacuous semantics τ to gain another semantics behaving again differently from undisputed and strongly undisputed semantics. We can repeat this for any finite number of steps, which yields infinitely many semantics all belonging to the same subclass of vacuous reduct semantics. For every natural number n we define a corresponding n-th order vacuous reduct semantics with n being the number of times we have the cf-semantics in the base before in the final step we check for adm-vacuousness. Let us demonstrate this idea with a quick example.

Example 19. As explained in Ex. 9 the AF from Fig. 2 has no strongly undisputed extension i. e. $vac^2_{cf}(ad)(F) = \emptyset$. For the next level, $vac^3_{cf}(ad)$-semantics, the empty set therefore satisfies the vacuity condition, we have $\emptyset \in cf(F)$ and $vac^2_{cf}(ad)(F^\emptyset) = \emptyset$, so $\emptyset \in vac^3_{cf}(ad)(F)$. On the other hand, the undisputed extensions $\{a\}, \{b\}, \{c\}$ are no $vac^3_{cf}(ad)$-extensions, because for, e. g., the reduct $F^{\{a\}}$ we have $\{c\} \in vac^2_{cf}(ad)(F^{\{a\}})$ as the reader may convince herself. Thus, for this very example the $vac^3_{cf}(ad)$-semantics is a refinement wrt. to its two predecessors in that it is stricter than undisputed semantics while yielding more extensions (to be precise one more) than the strongly undisputed semantics.

The example hints at the possibility of reaching more fine-grained weak semantics by adding more steps of cf-semantics before checking for ad-vacuousness. As a working ground for the formal investigation of the matter we will use the following equivalent characterisation of $vac^n_{cf}(ad)$-semantics.

Proposition 20. *Let $F = (A, R)$ be an AF. For any $n \in \mathbb{N}, n \geq 1$ it holds that:*

$$vac^n_{cf}(ad)(F) = vac^n_{cf}(pr)(F)$$

In particular the undisputed semantics is the $vac^1_{cf}(pr)$-semantics and the strongly undisputed semantics the $vac^2_{cf}(pr)$-semantics.

From here on we will refer to the preferred semantics as the vacuity condition for the $vac^n_{cf}(ad)$-class of semantics instead of admissibility. We follow (Thimm 2023) here where the undisputed semantics is introduced as a weakening of preferred semantics. Other benefits of this choice are the fact that $pr = vac_{ad}(ad)$ is a vacuous reduct semantics itself and furthermore, from a certain point of view even a fixpoint of vacuous reduct semantics, $pr = vac_{pr}(pr)$ being the case. We will discuss this in more detail in the next section. For now, we set the preferred semantics as our base case and assume $vac^0_{cf}(ad) = pr$ from which we move further and further away with each application of cf-semantics before it.

Within the class of $vac^n_{cf}(ad)$-semantics an observation made in the previous section takes concrete shape. We pointed out that having a weak semantics as a vacuity condition produces a respectively stricter vacuous reduct semantics. Conversely, having a strict semantics as a vacuity condition should result in a weaker semantics in total. By this reasoning, for the chain of $vac^n_{cf}(ad)$-semantics we will end up switching from a strict to a weak semantics and back to a strict semantics every two steps. And, indeed, looking at Ex. 19 we observe $vac^0_{cf}(ad)(F) \subset vac^1_{cf}(ad)(F) \supset vac^2_{cf}(ad)(F) \subset vac^3_{cf}(ad)(F)$ for the AF in Fig. 2. We will now prove this oscillation between predecessors and successors is a universal property of the class of $vac^n_{cf}(ad)$-semantics.

Theorem 21. *Let $F = (A, R)$ be an AF, $k \in \mathbb{N}$. Then*

1. $vac^{2k}_{cf}(ad)(F) \subseteq vac^{2k-1}_{cf}(ad)(F)$
2. $vac^{2k}_{cf}(ad)(F) \subseteq vac^{2k+1}_{cf}(ad)(F)$

Proof. Proof by induction over k, simultaneously for both statements. Let us start with the base case demonstrated in (Thimm 2023), there refer to Th. 1, Prop. 6:

$$\text{(2. for } k=0) \quad vac^0_{cf}(ad)(F) = pr(F) \subseteq ud(F) = vac^1_{cf}(ad)(F)$$

$$\text{(1. for } k=1) \quad vac^2_{cf}(ad)(F) = sud(F) \subseteq ud(F) = vac^1_{cf}(ad)(F)$$

For the induction step suppose $E \in vac^{2k}_{cf}(ad)$ (if no such E exists, both (1) and (2) hold trivially). Then by Prop. 20 $vac^{2k-1}_{cf}(ad)(F^E) \subseteq \{\emptyset\}$. With the induction hypothesis for statement (2), i. e. $vac^{2k-2}_{cf}(ad)(F^E) \subseteq vac^{2k-1}_{cf}(ad)(F^E)$ it follows that $vac^{2k-2}_{cf}(ad)(F^E) \subseteq \{\emptyset\}$, so $E \in vac^{2k-1}_{cf}(ad)(F)$ and (1) holds.

Having established (1) for $2k$, we can argue that for any $E \in vac^{2k}_{cf}(ad)(F)$ the fact that $vac^{2k-1}_{cf}(ad)(F^E) \subseteq \{\emptyset\}$ holds, implies $vac^{2k}_{cf}(ad)(F^E) \subseteq \{\emptyset\}$, because by (1) $vac^{2k}_{cf}(ad)(F^E) \subseteq vac^{2k-1}_{cf}(ad)(F^E)$. Therefore, since E is conflictfree, $E \in vac^{2k+1}_{cf}(ad)(F)$, so (2) holds, too. □

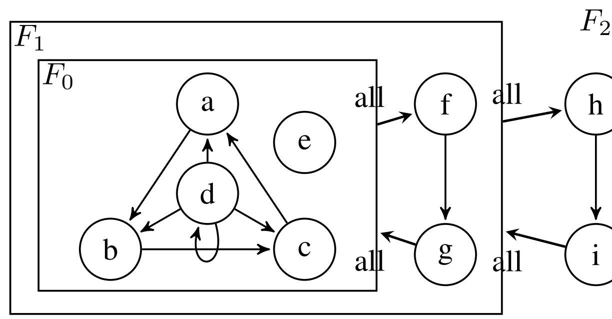

Figure 4: $vac^n_{cf}(ad)$-semantics have no bounds

Theorem 21 gives the impression that the sequence of $vac^n_{cf}(ad)$-semantics converges towards a certain set of extensions, alternating between a sort of set-theoretic upper and a lower bound, much like the convergence of a sandwich-sequence towards a limit in elementary analysis. We postpone the investigation of the existence of a "limit semantics" until the next section. It proves to be wrong, though, that the higher the odd index the less arguments are accepted (nor the other way around, nor does this hold in the even case). Consider the following counterexample.

Example 22. Using the results from Ex. 19, we can observe the following for the AFs depicted in Fig. 4.

1. $\{e\}$ is a $vac^4_{cf}(ad)$-extension of F_0 but not in $vac^2_{cf}(ad)(F_0)$ because the reduct $F^{\{e\}}$ is isomorphic to the AF in Fig. 2 and thus has non-empty $vac^1_{cf}(ad)$-extensions. No other $vac^2_{cf}(ad)$-extension of the AF in Fig. 4 exists. So neither the set of extensions nor the set of credulously accepted arguments by the $vac^2_{cf}(ad)$-semantics is an upper bound for the $vac^4_{cf}(ad)$-semantics.
2. $\{e, a\}$ is a $vac^1_{cf}(ad)$-extension of F_0 but not in $vac^3_{cf}(ad)(F_0)$ and the argument a is not credulously accepted by the $vac^3_{cf}(ad)$-semantics. So $vac^1_{cf}(ad)$ is no lower bound for $vac^3_{cf}(ad)$.
3. $\{f\}$ is a $vac^3_{cf}(ad)$-extension of F_1, where $F_1 = (A_{F_0} \cup \{f, g\}, R_{F_0} \cup \{(f, g)\} \cup \{(a, f), (g, a) \mid a \in A_{F_0}\})$, but not a $vac^1_{cf}(ad)$-extension nor is f credulously accepted by the $vac^1_{cf}(ad)$-semantics because of $\{e\}$ remaining preferred in the reduct. So $vac^1_{cf}(ad)$ is no upper bound of $vac^3_{cf}(ad)$.
4. $\{h\}$ is a $vac^2_{cf}(ad)$-extension of F_2 (defined analogously to F_1), but not a $vac^4_{cf}(ad)$-extension and h is not credulously accepted by the $vac^4_{cf}(ad)$-semantics. So $vac^2_{cf}(ad)$ is no lower bound for $vac^4_{cf}(ad)$.

When computing higher order $vac^n_{cf}(ad)$-extensions the distinguishability between different values for n is limited by the number of reduct steps possible. As soon as the reduct stops containing any extensions satisfying the base condition the iteration stops. For instance, the singleton g in $F_1^{\{e\}}$ of Fig. 4 attacks all remaining arguments and $F_1^{\{g\}^{\{e\}}}$ is empty, so $\{g\} \in vac^n_{cf}(ad)(F_1^{\{e\}})$ for any $n \geq 1$. Whether a set of arguments is a $vac^n_\sigma(\tau)$ extension may therefore be determined in many cases without even getting to check the vacuity condition τ.

Proposition 23. *Let $F = (A, R)$, $E \subseteq A$. If $E \in \sigma(F)$ and $\sigma(F^E) = \{\emptyset\}$ then $E \in vac^n_\sigma(\tau)(F)$ for any $n > 1$.*

Proof. Since for any n a set of arguments E is only a $vac^n_\sigma(\tau)$-extension if it is a σ-extension, F^E cannot contain any non-empty $vac^{n-1}_\sigma(\tau)$-extension if it has no non-empty σ-extension, so E is a $vac^n_\sigma(\tau)$-extension. □

For the class of $vac^n_{cf}(ad)$-semantics this result means we need multiple nested conflicts like in Ex. 22 for constructing non-trivial extensions of higher order which gives us a first hint at the increasing complexity of these semantics the larger n becomes. We will go into more detail about that in Sec. 6. Before that, we conclude our general treatment of vacuous reduct semantics by examining the case of infinite base repetitions in the next section.

5 Infinite Vacuousness

Before we take a look at vacuous reduct semantics with an infinite number of reduct applications, we want to shed some light on fixed points of the vacuous reduct scheme. By that we mean argumentation semantics which can be characterized as follows.

Definition 24. Let σ be an argumentation semantics. A vacuous reduct fixed point semantics to the base of σ is any semantics τ which satisfies

$$\tau = vac_\sigma(\tau)$$

If τ is uniquely determined by σ, we define $\tau^{fp}_\sigma = \tau$.

We will leave the question of the existence and uniqueness of τ^{fp}_σ wrt. a given σ in the general case for future work. For starters, have a look at the problem $\tau = vac_{ad}(\tau)$. By Prop. 6 we know $pr = vac_{ad}(ad)$ and with Prop. 10 we get $pr = vac_{ad}(pr)$. The same line of reasoning gives us a solution for $\tau = vac_{ad^w}(\tau)$, namely $\tau = pr^w$. After the previous sections these results for the (weakly) preferred semantics come as no surprise. Far more intriguing is the question whether there is a (unique) solution for $\tau = vac_{cf}(\tau)$. In this case we have as the base semantics the conflictfree semantics and for the vacuous part we refer to the semantics itself, as mentioned above. Let us begin with two basic observations regarding such a semantics, if it exists:

Lemma 25. *Let $F = (A, R)$ be an AF and suppose an argumentation semantics τ exists such that $\tau = vac_{cf}(\tau)$. Then*

1. *$E \in \tau(F)$ implies that E is conflictfree.*
2. *$\emptyset \in \tau(F)$ if and only if $\tau(F) = \{\emptyset\}$*

Proof. The first is clear, for the second note for the left to right direction that the empty set is always conflictfree and $F^\emptyset = F$, so by the definition of a τ-vacuous cf-extension the empty set has to be the only extension if it is an extension. So if and only if no non-empty extensions exist the empty set becomes an extension meaning for our definition we can look at the non-empty conflictfree sets, only. □

We will now examine some base cases:

Example 26. 1. Let $F = (\emptyset, \emptyset)$ then $\tau(F) = \emptyset = vac_{cf}(\tau)(F) = cf(F)$ as argued above.

2. Let $F = (\{a\}, \emptyset)$ then $cf(F) = \{\emptyset, \{a\}\}$. Since $F^{\{a\}} = (\emptyset, \emptyset)$ we get the first base case $\tau(F^{\{a\}}) = \{\emptyset\}$. So $\{a\}$ is a τ-extension and therefore the empty set is not an extension, $\tau(F) = \{\{a\}\} = vac_{cf}(\tau)(F)$.
3. Let $F = (\{a\}, \{(a, a)\})$ then $cf(F) = \{\emptyset\}$. Every τ-extension has to be conflictfree, so $\tau(F) \subseteq \{\emptyset\}$. Now if we had $\tau(F) = \emptyset$, then $\tau(F^\emptyset) = \emptyset$ so $\emptyset \in \tau(F)$. Contradiction. Therefore $\tau(F) = \{\emptyset\}$.

These three cases will cover the final step in computing τ after building the reduct with conflictfree sets, in fact, the first and the third suffice. The third can be generalized to cases with multiple self-attackers remaining, an AF with only self-attackers will have the empty set and only the empty set as its τ-extension for the same reason as in the single argument case. As a next step we need a semantics which recursively traces back through an AF to these base cases. Towards this end we define the following infinite vacuous reduct semantics.

Definition 27. Let $F = (A, R)$ be an AF.

$$^\infty cf(F) = \begin{cases} \{\emptyset\} & \text{if } \forall E \in cf(F)\setminus\{\emptyset\} : {}^\infty cf(F^E) \notin \{\emptyset, \{\emptyset\}\} \\ \{E \in cf(F)\setminus\{\emptyset\} \mid {}^\infty cf(F^E) \subseteq \{\emptyset\}\} & \text{otherwise} \end{cases}$$

This is well-defined because the reduct of non-empty argument sets is always a smaller AF, so at some point we hit the base cases, which are covered by this definition as well. For example, clearly the condition $\forall E \neq \emptyset,\ E \in cf(F) : {}^\infty cf(F^E) \notin \{\emptyset, \{\emptyset\}\}$ is satisfied by $F = \{\emptyset, \emptyset\}$. Due to its infinite application of the reduct scheme with the conflictfree semantics as its base condition, the $^\infty cf$-semantics can serve as a limit for the class of $vac^n_{cf}(ad)$-semantics, i. e. the $vac^n_{cf}(ad)$-semantics coincide for large enough n on small enough AFs with the $^\infty cf$-semantics.

Proposition 28. *Let $F = (A, R)$ an AF with $|A| = n$. Then for all $m > n$ $vac^m_{cf}(ad)(F) = {}^\infty cf(F)$.*

Corollary 29. *Let $F = (A, R)$ an AF with $|A| = n$. Then for all $m, l > n$: $vac^m_{cf}(ad)(F) = vac^l_{cf}(ad)(F)$.*

From Cor. 29 we can already see that some kind of fixed point is reached when applying the $vac^n_{cf}(ad)$-semantics with sufficiently large n to an AF. The following theorem formally establishes the $^\infty cf$-semantics as the vacuous reduct fixed point semantics for $\sigma = cf$.

Theorem 30. *There exists a unique argumentation semantics τ^{fp}_{cf} satisfying $\tau^{fp}_{cf} = vac_{cf}(\tau^{fp}_{cf})$ and that is $\tau^{fp}_{cf} = {}^\infty cf$.*

Proof. (Part One) $^\infty cf = vac_{cf}(^\infty cf)$. This follows from the definition of $^\infty cf$. By Def. 27 the semantics $vac_{cf}(^\infty cf)$ is defined as $vac_{cf}(^\infty cf) = \{E \in$

$cf(F) \mid {}^{\infty}cf(F^E) \subseteq \{\emptyset\}\}$. For $E \neq \emptyset$ this is exactly the condition from Def. 27. For $E = \emptyset$ it holds that $\emptyset \in cf(F)$ and $F^{\emptyset} = F$. So $\emptyset \in {}^{\infty}cf(F)$ iff ${}^{\infty}cf(F) = \{\emptyset\}$ i.e. iff $\forall E \neq \emptyset,\ E \in cf(F) : E \notin {}^{\infty}cf(F)$ which is equivalent to the first part of Def. 27 $\forall E \neq \emptyset,\ E \in cf(F) : {}^{\infty}cf(F^E) \notin \{\emptyset, \{\emptyset\}$.

(Part Two) Proof of uniqueness. For any semantics τ if $\tau = vac_{cf}(\tau)$ then for any $F = (A, R)$ it holds that $\tau(F) = {}^{\infty}cf(F)$. Proof by induction over the number of arguments $|A| = n \in \mathbb{N}$.

(Base Case) Let $n = 0$ and thus $F = (\emptyset, \emptyset)$. Then $cf(F) = \emptyset$. Because $\tau = vac_{cf}(\tau)$ we thus know $\tau(F)$ is either the empty set or $\{\emptyset\}$. If $\tau = \{\emptyset\}$ then $\tau = {}^{\infty}cf$ and since $\tau(F^{\emptyset}) = \tau(F) = \{\emptyset\}$ we know $\emptyset \in vac_{cf}(\tau)(F)$ holds. so $\tau(F) = vac_{cf}(\tau)(F)$. Suppose $\tau = \emptyset$ on the other hand, then for the empty set we get $\tau(F^{\emptyset}) = \tau(F) = \emptyset \subseteq \{\emptyset\}$ so $\emptyset \in cf^{\tau}(F)$ and therefore $\tau(F) \neq cf^{\tau}(F)$. Contradiction with the assumption that $\tau = vac_{cf}(\tau)$. For $F = (\emptyset, \emptyset)$ we thus get the unique $\tau(F) = {}^{\infty}cf(F) = \{\emptyset\}$.

(Induction Step) Let $F = (A, R)$ with $|A| = n$. By the induction hypothesis for all $F' = (A', R')$ with $|A| < n$ it holds that $\tau(F') = {}^{\infty}cf(F')$ for any τ satisfying $\tau = vac_{cf}(\tau)$. We distinguish three cases. First, if $cf(F) = \{\emptyset\}$, then as explained in the base case $vac_{cf}(\tau)(F) \subseteq \{\emptyset\}$ so $E \in vac_{cf}(\tau)(F)$ so $\tau(F) = \{\emptyset\} = {}^{\infty}cf(F)$ is unique. Second, suppose some $E \in cf(F),\ E \neq \emptyset$ exists. Then F^E has less arguments than F and thus satisfies the induction hypothesis, so $\tau(F^E) \subseteq \{\emptyset\}$ iff ${}^{\infty}cf(F^E) \subseteq \{\emptyset\}$. This implies $E \in vac_{cf}(\tau)(F) = \tau(F)$ iff $E \in {}^{\infty}cf(F)$ so τ is unique wrt. non-emtpy extensions. For the last case we examine the empty set. It holds that $F^{\emptyset} = F$ and by the second case there is no nonempty $E \in \tau(F)$ iff there is no such E in ${}^{\infty}cf(F)$. So $\tau(F) \subseteq \{\emptyset\}$ iff ${}^{\infty}cf(F) \subseteq \{\emptyset\}$. Therefore $\emptyset \in \tau(F)$ iff $\emptyset \in {}^{\infty}cf(F)$, and in that case $\tau(F) = {}^{\infty}cf(F) = \emptyset$ follows. □

What follows is one of the main results of this paper. It turns out, the ${}^{\infty}cf$-semantics coincides with the weakly preferred semantics.

Theorem 31. ${}^{\infty}cf = pr^w$.

Proof. By induction over the size of the AF $|A| = n$.
(Base case) Let $F = (\emptyset, \emptyset)$. By definition ${}^{\infty}cf(AF) = \{\emptyset\}$. For the weakly preferred semantics we have only one candidate for an extension: $\emptyset$ is conflictfree and has no attackers, so it is weakly admissible. It also has no weakly admissible superset, so $pr^w(F) = \{\emptyset\} = {}^{\infty}cf(F)$.
($|A| = n$) We show that for any $E \subseteq A$ it holds that $E \in {}^{\infty}cf(F) \Leftrightarrow E \in pr^w(F)$. Let us first assume $E \neq \emptyset$.
($\Rightarrow$) Let $E \in {}^{\infty}cf(F)$ then E is conflict-free and ${}^{\infty}cf(F^E) \subseteq \{\emptyset\}$. Since E is not empty, $F^E \subset F$ is a proper subset with less arguments, so by the induction hypothesis $pr^w(F^E) = {}^{\infty}cf(F^E) = \{\emptyset\}$. The weakly preferred extensions are the maximal weakly admissible extensions so every weakly admissible attacker of E in F^E would be part of some $E' \in pr^w(F^E)$. We thus can conclude E has no attacker $y \in \bigcup ad^w(F^E)$, and since E is conflictfree this makes E weakly admissible. By modularization E weakly-admissible and $pr^w(F^E) = \{\emptyset\}$ imply $E \in pr^w(F)$.
($\Leftarrow$) $E \in pr^w(F)$ means E conflict-free and $pr^w(F^E) = \{\emptyset\}$ because of modularization (Baumann, Brewka, and Ulbricht 2020b). Since E is not empty, $F^E \subset F$ is a proper subset, so we can again apply the induction hypothesis to get ${}^{\infty}cf(F^E) = pr^w(F^E) = \{\emptyset\}$. So E satisfies both conflict-freeness and ${}^{\infty}cf(F^E) = \{\emptyset\}$, therefore $E \in {}^{\infty}cf(F)$.

For $E = \emptyset$ we have the problem that $F^{\emptyset} = F$, so the induction hypothesis does not work directly. However, we can use that we have already shown $E \in {}^{\infty}cf(F) \Leftrightarrow E \in pr^w(F)$ for nonempty E which is equivalent to $E \notin {}^{\infty}cf(F) \Leftrightarrow E \notin pr^w(F)$. Therefore, no nonempty ${}^{\infty}cf$-extension of F exists if and only if no nonempty weakly preferred extension exists, i. e. ${}^{\infty}cf(F) \subseteq \{\emptyset\} \Leftrightarrow pr^w(F) \subseteq \{\emptyset\}$. Since the empty set is always weakly admissible, we get $pr^w(F) = \{\emptyset\}$ whenever there is no nonempty weakly preferred extension. Analogously, if there is no nonempty ${}^{\infty}cf$-extension, the empty set satisfies both conflict-freeness and ${}^{\infty}cf(F^{\emptyset}) \subseteq \{\emptyset\}$, so in the case of no nonempty extensions it holds that ${}^{\infty}cf(F) = \{\emptyset\} = pr^w(F)$. □

Theorem 31 amounts to a fixed point characterization of another weak semantics after weakly complete extensions were characterized as fixed points of the weak defense operator in (Blümel and Ulbricht 2022). Note that the two vacuous reduct fixed point semantics recovered so far are the classic and weakly *preferred* semantics, resp., i. e. the $\subseteq$-maximal extensions wrt. some notion of admissibility. Whether this is a characteristic of fixed points of the vacous reduct scheme in general is an interesting starting point for further research. The above characterization makes a simplification of weakly preferred semantics possible. The notion of weak admissibility has become superfluous, all we need are conflict-freeness and the reduct. On the other hand, having the well-studied weakly preferred semantics as the infinite vacuous reduct semantics to base semantics cf ensures us that ${}^{\infty}cf$ satisfies various desirable properties like I-maximality, unattack-inclusion, directionality (Baumann, Brewka, and Ulbricht 2020b). Notably, Theorem 31 guarantees the existence of a ${}^{\infty}cf$-extension for any AF.

Corollary 32. *For any $F = (A, R)$ there exists at least one ${}^{\infty}cf$-extension.*

The initial motivation behind the definition of undisputed semantics and subsequently $vac^n_{cf}(ad)$-semantics in general was to define computationally less expensive semantics, which behave similar to the weak semantics. By making use of Prop. 28 we can demonstrate that the $vac^n_{cf}(ad)$-semantics approximate the weakly preferred semantics in the sense that they coincide with pr^w for sufficiently large n wrt. the size of the AF. The $vac^n_{cf}(ad)$-semantics can be said to approximate the weakly preferred semantics in a second sense, i. e., in terms of their computational complexity. The following section is dedicated to this relationship.

6 Complexity Results

In the previous section we showed that for small enough AFs resp. large enough n the ${}^{n}cf$-semantics coincides with the weakly preferred semantics. We will now show how the class of ${}^{n}cf$-semantics approximates the weakly preferred semantics from a computational complexity perspective. In order to do this let us recall some basic definitions (Dvorák and Dunne 2018). For better comparability we use the notation from (Thimm 2023). We assume the reader is familiar with the standard complexity classes P, NP, coNP and PSPACE as well as the polynomial hierarchy.

Definition 33. We define $\mathsf{NP} = \Sigma_1^p$, $\mathsf{coNP} = \Pi_1^p$ and for any $i \in \mathbb{N}$

$$\Sigma_i^p = \mathsf{NP}^{\Sigma_{i-1}^p} \quad \Pi_i^p = \mathsf{coNP}^{\Sigma_{i-1}^p}$$

to be the class of all problems which can be solved in NP resp. coNP with access to a Σ_{i-1}^p-oracle.

Within the research field of abstract argumentation the following decision problems are usually considered for a semantics σ.

Ver_σ Deciding whether a given set E is in $\sigma(F)$

$Exists_\sigma$ Deciding whether $\sigma(F) \neq \emptyset$.

$Exists_\sigma^{\neg\emptyset}$ Deciding whether a non-empty σ-extension $E \in \sigma(F),\ E \neq \emptyset$ exists.

$Cred_\sigma$ Deciding whether an argument $a \in A$ is credulously accepted wrt. to σ.

$Skept_\sigma$ Deciding whether an argument $a \in A$ is skeptically accepted wrt. to σ.

Building on the results for the undisputed semantics $vac_{cf}^1(ad)$ and strongly undisputed semantics $vac_{cf}^2(ad)$ from (Thimm 2023) we can derive the complexity of the above decision problems for $vac_{cf}^n(ad)$-semantics wrt. the choice of n.

Theorem 34. *1. $Ver_{vac_{cf}^n(ad)}$ is Π_n^P-complete.*

2. $Exists_{vac_{cf}^n(ad)}^{\neg\emptyset}$ is $\sum_{n+1}^P$-complete.

3. $Cred_{vac_{cf}^n(ad)}$ is $\sum_{n+1}^P$-complete.

4. $Skep_{vac_{cf}^n(ad)}$ is Π_{n+1}^P-complete.

5. $Exists_{vac_{cf}^n(ad)}$ is $\sum_{n+1}^P$-complete for $n \neq 0$ even and trivial for n odd.

With the $vac_{cf}^n(ad)$-semantics we have thus defined a class of semantics for which the existence and verification problem together span the polynomial hierarchy completely. In the light of these results the PSPACE-completeness of all these problems for the weakly preferred semantics (Dvorák, Ulbricht, and Woltran 2021) is a natural outcome. For a fixed n the above problems are of different complexity with verification being the simplest (apart from the trivial existence problem for n odd) and skeptical acceptance the hardest. Due to the correlation between $vac_{cf}^n(ad)$-semantics and the weakly preferred semantics shown in Sec. 5 we can therefore expect differences in the performance when solving these problems for weakly preferred semantics. It will be interesting to see how well $vac_{cf}^n(ad)$-semantics can approximate the weakly preferred semantics on a representative selection of AFs, e. g., the datasets of the ICCMA competitions[5].

7 Discussion and Conclusion

The main objective of our paper is to give an overview on semantical constructions with the vacuous reduct scheme. The vacuous reduct scheme is following in the footsteps of works like (Baroni, Giacomin, and Guida 2005) and more recently (Cramer and van der Torre 2019; Blümel and Ulbricht 2022). Apart from introducing new argumentation semantics they present generalized construction methods for representing and refining existing semantics. In contrast to the principle-based approach (Baroni and Giacomin 2007) with its focus on intuitive properties to be satisfied by the extensions of a semantics these works lead to a further understanding of the technical aspects of abstract argumentation semantics. Our contributions to this line of research can be summarized as follows. We generalize the notion of vacuous reduct semantics from (Thimm 2023) to nth-order vacuous reduct semantics with a finite number of base repetitions and present results on the infinite case. To illustrate our findings for the general case we provide vacuous reduct representations for a number of semantics from the literature. We give a fixpoint characterization of the weakly preferred semantics solely based on conflict-freeness and the reduct. We also define a class of semantics which approximates the weakly preferred semantics complexity-wise at any desired level of the polynomial hierarchy and discuss its behaviour.

Future work directions include 1) a principal-based analysis of the newly introduced semantics and a principle-based investigation of vacuous reduct semantics in general, e. g., which principles are inherited from the base semantics, which can be derived from the vacuity condition and which are inherent; 2) comparing the class of $vac_{cf}^n(ad)$-semantics with the not yet discussed class of $vac_{cf}^n(cf)$-semantics and other conflict-free resp. naive-based approaches from the literature; 3) investigating subclasses of AFs on which these classes of semantics collapse, e. g., cases where the weakly preferred semantics is in P (Baumann, Brewka, and Ulbricht 2020b; Dvorák, Ulbricht, and Woltran 2021). An interesting open topic is the relation between stable semantics and the reduct, starting from the question which semantics can be derived with the stable semantics as the vacuity condition up to the question whether vacuous reduct semantics are generalized stable semantics in the sense of (Blümel and Ulbricht 2022).

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