Projection of Belief in the Presence of Nondeterministic Actions and Fallible Sensing

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Abstract

We recently presented a Situation Calculus-based framework for modelling an agent who has incomplete or inaccurate knowledge about its environments, whose actions are nondeterministic, and whose sensors might give incorrect results. Generalizing earlier proposals, the approach represented the agent’s epistemic state by a set of situations ranked by their respective plausibility, and which would then be updated by modifying the plausibility ranks accordingly. Here we extend our earlier work by considering projection in this framework, i.e. the question whether a certain (epistemic) formula will hold after a given sequence of actions. We present results on both regression, where the query is transformed into an equivalent one about the initial situation, as well as progression, where the knowledge base is updated to reflect the situation after executing the action sequence in question.

1 Introduction

In a recent paper (Claßen and Delgrande 2021), we presented a framework based on the Situation Calculus (McCarthy and Hayes 1969; Reiter 2001) for representing an agent that has incomplete or inaccurate knowledge about its environment, whose actions are non-deterministic, and whose sensors might be fallible. The approach generalizes earlier proposals (Shapiro et al. 2011; Delgrande and Levesque 2012; Delgrande and Levesque 2019) where the agent’s epistemic state is presented by a set of situations ranked by their respective plausibility, and which would then be updated according to the executed action and its sensing outcome. Two notions of belief were distinguished, an extensional “bird’s eye” view, and an intensional one from the agent’s perspective. Here we extend this work by considering the projection problem, which means deciding whether a formula will hold after a given sequence of actions, which is in particular needed for the purpose of planning. We present results on both regression, where the query is transformed into an equivalent one about the initial situation, as well as progression, where the knowledge base is updated to reflect the situation after executing the action sequence in question. For brevity, we only discuss the extensional case, but it is straightforward to adapt our results to the intensional one.

As a motivating example, we adapt the extended litmus paper problem from (Hunter and Delgrande 2011). The original example is due to Moore (1985): There is a liquid in a beaker that is either an acid or a base, and the agent has a piece of litmus paper. After dipping the paper into the liquid and observing its colour, the agent will learn that the liquid is an acid, if the paper turned red, and a base, if the paper turned blue. In the extended version, the agent most plausibly believes that the paper is litmus, however also allows for the possibility that it is just a plain piece of white paper. After dipping it and observing that it remained white, the agent will revise its beliefs so that it now most plausibly believes that the paper is (and initially was) in fact not litmus paper.

We present preliminaries in Section 2, discuss regression in Section 3, progression in Section 4, and then conclude.

2 Preliminaries

2.1 The Situation Calculus

The Situation Calculus is a dialect of first-order logic, with some second-order features, for reasoning about action and change. There is a sort for actions, one for situations, and one for objects (everything else, including numbers). Situations describe possible sequences of actions, where a term do(α; σ) denotes the situation resulting from applying action α in situation σ, and S0 is a constant representing the initial situation. We will use σ (possibly with decorations) to denote terms of sort situation, and s (possibly with decorations) as situation variables. Similarly, α and β will refer to action terms, whereas α is used as action variable. For convenience, we may use sequence notation for do where do(β, do(α, S0)) can be written as do(⟨α; β⟩, S0). Changing properties are represented by fluents, which are predicates that take a situation as their last argument, e.g. Re(s) to express that the litmus paper is red in situation s. A formula is called uniform in σ if the only situation term it mentions is σ (ruling out quantification and equalities over situations).

2.2 Nondeterminism, Fallible Sensing, Belief

Here we only give a brief summary of our framework for nondeterministic actions, fallible sensing, and extensional belief, and refer to (Claßen and Delgrande 2021) for details. Nondeterminism is handled by distinguishing two types of action parameters: the ones under the control of the agent, and those under “nature’s” control (we henceforth omit the scare quotes). For example, flip(x, y) might denote the action of flipping a coin x, for which nature then determines...
outcome \( y \) (e.g., heads, tails, or some form of failure).

We use three special predicates: \( \text{Ieq}(a, a') \) expresses that actions \( a \) and \( a' \) are \textit{intensionally equivalent}, meaning they are identical except maybe for arguments set by nature. \( \text{Apl}(a, p, s) \) says that action \( a \) has \textit{argument plausibility} \( p \) in situation \( s \), where \( p = 0 \) corresponds to the most plausible outcome(s), and higher values to less plausible ones. The \textit{sensing fluent} \( SF(a, s) \) holds iff action \( a \) reports sensing result “true” in situation \( s \). Falsible sensing is again handled by a special argument set by nature, e.g., indicating that a light sensor works correctly (\( sL(\text{ok}) \)), or has a fault where it transiently reports the light being on (\( sL(\text{on}) \)) or off (\( sL(\text{off}) \)).

For beliefs, we allow for multiple initial situations \( s \), denoted by \( \text{Init}(s) \), in addition to \( S_0 \). We use an accessibility predicate \( B(s', p, s) \) to express that in situation \( s \), the agent considers situation \( s' \) possible with plausibility \( p \). A formula \( \phi \) is believed in situation \( s \), written \( Bel(\phi, s) \), if it is true in all most plausible (\( MP \)) situations \( s' \) accessible from \( s \):

\[
MP(s', s) \equiv \exists p. B(s', p, s) \land \forall s'' p'[B(s'', p'[s' \supset p \leq p')] \\
Bel(\phi, s) \equiv \forall s'. MP(s', s) \supset \phi[s'].
\]

(As convention, free variables are assumed to be universally quantified from the outside.) A successor state axiom for \( B \) then determines what is believed after executing an action \( a \):

\[
B(s', n, do(a, s)) \equiv \exists p. s', n*, a*, p'. B(s*, n*, a*, p') \land (1) \text{Apl}(a, p', s) \land \text{Ieq}(a, a') \land \text{Apl}(a', p*, s') / s = do(a*, s') \\
\land \left( SF(a*, s') \equiv SF(a, s) \right) \land n = n' + p' + p*
\]

We call formulas that don’t mention \( SF, Apl, \text{Ieq}, \) or \( B \) \textit{objective} formulas. A formula is \textit{situation-independent} if it does not mention any fluents (including special ones such as \( B \)), but perhaps includes equalities over situation terms. Finally, we say that a (non-objective) formula is \textit{about a ground situation term} \( \sigma = do(\vec{a}, S_j) \) if: (a) only \( \sigma \) occurs as right-hand side argument of \( B \); (b) every other situation term in it is either a variable or a ground situation \( do(\vec{\beta}, S_j) \) with \( |\vec{\beta}| = |\vec{\alpha}| \); (c) if a fluent atom \( F(\vec{f}, s) \) has a variable \( s \) as its situation argument, \( s \) is restricted to range only over situations such that \( B(s, p, \sigma) \) for some \( p \) or \( \text{Init}(s) \) holds. In particular any \( Bel(\psi, \sigma) \) formula then is about \( \sigma \).

### 2.3 Basic Action Theories

In order to describe a dynamic domain, we use action theories very similar to the ones proposed by Reiter (2001), but with additional axioms for the new special predicates:

**Definition 1 (Basic Action Theory).** A basic action theory (BAT) consists of the union of the following sets of formulas:

- \( \Sigma_\text{Init} \) is an initial theory that is given by a finite set of sentences about \( S_0 \), satisfying that initial situations are only \( B \)-related to other initial situations, and that plausibility values are a function of the pair of situations:

\[
\text{Init}(s) \land B(s', n, s) \supset \text{Init}(s') \\
\text{Init}(s) \land B(s', n_1, s) \land B(s', n_2, s) \supset n_1 = n_2
\]

- \( \Sigma_\text{post} \) is a finite set of successor state axioms, one for each fluent predicate \( F(\vec{x}, s) \), of the form

\[
F(\vec{x}, do(a, s)) \equiv \gamma_F(\vec{x}, a, s)
\]

where \( \gamma_F \) is an objective formula uniform in \( s \) whose free variables are among \( \vec{x} \), \( a \), and \( s \).

- \( \Sigma_\text{sense} \) is a singleton sentence of the form

\[
SF(a, s) \equiv \varphi_{SF}(a, s),
\]

where \( \varphi_{SF} \) is an objective formula uniform in \( s \) whose free variables are among \( a \) and \( s \);

- \( \Sigma_\text{eq} \) is a singleton sentence of the form

\[
\text{Ieq}(a_1, a_2) \equiv \varphi_{\text{Ieq}}(a_1, a_2),
\]

where \( \varphi_{\text{Ieq}}(a_1, a_2) \) is a situation-independent objective formula whose free variables are among \( a_1 \) and \( a_2 \);

- \( \Sigma_\text{Apl} \) is a singleton sentence of the form

\[
\text{Apl}(a, p, s) \equiv \varphi_{\text{Apl}}(a, p, s),
\]

where \( \varphi_{\text{Apl}} \) is an objective formula uniform in \( s \) whose free variables are among \( a, p, \) and \( s \);

- \( \Sigma_\text{uni} \) is a set of unique names axioms for actions;

- \( \Sigma_\text{fnd} \) is the set of foundational axioms that defines the set of situations to form a forest of isomorphic trees rooted at initial situations, where \( \text{Init}(s) \equiv \neg\exists a. s'. s = do(a, s') \).

We ignore preconditions here for brevity, but they would be straightforward to include. Compared to (Clàrchen and Delgrande 2021), we also make the simplifying assumption that axioms for \( \text{Ieq}, \text{Apl}, \) and \( SF \) are given as a single equivalence each, instead of multiple guarded axioms. It is conceivable, though significantly more involved, to extend our projection results to guard axioms along the lines of what is described in (De Giacomo and Levesque 1999).

**Example 1.** We formalize the extended litmus scenario as follows. Fluents \( \text{Ac}(s), \text{Li}(s), \text{Re}(s), \) and \( \text{Bl}(s) \) denote that in situation \( s \), the liquid is an acid, the paper is litmus paper, the paper is red, and the paper is blue, respectively. The action of dipping the paper in the liquid is denoted by \( \text{dip} \), and the action of sensing whether the paper is white by \( \text{sw}(y) \).

- \( \Sigma_0 \): Initially, the agent considers four situations possible:

\[
\text{Init}(s) \equiv s = S_0 \lor s = S_1 \lor s = S_2 \lor s = S_3, \\
B(s, p, S_0) \equiv (s = S_1 \land p = 0) \lor (s = S_2 \land p = 0) \lor (s = S_0 \land p = 1) \lor (s = S_3 \land p = 1), \\
\neg\text{Ac}(S_0) \land \neg\text{Li}(S_0) \land \neg\text{Re}(S_0) \land \neg\text{Bl}(S_0), \\
\text{Ac}(S_1) \land \text{Li}(S_1) \land \neg\text{Re}(S_1) \land \neg\text{Bl}(S_1), \\
\neg\text{Ac}(S_2) \land \text{Li}(S_2) \land \neg\text{Re}(S_2) \land \neg\text{Bl}(S_2), \\
\text{Ac}(S_3) \land \neg\text{Li}(S_3) \land \neg\text{Re}(S_3) \land \neg\text{Bl}(S_3).
\]

- \( \Sigma_\text{post} \): In this example, \( \text{Ac}(s) \) and \( \text{Li}(s) \) are rigid predicates, and don’t change their value even if any action, \( \text{Re}(\text{do}(a, s)) \) will be true iff \( a \) is a dip action, the liquid is acid, and the paper is litmus. Similar for \( \text{Bl} \).

\[
\text{Ac}(\text{do}(a, s)) \equiv \text{Ac}(s), \text{Li}(\text{do}(a, s)) \equiv \text{Li}(s), \\
\text{Re}(\text{do}(a, s)) \equiv a = \text{dip} \land \text{Ac}(s) \land \text{Li}(s) \lor \text{Re}(s), \\
\text{Bl}(\text{do}(a, s)) \equiv a = \text{dip} \land \neg\text{Ac}(s) \land \text{Li}(s) \lor \text{Bl}(s).
\]
sequences of action variables of the same length, and prefix a sequence of ground actions of length the form sequences rewrite it to $P$ that cannot immediately be regressed further, unless we may end up with a formula such as $\cdot$

Due to being deterministic, the dip action always has plausibility 0. The sensing action most plausibly returns the correct result, less plausibly has a transitory off failure, and even less plausibly a transitory on failure.

$$\begin{align*}
    Ap\ell(a, p, s) &\equiv \\
    & (a = dip \land p = 0) \lor (a = sW(ok) \land p = 0) \lor \\
    & (a = sW(off) \land p = 1) \lor (a = sW(on) \land p = 2)
\end{align*}$$

3 Regression

The idea behind regression is to turn a query formula about a future situation into an equivalent one about the initial situation, and then test it against the initial theory of the BAT.

**Definition 2** (Regression). Let $\Sigma$ be a BAT, $\bar{a}$ a sequence of ground action terms, and $\psi$ a formula about $do(\bar{a}, S_0)$. Then a formula $\phi$ is a regression of $\psi$ wrt $\Sigma$ if it is about $S_0$ and $\Sigma \models \psi$ iff $\Sigma_0 \cup \Sigma_{\text{una}} \models \phi$.

Reiter (2001) presented a regression operator $R$ that achieves this for objective formulas by iteratively replacing subexpressions deemed equivalent by the BAT. It is straightforward to extend it to the new $SF$, $Ieq$, and $Ap\ell$ predicates:

- $R[F(t, do(\bar{a}, \sigma))] = R[F(t, do(\bar{a}, \sigma))], \text{if } F \text{ is a fluent};$
- $R[P(t)] = R[P(t)] \text{ for } P \in \{SF, Ieq, Ap\ell\};$
- $R[\phi] = \phi \text{ for every atomic formula } \phi;$
- $R$ distributes over $\neg, \land, \lor$, e.g. $R[\phi \land \psi] = R[\phi] \land R[\psi].$

In principle, formulas involving $B$ and $Bel$ could be handled similarly, given that (1) is a successor state axiom for $B$. A technical problem is that the axiom as well as $Bel$ formulas use quantification over situations, and due to nesting, we may end up with a formula such as $\exists s'. s' = do(a, s) \land F(s')$ that cannot immediately be regressed further, unless we rewrite it to $F(do(a, s))$. The solution below makes use of this idea, together with the fact that if situations $\sigma$ and $\sigma'$ are $B$-related, then they are the result of applying action sequences of the same length to $B$-related initial situations.

We here restrict ourselves to regressing belief atoms of the form $Bel(\phi, do(\bar{a}, S_0))$, where $\bar{a} = \langle a_0, \ldots, a_{k-1} \rangle$ is a sequence of ground actions of length $k$. Let $\vec{a}, \vec{a}'$ denote sequences of action variables of the same length, and $\bar{a}_i$ the prefix $\langle a_0, \ldots, a_i \rangle$, with $\bar{a}_{i-1} = \emptyset$. We add the rules

- $R[\bar{a}] = \bar{a}$,
- $R[\phi \land \psi] = R[\phi] \land R[\psi].$

During further evaluation, many subexpressions can be simplified. For example, $SF(wok, \psi)$ reduces to $true$, and $Ieq(dip, \bar{a}'_0)$ to $\bar{a}' = \bar{a}$. We thus obtain

$\exists n_0. B(s'_0, n_0, S_0) \land a'_0 = \text{dip} \land$

$\exists s_0. s_0 = sW(ok) \land n = n_0 \land Li(s_0) \lor$

$\exists a'_1, a'_0. a'_1 = sW(on) \land n = n_0 + 2$. 

$MP(\ldots)$ now describes those $do(\bar{a}'_2, \bar{a}'_1, s'_0)$ situations with minimal $n$, and $Bel(\ldots)$ requires $Li$ to be false there. Both $do(\bar{a}'_2, \bar{a}'_1, s'_0)$ and $do(\bar{a}'_2, \bar{a}'_1, s'_0)$ satisfy this with $n = 1$. The agent hence believes the paper is not litmus after dipping and sensing that it is white.

4 Progression

We follow (Vassos and Levesque 2013) and define:

**Definition 3** (Progression). Let $\Sigma$ be a BAT, $\bar{a}$ a ground action term, and $\Sigma_0$ a set of sentences about $do(\bar{a}, S_0)$. Then $\Sigma_1$ is a progression of $\Sigma_0$ wrt $\Sigma$, $\Sigma$ iff for every sentence $\phi$ about $do(\bar{a}, S_0)$, $\Sigma_1 \models \phi$.
Lin and Reiter (1997) showed that even in the objective case, the progression of a theory may require second-order logic. However, they also presented special cases for which progression is first-order definable, one of them being the relatively complete databases. Here, we generalize this class to the case of extensional belief. Let us call a BAT relatively complete if $\Sigma_0$ comprises exactly the following formulas:

1. The formula

$$Init(s) \equiv s = S_0 \lor \cdots \lor s = S_k,$$

where $S_0, \ldots, S_k$ denote all and only initial situations;

2. one formula of the form

$$B(s, p, S_0) \equiv \Pi^{S_0}_{B}(s, p),$$

where $\Pi^{S_0}_{B}$ is a situation-independent objective formula whose free variables are among $s, p$;

3. for each fluent $F$ and each $S_i$, one formula of the form

$$F(\bar{x}, S_i) \equiv \Pi^{S_i}_{F}(\bar{x}),$$

where $\Pi^{S_i}_{F}(\bar{x})$ is a situation-independent objective formula whose free variables are among $\bar{x}$.

Also assume that a ground action $\alpha$ has only finitely many $\beta$ with $\Sigma_{\text{una}} \models \text{ieq}(\alpha, \beta)$. Let $\mathcal{Q}$ be an operator replacing $B$ and $F$ subformulas by corresponding $\Pi$ expressions:

- $\mathcal{Q}[B(\sigma, t, S_0)] = \Pi^{S_0}_{B}(\sigma, t)$;
- $\mathcal{Q}[F(\bar{t}, \sigma)] = \bigvee_{0 \leq i \leq k} (\sigma = S_i) \land \Pi^{S_i}_{F}(\bar{t})$;
- $\mathcal{Q}[P(\bar{t})] = \mathcal{Q}[\phi_F(\bar{t})]$ for $P \in \{SF, \text{ieq}, ApI\}$;
- $\mathcal{Q}[\phi] = \phi$ for every other atomic formula $\phi$;
- $\mathcal{Q}$ distributes over $\neg, \land, \lor$, e.g. $\mathcal{Q}[\phi \land \psi] = \mathcal{Q}[\phi] \land \mathcal{Q}[\psi]$.

**Theorem 2.** Let $\Sigma$ be a relatively complete BAT, $\alpha$ a ground action, and $\gamma_B(s', p, \alpha, s)$ the right-hand side of axiom (1). We construct a set $\Sigma_\alpha$ consisting of the following formulas:

1. The formula

$$B(s, p, \text{do}(\alpha, S_0)) \equiv \mathcal{Q}[\gamma_B(s, p, \alpha, S_0)].$$

2. For each fluent $F$, each $S_i$, and each ground action $\beta$ such that $\Sigma_{\text{una}} \models \text{ieq}(\alpha, \beta)$, the formula

$$F(\bar{x}, \text{do}(\beta, S_i)) \equiv \mathcal{Q}[\gamma_F(\bar{x}, \beta, S_i)].$$

Then $\Sigma_\alpha$ is a progression of $\Sigma_0$ through $\alpha$ wrt $\Sigma$.

**Proof.** (Sketch) $\Sigma_\alpha$ obviously has the right form. Soundness follows from the fact that we only replace equivalent subformulas, completeness from the fact that the set gives a complete description for all fluents and situations.

Notice that $\Sigma_\alpha$ has the right form to apply progression repeatedly, if we treat the finitely many $\text{do}(\beta, S_i)$ as new initial situations (e.g. by replacing each by a new constant $S'_\beta$).

**Example 3.** Suppose we want to progress the BAT from Example 1 through $\langle \text{dip}, sW(\text{ok}) \rangle$, i.e. determine what the agent believes after dipping the paper and then performing a sense-white action. Notice that the BAT has the right form if we understand expressions like $\neg \text{Ac}(S_2)$ as shorthand for $\text{Ac}(S_2) \equiv \text{FALSE}$. After the dip action, each initial situation is updated by the results of the dip (let $S'_2 = \text{do}(\text{dip}, S_2)$):

$$\neg \text{Ac}(S'_0) \land \neg \text{Li}(S'_0) \land \neg \text{Re}(S'_0) \land \neg \text{Bl}(S'_0)$$

$$\text{Ac}(S'_1) \land \text{Li}(S'_1) \land \text{Re}(S'_1) \land \neg \text{Bl}(S'_1)$$

$$\neg \text{Ac}(S'_2) \land \text{Li}(S'_2) \land \neg \text{Re}(S'_2) \land \text{Bl}(S'_2)$$

$$\text{Ac}(S'_3) \land \neg \text{Li}(S'_3) \land \neg \text{Re}(S'_3) \land \neg \text{Bl}(S'_3)$$

The situations’ plausibilities remain unchanged, but for those situations where $\text{Li}$ is true, the agent will believe the paper is red if the liquid is acid, and blue if base. Consequently the agent believes the paper is either red or blue.

$$B(s, p, S'_0) \equiv (s = S'_1 \land p = 0) \lor (s = S'_2 \land p = 0) \lor (s = S'_3 \land p = 1) \lor (s = S'_3 \land p = 1)$$

After the sense-white action, the physical fluents retain their previous value, but the belief state changes as follows:

$$B(s, p, \text{do}(\langle \text{dip}, sW(\text{ok}) \rangle, S_0)) \equiv$$

$$(s = \text{do}(\langle \text{dip}, sW(\text{ok}) \rangle, S_0) \land p = 1) \lor$$

$$(s = \text{do}(\langle \text{dip}, sW(\text{ok}) \rangle, S_0) \land p = 1) \lor$$

$$(s = \text{do}(\langle \text{dip}, sW(\text{on}) \rangle, S_1) \land p = 2) \lor$$

$$(s = \text{do}(\langle \text{dip}, sW(\text{on}) \rangle, S_2) \land p = 2) \lor$$

$$(s = \text{do}(\langle \text{dip}, sW(\text{on}) \rangle, S_3) \land p = 3) \lor$$

$$(s = \text{do}(\langle \text{dip}, sW(\text{on}) \rangle, S_4) \land p = 3)$$

The two situations with minimum ($p = 1$) implausibility are those where sensing worked correctly. In both, the paper is not litmus, but $\text{Ac}$ only holds in one. Other situations ($p > 1$) record less plausible circumstances where the sensing action failed. Hence the agent believes the paper is not litmus and does not know whether the liquid is acid or base.

## 5 Conclusion

We presented results on both regression and progression in a Situation Calculus-based framework for reasoning about the beliefs of an agent who is dealing with conflicting information, nondeterministic actions, and fallible sensing. As the two main solutions to the projection problem, they are paramount ingredients for any practical application of the formalism. A prototype implementation of our methods in SWI-Prolog with the examples from this paper is available on GitHub (Claßen 2022). For future work, we want to investigate their use as basis for an epistemic planner, possibly in combination with first-order knowledge compilation techniques (Claßen 2018). Without going into detail, we believe that our approach lends itself very well for this purpose when compared to similar work (Fang, Liu, and Wen 2015; Schwering and Lakemeyer 2015; Arenas et al. 2016). While (Shapiro et al. 2011) do not even consider projection, (Hunter and Delgrande 2011) only keep track of the most plausible set of situations, albeit in an action language (Gelfond and Lifschitz 1998) framework. In case of a contradiction, they regress to the initial situation, carry out belief revision, and progress through the action sequence again. We avoid this by keeping track of all possible outcomes, and in case of a sensing conflict, only need to look at the most plausible situations compatible with the sensing outcome.
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