Proceedings of the 18th International Conference on Principles of Knowledge Representation and Reasoning
Main Track

Belief Contraction in Non-classical logics as Hyperintensional Belief Change

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Abstract

AGM's belief revision is one of the main paradigms in the study of belief change operations. Despite its popularity and importance to the area, it is well recognised that AGM's work relies on a strong idealisation of the agent's capabilities and the nature of beliefs themselves. Particularly, it is recognised in the literature that Belief and Knowledge are hyperintensional attitudes, i.e. they can differentiate between contents that are necessarily equivalent, but to our knowledge, only a few works have explicitly considered how hyperintensionality affects belief change. This work investigates abstract operations of hyperintensional belief change and their connection to belief change in non-classical logics, such as belief contraction operations for Horn Logics and Description Logics. Our work points to hyperintensional belief change as a general framework to unify results in belief change for non-classical logics.

1 Introduction

Belief Change is the area that studies how doxastic agents change their minds after acquiring new information. One of the most influential approaches to Belief Change in the literature is the AGM paradigm (Alchourrón, Gärdenfors, and Makinson 1985).

Although the AGM approach has led to profound developments in belief dynamics, it has received criticism, particularly concerning the idealised nature of doxastic agents in their work (Hansson 1991; Rott and Pagnucco 1999; Hansson and Wassermann 2002). Namely, an agent's belief state is characterised in their work by a consequentially closed set of beliefs, and a belief revision is an intensional operator, i.e. based on the language semantics/proof theory. In fact, Gärdenfors (1988, p. 9), acknowledges that AGM's notion of belief is but merely an idealisation "judged in relation to the rationality criteria for the epistemological theory".

Hansson (1992) argues in his criticism of using consequentially closed belief sets that, on a dynamic level, the agent's belief state depends not only on the meaning of their beliefs but also on something else, which the author identifies with its syntactic structure. More yet, it has been argued in the literature (Halpern and Pucella 2011; Wansing 1990; Bjerring 2013) that resource-bounded agents are not required to believe all consequences of their currently held beliefs, even if they are logically capable of rational inquiry, since an agent can fail to reach the conclusion of a reasoning process due to a lack of cognitive resources.

We call attitudes that depend on sentential contents finer-grained than sentential intensions of *hyperintensional attitudes* (Cresswell 1975). In other words, these attitudes can draw distinctions between necessarily equivalent contents. For example, while the sentences "*3 is a prime number*" and "*3068 is divisible by 13*" have the same intension, as mathematical necessities, they certainly cannot be transparently substituted for the other in the sentence "Alice believes that *3 is a prime number*."

As Özgün and Berto (2020) argue, it is well-known that mental attitudes, such as believing, are sensitive to hyperintensional distinctions between equivalent sentences and that these distinctions are connected to well-studied problems, such as logical omniscience (Halpern and Pucella 2011; Bjerring 2013; Rantala 1982).

Similarly, several non-classic logics, such Horn Logic (Delgrande 2008) and Description Logics (Baader et al. 2003) can be understood as sublanguages of some classical logic and, as such, impose hyperintensional differences within the latter. Thus, understanding the interplay between the hyperintensional nature of beliefs and their dynamics has significance from a philosophical point of view and may provide a framework to study belief change in several non-classical logics of interest to Knowledge Representation and Artificial Intelligence.

Based on the work of Santos et al. (2018) and of Berto (2019), Souza (2020) studies general notions of hyperintensional belief contraction based on an Abstract Logic framework. In this work, we employ Souza's hyperintensional belief contractions to study belief change in non-classical logics, with applications to belief change in Horn Logic and Description Logics. We also extend Souza's framework by introducing a new family of hyperintensional belief contraction operations based on Hansson's (1991) belief base contraction, which is a generalization of Souza's previous proposals, as well as correct the characterisations presented in that work.

This work is structured as follows: first, in Section 2, we discuss the literature related to hyperintensional belief change and frameworks for non-classical logic that can be connected to this notion; in Section 3, we present the background results of AGM belief change and Hansson's belief base change; we introduce, in Section 4, three hyperintensional contraction operations, based on the work of Souza (2020) and characterise these operations through postulates; in Section 5, we apply these operations to study examples in the literature of belief change for non-classical logics, such as Delgrande and Wassermann's (2013) Horn Contractions and Ribeiro and Wassermann's (2009) contractions for Description Logics; finally, in our Final Considerations, we discuss the importance of hyperintensional belief change as a general framework of belief change in non-classical logics and possible future developments for this framework.

2 Related Work

Extensive work has been published on general notions of belief change not constrained by the laws of classical logic, such as the work on belief change for non-classical (Delgrande 2008; Ribeiro 2012; Ribeiro et al. 2013; Gabbay, Rodrigues, and Russo 2008), paraconsistent (Girard and Tanaka 2016), or substructural logics (Aucher 2015). We will focus on work connected to, or that can be used to study, hyperintensional notions of belief change.

Girard and Tanaka (2016) have investigated dynamic belief change operators for many-valued logics, which in principle could be used to model some hyperintensional notions, as classical consequences need not be valid in such logics. While we believe this work proposes an interesting and general framework for non-classical notions of belief change, including *some* notion of hyperintensionality, it is not clear how hyperintensional contexts can be encoded in it. Particularly, since their models are based on intensional interpretations of connectives, it is not directly clear how to construct (a class of) models that can differentiate two (classically) equivalent formulas φ and ψ without degenerating the interpretation of these connectives.

Berto (2019) proposes a topic-sensitive hyperintensional logic of conditional beliefs, in which belief revision is interpreted as conditionalisation. In this logic, the author shows that the notion of conditional belief satisfies minimal desiderata for logics of belief change (Board 2004) and non-monotonic reasoning (Gabbay 1985). To our knowledge, this is the first work explicitly proposing the integration of hyperintensional phenomena within a theory of belief change and, in fact, it constitutes the main inspiration for our work. Unlike his work, ours investigates how a general notion of hyperintensional belief change can be defined, based on the AGM approach, that can be connected to different semantic frameworks for hyperintensional reasoning.

It is interesting to notice that, since impossible-world semantics has been a popular approach to model the hyperintensional nature of beliefs (Cresswell 1975; Rantala 1982), work on belief change based on impossible-world semantics can be connected to our work and may present tools and properties with which we can evaluate our contribution.

Badura and Berto (2019) propose the use an impossible-world semantics framework to overcome several limitations of Lewis' modal analyses of *Truth in Fiction*. While their interpretation of the notion of *truth in fiction* statements is conceptually connected to belief revision, the authors also do not establish the connection with the AGM approach, as this was not their goal in the first place.

Fermé and Wassermann (2018) have also proposed the use of impossible worlds semantics as a framework to study iterated belief expansion. The authors extend Grove's models (Grove 1988) for classical propositional logic with one impossible world $w_\perp$ satisfying all formulas of the language. This world is, however, included in the model merely as a technical device to represent the belief state in which the agent has inconsistent belief and not to model any kind of hyperintensional property of beliefs. However, if their approach was to be extended, we believe such a framework can be used to define hyperintensional notions of revision, contraction and expansion, based on the authors encoding of such operations.

Perhaps the works closest to ours are that of Santos et al. (2018) on pseudo-contractions and of Souza (2020) on hyperintensional belief contractions. Given a logic $\mathcal{L}$, Santos et al. (2018) investigate how these operations can be defined for sublogics of $\mathcal{L}$, thus studying how contraction-like operators can be defined for consequence operators which can take into consideration (possibly) hyperintensional differences between formulas in the reasoning process. While their work does investigate the connections between hyperintensional beliefs and belief change, it does not do so explicitly. Souza (2020), on the other hand, based on the work of Santos et al. (2018) and of Berto (2019), proposed general notions of hyperintensional belief contraction. We extend this latter work by proposing a new operation, namely C-Base contractions, which can be interpreted as a generalisation of Hansson's (1991) belief base contractions, and applying these operations to obtain belief contraction operators for non-classical logics, such as Horn Logic and Description Logics.

3 Preliminaries

Let us consider a logic $\mathcal{L} = \langle L, Cn \rangle$ where L is the logical language and $Cn : 2^L \rightarrow 2^L$ is a consequence operator. In AGM's approach, the belief state of an agent is represented by a belief set, i.e. a consequentially closed set $K = Cn(K) \subseteq L$ of $\mathcal{L}$-formulas.

On a belief set, AGM investigate three basic belief change operators: expansion, contraction and revision. Belief expansion blindly integrates a new piece of information into the agent's beliefs. Belief contraction removes a currently believed sentence from the agent's set of beliefs, with minimal alterations. Finally, belief revision is the operation of integrating new information into an agent's beliefs while maintaining consistency. Among these basic operations, only expansion can be univocally defined. The other two are defined by a set of rational constraints or postulates, usually referred to as the AGM postulates. These postulates define a class of suitable change operators representing different rational ways in which an agent can change their beliefs.

AGM require the foundational logic $\mathcal{L}$ to satisfy some properties in order for belief change operations to be definable. These properties are the following:

- **inclusion**: $\Gamma \subseteq Cn(\Gamma)$.
- **idempotence**: $Cn(\Gamma) = Cn(Cn(\Gamma))$.
- **monotonicity**: If $\Gamma \subseteq \Gamma'$ then $Cn(\Gamma) \subseteq Cn(\Gamma')$.
- **tarskianicity**: If Cn satisfies inclusion, idempotence and monotonicity.
- **compactness**: for any $\varphi \in Cn(\Gamma)$, there is some finite $\Gamma' \subseteq \Gamma$ s.t. $\varphi \in Cn(\Gamma')$.
- **supraclassicality**: Let Cn_0 be the classical propositional logic consequence operator, $Cn_0(\Gamma) \subseteq Cn(\Gamma)$ for any $\Gamma \subseteq L$.
- **deduction theorem**: If $\beta \in Cn(\Gamma \cup \{\alpha\})$ then $\alpha \to \beta \in Cn(\Gamma)$.

To construct contraction operations, AGM (1985) introduce the notion of partial meet contractions. Let $K \subseteq L$ be a set of formulas and $\varphi \in L$ be a formula of L, the remainder set $K \bot_{\mathcal{L}} \varphi$ is the set of sets K' satisfying:

- $K' \subseteq K$
- $\varphi \notin Cn(K')$
- $K' \subset K'' \subseteq K$ implies $\varphi \in Cn(K'')$.

When it is clear to which logic $\mathcal{L}$ we are referring, we will denote $K \bot_{\mathcal{L}} A$ by $K \bot A$.

A partial meet contraction $-$ is an operation for which there is a selection function γ, s.t. for any K and φ

$$K - \varphi = \bigcap \gamma(K \bot \varphi).$$

By selection function, we mean that the function γ satisfies (i) $\emptyset \neq \gamma(K \bot A) \subseteq K \bot A$ if $K \bot A \neq \emptyset$ and (ii) $\gamma(K \bot A) = \{K\}$ otherwise.

Hansson and Wassermann (2002) have shown that for any monotonic and compact logic, an operation $-$ is a partial meet contraction if and only if it satisfies the following postulates:

$(\mathbf{success})$ If $\varphi \notin Cn(\emptyset)$, then $\varphi \notin Cn(K - \varphi)$

$(\mathbf{inclusion})$ $K - \varphi \subseteq K$

$(\mathbf{relevance})$ If $\beta \notin K \backslash K - \varphi$, then there is some $K' \subseteq K$ s.t. $K - \varphi \subseteq K'$, $\varphi \notin Cn(K')$, and $\varphi \in Cn(K' \cup \{\beta\})$

$(\mathbf{uniformity})$ If for any $K' \subseteq K$, $\varphi \in Cn(K')$ iff $\psi \in Cn(K')$, then $K - \varphi = K - \psi$

This study has spammed several investigations on the generalisability of AGM belief change to other logics, such as Horn Logics (Delgrande 2008), Description Logics (Flouris, Plexousakis, and Antoniou 2005; Ribeiro and Wassermann 2009), and non-compact logics (Ribeiro, Nayak, and Wassermann 2018). Given the more general nature of Hansson and Wassermann's (2002) characterisation, we will focus on these postulates to define hyperintensional belief contractions to a broader class of logics instead of the more restricted AGM postulates. In the following sections, we will investigate how the previous postulates and constructions need to be altered to account for the hyperintensional nature of beliefs.

4 Hyperintensional Belief Contraction

The term hyperintensionality was introduced by Creswell (1975) to describe attitudes, such as beliefs, that can draw distinctions between necessarily equivalent formulas or having the same intension. As such, hyperintensionality is commonly explained by means of the relation between the contents of a sentence[1] and its intension, with respect to a standard semantics. For example, in possible-world semantics, a sentence's intension is commonly understood as the set of possible worlds in which this sentence holds. Thus, hyperintensional semantic differences can be established in relation to this fixed semantic framework.

In the remainder of this work, we will represent hyperintensional reasoning - i.e. reasoning that considers hyperintensional contexts - in an abstract form by means of the relationship between two consequence operators over a given language. Let us define this notion formally.

Definition 1. *Let $\mathcal{L} = \langle L, Cn \rangle$ be a logic, and $C : 2^L \to 2^L$ be a consequence operator. We say that:*

- *C is $\mathcal{L}$-sound, if for every $\Gamma \subseteq 2^L$, $C(\Gamma) \subseteq Cn(\Gamma)$*
- *C is $\mathcal{L}$-complete, if for every $\Gamma \subseteq 2^L$, $Cn(\Gamma) \subseteq C(\Gamma)$*

We believe $\mathcal{L}$-sound consequences are useful to study hyperintensional attitudes. For example, they are easily applicable for specifying *slow thinking* cognitive processes (Solaki, Berto, and Smets 2019; Bjerring and Skipper 2019) which can be used to describe the cognitive process of deduction and explain why knowing that "*3 is a prime number*" is valid is not enough for one to know that "*3068 is divisible by 13*" also is, even though they are intensionally equivalent. $\mathcal{L}$-complete consequence operators, on the other hand, can be useful to understand natural inferences that lie outside the realm of the language's semantics and usually associated with pragmatic phenomena, such as conversational implicatures.

In the following, we explore some different $\mathcal{L}$-sound hyperintensional contraction operations, i.e. hyperintensional contractions based on a $\mathcal{L}$-sound consequence operator C, and characterise them through appropriate postulates, as commonly pursued by the AGM tradition. Throughout this section, we will denote by $K \bot \varphi$ the remainder set $K \bot_{\mathcal{L}} \varphi$ according to the foundational logic $\mathcal{L}$.

4.1 C-Dependent Contractions

The first example of a $\mathcal{L}$-sound hyperintensional contraction operation that we are aware of was proposed by Santos et al. (2018) in their study of pseudo-contraction operations for non-classic logics. The authors propose the notion of C-dependent partial meet contractions[2], which consist of maintaining all information that can be deduced by a $\mathcal{L}$-sound consequence operator, i.e. by hyperintensional reasoning, while removing a formula from the agent's beliefs. The operation can be formally defined as follows.

[1]The is an ongoing debate on the nature of the sentential contents involved in drawing appropriate hyperintensional distinctions, see for example the work of Cresswell (1975) and Jago (2014)

[2]In their terminology, Cn^*-pseudo-partial meet contractions.

Definition 2. *Let $\mathcal{L}$ be a monotonic and compact logic, let C be a $\mathcal{L}$-sound consequence operator, and $K \subseteq L$ be a set of formulas. We say an operator $- : 2^L \times L \to 2^L$ is a C-dependent partial meet contraction (or C-dependent contraction) on K if there is some selection function γ s.t. for any formula $\varphi \in L$ it holds that*

$$K - \varphi = \bigcap \gamma(C(K) \bot \varphi)$$

To characterise these operations, we employ the following postulates.

(C-logical inclusion) $K - \varphi \subseteq C(K)$.

(success) $\varphi \notin Cn(\varnothing)$ then $\varphi \notin Cn(K - \varphi)$

(C-uniformity) If for all $K' \subseteq C(K)$ it holds that $\varphi \in Cn(K')$ iff $\psi \in Cn(K')$, then $K - \varphi = K - \psi$.

(C-logical relevance) If $\beta \in C(K) \backslash K - \varphi$, then there is some $K' \subseteq C(K)$ s.t. $K - \varphi \subseteq K'$, $\varphi \notin Cn(K')$, and $\varphi \in Cn(K' \cup \{\beta\})$.

We can easily generalise Santos et al.'s (2018) characterisation, which assumes the consequence operator Cn to be tarskian, to monotonic and compact logics $\mathcal{L}$ and, thus, to C-dependent contractions in general.

Theorem 3. *Let $\mathcal{L}$ be a monotonic and compact logic, and C be a $\mathcal{L}$-sound consequence operator. An operator $-$ is a C-dependent contraction on a set $K \subseteq L$ and a formula $\varphi \in L$ iff $-$ satisfies (C-logical inclusion), (success), (C-uniformity), and (C-logical relevance)*

Proof. (Construction to Postulates): The satisfaction of the postulates can be trivially obtained from the fact that there is some partial meet contraction $\ominus$ in $\mathcal{L}$ s.t. $K - \varphi = C(K) \ominus \varphi$ and $\ominus$ satisfies *(success)*, *(inclusion)*, *(relevance)*, and *(uniformity)*.

(Postulates to Construction): Let us construct the function γ s.t. $\gamma(C(K) \bot \varphi) = \{K' \in C(K) \bot \varphi \mid K - \varphi \subseteq K'\}$ if $\varphi \notin Cn(\varnothing)$ and $\gamma(C(K) \bot \varphi) = \{C(K)\}$, otherwise.

It is easy to see that the γ function is well-defined, i.e. if $C(K) \bot \alpha = C(K) \bot \beta$, then $\gamma(C(K) \bot \alpha) = \gamma(C(K) \bot \beta)$ from *(C-uniformity)*. The fact that γ is a selection function follows from *(success)*, since Cn is monotonic and compact.

Finally, we must show that $K - \varphi = \bigcap \gamma(C(K) \bot \varphi)$. We have two cases: if $\varphi \in Cn(\varnothing)$ and $\varphi \notin Cn(\varnothing)$.

In the first case, by construction, $\bigcap \gamma(C(K) \bot \varphi) = C(K)$ and, by *C-logical inclusion*, $K - \varphi \subseteq C(K) = \bigcap \gamma(C(K) \bot \varphi)$. As $\varphi \in Cn(\varnothing)$, then there is no $K' \subseteq C(K)$ s.t. $\varphi \notin Cn(K')$, as Cn is monotonic. Thus, by *(C-logical relevance)* of $-$, there is no $\beta \in C(K) \backslash K - \varphi$, i.e. $C(K) = K - \varphi$.

In the second case, we have that $\varphi \notin Cn(\varnothing)$, thus $C(K) \bot \varphi \neq \varnothing$ and $\gamma(C(K) \bot \varphi) \neq \varnothing$. As $K - \varphi$ is a subset of any $K' \in \gamma(C(K) \bot \varphi)$, then $K - \varphi \subseteq \bigcap \gamma(C(K) \bot \varphi)$. To obtain the reverse inclusion, take $\beta \notin K - \varphi$. If $\beta \notin C(K)$, then $\beta \notin \bigcap \gamma(C(K) \bot \varphi)$; on the other hand, if $\beta \in C(K) \backslash K - \varphi$, by *($C$-logical relevance)*, there is some K' s.t. $K - \varphi \subseteq K \subseteq C(K')$, $\varphi \notin Cn(K')$, but $\varphi \in Cn(K' \cup \{\beta\})$. Since Cn is monotonic and compact, there is some maximal $K'' \in C(K) \bot \varphi$ s.t. $K' \subseteq K''$. As $K - \varphi \subseteq K''$, then $K'' \in \gamma(C(K) \bot \varphi)$ and, thus, $\beta \notin \bigcap \gamma(C(K) \bot \varphi)$. □

As Santos et al. (2018) discuss, C-dependent contractions define intermediary levels of information preservation for a contraction operation, encoded by the *(C-logical relevance)* postulate, ranging from belief base contractions - which are completely dependent on the structure of the belief base - to AGM contractions - which are completely independent of that structure. In fact, it is easy to see that if $C(K) = K$, the notion of C-dependent contraction is reduced to that of partial meet contraction.

Corollary 4. *Let $\mathcal{L}$ be a monotonic and compact logic, C be a $\mathcal{L}$-sound consequence operator s.t. $C(K) = K$. An operator $-$ is a C-dependent contraction iff $-$ satisfies (success), (inclusion), (relevance) and (uniformity).*

4.2 C-Sensitive Contraction

Berto (2019) defines their topic-sensitive conditional beliefs, which the author interprets as belief revision, as the result of revising the agent's beliefs by some new information and selecting all hyperintensional consequences of the agent's beliefs. Based on the Harper identity (Alchourrón, Gärdenfors, and Makinson 1985), which connects the operations of revision and contraction, Souza (2020) proposes the notion of C-sensitive contraction, generalising Berto's topic-sensitive belief change operations.

Definition 5. *Let $\mathcal{L}$ be a monotonic and compact logic, C be a $\mathcal{L}$-sound consequence operator, $K \subseteq L$ a set of formulas. We say an operator $- : 2^L \times L \to 2^L$ is a C-sensitive partial meet contraction (or C-sensitive contraction) on K if there is some selection function γ s.t. for any formula $\varphi \in L$, it holds that*

$$K - \varphi = C(\bigcap \gamma(K \bot \varphi))$$

It is important to notice that C-sensitive contractions return a consequentially closed set, similar to AGM belief contractions. While in Definition 5, we do not require the input set K to be consequentially closed, as in AGM's operations, we will focus in our characterisation for contractions on consequentially-closed sets of beliefs, either regarding the operator Cn or the hyperintensional operator C. As such, depending on the properties of the consequence operator C, C-sensitive contractions satisfy a variation of the postulate *(closure)*. We will employ the following postulates in the characterisation of C-sensitive contractions, a revision of the postulates originally proposed by Souza (2020) regarding the *(C-sensitive relevance)* postulate.

(C-enforced closure) $K - \varphi = C(K - \varphi)$

(C-logical inclusion) $K - \varphi \subseteq C(K)$.

(success) If $\varphi \notin Cn(\varnothing)$ then $\varphi \notin Cn(K - \varphi)$

(uniformity) If for all $K' \subseteq K$, $\varphi \in Cn(K')$ iff $\psi \in Cn(K')$, then $K - \varphi = K - \psi$

(C-sensitive relevance) If $\beta \in C(K) \backslash Cn(K - \varphi)$ then there is some $K' \subseteq Cn(K)$ s.t. $K - \varphi \subseteq K'$, $\varphi \notin Cn(K')$, and $\varphi \in Cn(K' \cup \{\beta\})$.

It is clear from similar studies in belief change in non-classical logics (Hansson and Wassermann 2002; Flouris, Plexousakis, and Antoniou 2005; Ribeiro and Wassermann

2009; Santos et al. 2018) that the connection between postulates and constructions of operators may depend on which properties the foundational logic satisfies.

To provide the connection between the construction of C-Sensitive contractions in Definition 5 and the postulates presented above, beside those properties introduced in Section 3, in this this work, we will employ the following property for the consequence operator C:

- **locality**: If $\varphi \in C(K)$, for all $K' \subseteq K$ s.t. $\varphi \in Cn(K')$, $\varphi \in C(K')$.
- **local inclusion**: If $\varphi \in C(K)$, for all $K' \subseteq K$ s.t. $\varphi \in K'$, $\varphi \in C(K')$.

The locality property indicates that the consequence operator C is local, in the sense that it is restricted to a relevant subset of the language or domain of discourse. This property abstractly encodes forms of restricted (or resource-bounded) reasoning that we observe, for example, in Hansson and Wassermann's (2002) local implications and Souza's (2020) topic-sensitive consequences.

As such, we can characterise the following connections.

Proposition 6. *Let $\mathcal{L}$ be a tarskian and compact logic, C be a $\mathcal{L}$-sound consequence operator, $K \subseteq L$ be a set of formulas, and $-$ be a C-sensitive partial meet contraction on K. It holds that:*

1. *$-$ satisfies (uniformity)*
2. *If C satisfies idempotence, $-$ satisfies (C-enforced closure).*
3. *If C satisfies monotonicity, $-$ satisfies (success) and (C-logical inclusion)*
4. *If $K = Cn(K)$ and C satisfies monotonicity and locality, then $-$ satisfies (C-sensitive relevance)*
5. *If $K = C(K)$ and C satisfies monotonicity, local inclusion, and locality, then $-$ satisfies (C-sensitive relevance)*

Proof. The proof can be easily obtained from the fact that if $-$ is a C-sensitive partial meet contraction, there is a partial meet contraction $\ominus$ on $\mathcal{L}$ s.t. $K - \varphi = C(K \ominus \varphi)$. As such, we will only show the proof for the *(C-enforced closure)* and *(C-sensitive relevance)* postulate.

(C-enforced closure): As $K - \varphi = C(\bigcap \gamma(K \perp \varphi))$, then $C(K - \varphi) = C(C(\bigcap \gamma(K \perp \varphi)))$. By idempotence of C, $C(C(\bigcap \gamma(K \perp \varphi))) = C(\bigcap \gamma(K \perp \varphi))) = K - \varphi$. Thus $K - \varphi = C(K - \varphi)$.

(C-sensitive relevance): First we show for the case $K = Cn(K)$. Take $\beta \in C(K) \backslash Cn(K - \varphi)$, then $\beta \in C(K)$ and $\beta \notin Cn(C(K \ominus \varphi))$. By the inclusion property of Cn, we conclude that $\beta \notin C(K \ominus \varphi)$. As C satisfies monotonicity and locality, since $\beta \in C(K)$ and $K - \varphi \subseteq C(K)$, $\beta \notin Cn(K \ominus \varphi)$ and thus $\beta \notin K \ominus \varphi$. By *(relevance)* of the partial meet operator $\ominus$, as $\beta \in C(K) \subseteq Cn(K) = K$ and $\beta \notin K \ominus \varphi$, there is some $K' \subseteq K$ s.t. $K \ominus \varphi \subseteq K'$ and $\varphi \notin Cn(K')$ but $\varphi \in Cn(K' \cup \{\beta\})$. As Cn is monotonic and compact, there is some maximal element $K'' \subseteq K$ s.t. $K' \subseteq K$ and $\varphi \notin Cn(K'')$. As $K = Cn(K)$ and Cn is tarskian, it is easy to see that $K'' = Cn(K'')$, then $K - \varphi = C(K \ominus \varphi) \subseteq Cn(K \ominus \varphi) \subseteq Cn(K'') = K''$, $\varphi \notin K''$ and $\varphi \in Cn(K'' \cup \{\beta\})$, by monotonicity of Cn.

To show that it holds for the case $K = C(K)$, it suffices to see that if $K' \in K \perp \varphi$, then $K' = C(K)$. Take $K' \in K \perp \varphi$. As $K' \subseteq K = C(K)$, $K' \subseteq C(K')$ is immediate since C satisfies local inclusion. Let us show that the reverse inclusion holds. Take $\beta \in C(K')$. If $\beta \notin K'$ then, by maximality of K', $\varphi \in Cn(K' \cup \{\beta\})$. As $\beta \in C(K') \subseteq Cn(K')$, $K' \cup \{\beta\} \subseteq Cn(K')$. As Cn is tarskian, $Cn(K' \cup \{\beta\}) \subseteq Cn(Cn(K')) = Cn(K')$. Thus, $\varphi \in Cn(K')$ which is absurd due to the fact that $K' \in K \perp \varphi$. Then $\beta \in K'$ and $C(K') \subseteq K'$. The remaining arguments for showing *(C-sensitive relevance)* are similar to the case in which $K = Cn(K)$. □

With that, we can provide the following representation result for C-sensitive contractions.

Theorem 7. *Let $\mathcal{L}$ be a tarskian and compact logic, and C be a $\mathcal{L}$-sound consequence operator satisfying monotonicity, idempotence, and locality (monotonicity, idempotence, local inclusion and locality). Let yet $K \subseteq L$ be a belief set, i.e. set of formulas s.t. $K = Cn(K)$ ($K = C(K)$). An operator $-$ is a C-sensitive partial meet contraction on K iff $-$ satisfies (success), (C-enforced closure), (logical uniformity), (C-logical inclusion) and (C-sensitive relevance).*

Proof. Satisfaction of postulates is consequence of Proposition 6. The proof of construction follows a similar idea to the construction of partial meet contraction by AGM. Take

$$\gamma(K \perp \varphi) = \begin{cases} \{K' \in K \perp \varphi \mid K - \varphi \subseteq K'\} & \text{if } \varphi \notin Cn(\varnothing) \\ \{K\} & \text{otherwise} \end{cases}$$

It is easy to see that $\gamma(K \perp \varphi)$ is well-defined, as in the previous cases. As such, let us show that $K - \varphi = C(\bigcap \gamma(K \perp \varphi)$.

($\subseteq$): Since C is monotonic and $K - \varphi \subseteq \bigcap \gamma(K \perp \varphi)$, we have that $C(K - \varphi) \subseteq C(\bigcap \gamma(K \perp \varphi)$. By *($C$-enforced closure)* of $-$, we conclude that $K - \varphi \subseteq C(\bigcap \gamma(K \perp \varphi))$.

($\supseteq$): Take $\beta \in C(\bigcap \gamma(K \perp \varphi)$, then $\beta \in C(K)$, since C is monotonic. Suppose $\beta \notin Cn(K - \varphi)$, then by *($C$-sensitive relevance)* there is some $K' \subseteq K$ s.t. $K - \varphi \subseteq K'$, $\varphi \notin Cn(K')$ but $\varphi \in Cn(K' \cup \{\beta\})$. As Cn is monotonic and compact, there is some maximal $K'' \subseteq K$ s.t. $K - \varphi \subseteq K''$ and $\varphi \in Cn(K'' \cup \{\beta\})$. As K'' is maximal, $K'' \in K \perp \varphi$ and, thus, $K'' \in \gamma(K \perp \varphi)$. As $\beta \notin K''$, since $\varphi \notin Cn(K'')$, we conclude that $\beta \notin \bigcap \gamma(K \perp \varphi)$ and $\beta \notin Cn(\bigcap \gamma(K \perp \varphi))$, since Cn is tarskian. As C is $\mathcal{L}$-sound, we conclude that $\beta \notin C(\bigcap \gamma(K \perp \varphi)$, which is absurd. Then $\beta \in Cn(K - \varphi)$. As $\beta \in C(C(K)) = C(K)$ and $K - \varphi \subseteq C(K)$, then, by locality of C, $\beta \in C(K - \varphi) = K - \varphi$. □

4.3 C-Base Contraction

On the other way of the construction of C-sensitive contraction, Hansson (1991) shows that belief base contractions $-$ can be defined by means of a belief set contraction $\ominus$ as $K - \varphi = K \cap Cn(K) \ominus \varphi$. We can generalise this notion to hyperintensional belief contraction in the following way.

Definition 8. *Let $\mathcal{L}$ be a monotonic and compact logic, C be a $\mathcal{L}$-sound consequence operator, and $K \subseteq L$ be a set of formulas. We say an operator $- : 2^L \times L \to 2^L$ is a*

C-base partial meet contraction (or C-base contraction) on K if there is some selection function γ s.t. for any formula $\varphi \in L$ it holds that

$$K - \varphi = C(K) \cap \bigcap \gamma(Cn(K) \perp \varphi)$$

As C-base contractions are defined based on a generalisation of belief base contractions, we can provide a characterisation of our operations based on Hansson's characterisation of belief base contractions by the following modified postulates.

(C-logical inclusion) $K - \varphi \subseteq C(K)$.

(success) If $\varphi \notin Cn(\emptyset)$, then $\varphi \notin Cn(K - \varphi)$

(Cn uniformity) If for all $K' \subseteq Cn(K)$, it holds that $\varphi \in Cn(K')$ iff $\psi \in Cn(K')$, then $K - \varphi = K - \psi$

(C-local logical relevance) If $\beta \in C(K) \backslash K - \varphi$, then there is some $K' \subseteq Cn(K)$, s.t. $K - \varphi \subseteq K'$, $\varphi \notin Cn(K')$ and $\varphi \in Cn(K' \cup \{\beta\})$.

With that, we can provide the following representation result for C-sensitive contractions.

Theorem 9. *Let $\mathcal{L}$ be a monotonic and compact logic, C be a $\mathcal{L}$-sound consequence operator, and $K \subseteq L$ be a set of formulas. An operator $-$ is a C-base partial meet contraction on K iff $-$ satisfies (C-logical inclusion), (success), (Cn uniformity), and (C-local logical relevance).*

Proof. (Construction to postulates): satisfaction of *(C-logical inclusion)*, *(success)*, and *(Cn uniformity)* is immediate from the fact that $\mathcal{L}$ is monotonic and compact and, thus, there is a partial meet contraction $\ominus$ in $\mathcal{L}$ s.t. $K - \varphi = C(K) \cap (Cn(K) \ominus \varphi)$.

To prove satisfaction of *(C-local logical relevance)*, we need to see that if $\beta \in C(K) \backslash K - \varphi$, as $K - \varphi = C(K) \cap (Cn(K) \ominus \varphi)$, then $\beta \notin Cn(K) \ominus \varphi$. As $C(K) \subseteq Cn(K)$ and $\beta \in C(K)$, we conclude that $\beta \in Cn(K) \backslash Cn(K) \ominus \varphi$. By *(relevance)* of the partial meet contraction $\ominus$, we conclude that there is some $K' \subseteq Cn(K)$ s.t. $Cn(K) \ominus \varphi \subseteq K'$ and $\varphi \notin Cn(K')$ but $\varphi \in Cn(K' \cup \{\beta\})$. As, by definition, $K - \varphi \subseteq Cn(K) \ominus \varphi$, then there is some $K' \subseteq Cn(K)$ s.t. $K - \varphi \subseteq K'$ and $\varphi \notin Cn(K')$ but $\varphi \in Cn(K' \cup \{\beta\})$.

(Postulates to construction): The proof is similar to that of Theorem 3. Let us define $\gamma(Cn(K) \perp \varphi) = \{K' \in Cn(K) \perp \varphi \mid K - \varphi \subseteq K'\}$ if $\varphi \notin Cn(\emptyset)$, and $\gamma(Cn(K) \perp \varphi) = Cn(K)$, otherwise.

The only important changes from the proof of Theorem 3 concern the argumentation for the inclusion $C(K) \cap \bigcap \gamma(Cn(K) \perp \varphi) \subseteq K - \varphi$.

Suppose, by contradiction, that there is some $\beta \in C(K) \cap \bigcap \gamma(Cn(K) \perp \varphi) \backslash K - \varphi$, then $\beta \in C(K) \backslash K - \varphi$. By *($C$-local logical relevance)*, there is some $K' \subseteq Cn(K)$ s.t. $K - \varphi \subseteq K'$, $\varphi \notin Cn(K')$ but $\varphi \in Cn(K' \cup \{\beta\})$. As $\mathcal{L}$ is monotonic and compact, by the upper bound property, there is $K'' \in Cn(K) \perp \varphi$ s.t. $K' \subseteq K''$. As $K - \varphi \subseteq K' \subseteq K''$, we conclude that $K'' \in \gamma(Cn(K) \perp \varphi)$, by construction. As such, $\beta \notin K''$ and, thus, $\beta \notin \bigcap \gamma(Cn(K) \perp \varphi)$, which is a contradiction to the hypothesis that $\beta \in C(K) \cap \bigcap \gamma(Cn(K) \perp \varphi) \backslash K - \varphi$. Thus, $C(K) \cap \bigcap \gamma(Cn(K) \perp \varphi) \subseteq K - \varphi$ □

Notice that C-base contractions are generalisations of C-dependent contractions, for $\mathcal{L}$-sound operators C, since the postulate *(C-logical relevance)* implies the postulate *(C-local logical relevance)*. Moreover, it is not always the case that C-base contractions are C-dependent contractions. Let us examine the following example.

Example 10. *Let $L = \{a, b, c, d, e\}$ and let $Cn : 2^L \to 2^L$ and $C : 2^L \to 2^L$ be consequence operators such that:*

$Cn(\{a, c, d, e\}) = \{a, c, d, e\}$,
$Cn(\{a, c, d\}) = Cn(\{a, d, e\}) = Cn(\{a, c, d, e\})$,
$Cn(\{a, c\}) = \{a, c\}$,
$Cn(\{a, d\}) = \{a\}$,
$C(\{a, c, d\}) = \{a, d, e\}$,
$C(\{a, c\}) = \{a, c\}$,
$C(\{a, d\}) = \{a\}$.

Take $K = \{a, c, d\}$, we conclude that $Cn(K) \perp e = \{\{a, c\}, \{a, d\}\}$, while $C(K) \perp e = \{\{a, d\}\}$. Thus for some selection function γ, i.e. if $\gamma(Cn(K) \perp e) \neq \{a, d\}$, $\bigcap \gamma(C(K) \perp e) \neq C(K) \cap \bigcap \gamma(Cn(K) \perp e)$.

In the same way that C-sensitive contractions generalise the notion of AGM contraction to hyperintensional operators, C-base contractions generalise the notion of base contraction for the hyperintensional case. In fact, it is easy to see that the strong connection between AGM contractions and base contraction is reproduced in our framework.

Proposition 11. *Let $\mathcal{L}$ be a monotonic and compact logic satisfying inclusion and let C be a $\mathcal{L}$-sound consequence operator. An operator $- : 2^L \times L \to 2^L$ satisfies (C-sensitive relevance) only if it satisfies (C-local logical relevance).*

With that, we see that if the foundational logic $\mathcal{L}$ satisfies the suitable properties, every C-sensitive contraction is a C-base contraction.

5 Belief Change in Non-classical Logics as Hyperintensional Belief Change

In this section, we investigate some applications of our general notions of $\mathcal{L}$-sound hyperintensional contractions to examine belief change operations in the literature. We will focus on using hyperintensional belief contractions to examine Delgrande and Wassermann's (2013) belief contractions for Horn Logic, as well as for Ribeiro and Wassermann's (2009) belief contractions for Description Logics.

5.1 Belief Change in Horn Logic

Horn Logic is an important restriction on the syntax of classical propositional (or first-order) logic with several applications in Artificial Intelligence and Computer Science, in areas such as logic programming, truth maintenance systems, and databases. Horn Logic is also a framework to study limited reasoning and its applications.

The topic of belief change in Horn logic has, to our knowledge, firstly been tackled by the work of Delgrande (2008), which investigates the application of the AGM approach to defining belief change operators in such a restricted logic. The interest in studying belief change in this logic comes from its potential applications, for example, in evolving databases, but also in revealing the theoretical underpinnings of AGM belief change for weak logics not containing full classical propositional reasoning (Delgrande 2008).

In the remainder of this subsection, we will suppose our foundational logic to be the classical propositional logic $\mathcal{L}_0 = \langle L_0, Cn \rangle$ defined over some fixed countable propositional symbol set P.

Definition 12. *We define the language of Horn Formulas, the set $L_H \subset L_0$ containing the Horn formulas as given by:*

- *$p \in P$ is a Horn Clause;*
- *$a_1 \wedge a_2 \wedge \cdots \wedge a_n \rightarrow a$ is a Horn Clause, with $a_i, a \in P$ for $1 \leqslant i \leqslant n$;*
- *Every Horn Clause is a Horn Formula;*
- *If φ, ψ are Horn Formulas, so is $\varphi \wedge \psi$.*

We define Horn Logic as the logic constructed with the language of Horn Formulas and a consequence operator C_h - Horn consequence - over this language.

Definition 13. *We define the Horn consequence operator $C_h : 2^{L_0} \rightarrow 2^{L_0}$ as*

$$C_h(X) = Cn(X) \cap L_h$$

It is easy to see from the semantics of Horn Logic, presented in (Delgrande 2008; Delgrande and Wassermann 2013), that the consequence operator C_h actually defines consequence in Horn Logic. We can now introduce the notion of Horn Contraction, introduced by Delgrande (2008).

Definition 14. *(Delgrande 2008) Let $\mathcal{L}_h = \langle L_h, C_h \rangle$ be the Horn Logic, and $K \subseteq L_h$ be a closed set of Horn Formulas, i.e. $K = C_h(K)$. We say an operation $- : 2^{L_h} \times L_h \rightarrow 2^{L_h}$ is a Belief Set Horn Contraction operation on K iff there is a selection function γ s.t. for any formula $\varphi \in L_h$ it holds that*

$$K - \varphi = \bigcap \gamma(K \perp_{\mathcal{L}_h} \varphi)$$

Let us redefine our notion of C-dependent contractions to Horn Logic.

Definition 15. *Let $\mathcal{L}_0 = \langle L_0, Cn \rangle$ be the Classical Propositional Logic, and $K \subseteq L_0$ be a set of formulas. We say an operation $- : 2^{L_0} \times L_0 \rightarrow 2^{L_0}$ is a Horn-Dependent Contraction operation on K if there is a selection function γ s.t. for any formula $\varphi \in L_0$ it holds that*

$$K - \varphi = \bigcap \gamma(C_h(K) \perp \varphi)$$

It is easy to see that every Horn Belief Set Contraction is a Horn-Dependent Contraction.

Lemma 16. *Let $\mathcal{L}_h = \langle L_h, C_h \rangle$ be the Horn Logic and $K \subseteq L_h$ a closed set of Horn Formulas, i.e. $K = C_h(K)$. An operation $- : 2^{L_h} \times L_h \rightarrow 2^{L_h}$ is a Belief Set Horn Contraction operation on K iff there is a Horn-Dependent Contraction operation on K , $\ominus : 2^{L_0} \times L_0 \rightarrow 2^{L_0}$, s.t. for any $\varphi \in L_h$ it holds that*

$$K - \varphi = K \ominus \varphi$$

Proof. It suffices to see that, as $C_h(X) = Cn(X) \cap L_h$, if $K = C_h(K)$, then $K \perp_{\mathcal{L}_h} \varphi = C_h(K) \perp_{\mathcal{L}_0} \varphi = K \perp_{\mathcal{L}_0} \varphi$. Notice that, as C_h satisfies monotonicity, local inclusion and locality, if $K' \in C_h(K) \perp_{\mathcal{L}_0} \varphi$, then $K' = C_h(K')$. As C_h is compact, it is easy to see that $K' \in K \perp_{\mathcal{L}_h} \varphi$. The reverse inclusion is immediate from the fact that if $K' \in K \perp_{\mathcal{L}_h} \varphi$, then $K' \subseteq C_h(K) = K$ and it is the maximal element containing K' and not proving φ, given that C_n is compact. ☐

More yet, as C_h clearly satisfies monotonicity, idempotence, local inclusion and locality, it is easy to see that Horn Belief Set Contractions are also examples of C_h-sensitive contractions.

Lemma 17. *Let $\mathcal{L}_h = \langle L_h, C_h \rangle$ be the Horn Logic and $K \subseteq L_h$ be a closed set of Horn Formulas, i.e. $K = C_h(K)$. An operation $- : 2^{L_h} \times L_h \rightarrow 2^{L_h}$ is a Belief Set Horn Contraction operation on K iff there is a C_h-sensitive contraction operation on K, $\ominus : 2^{L_0} \times L_0 \rightarrow 2^{L_0}$, s.t. for any $\varphi \in L_h$, it holds that*

$$K - \varphi = K \ominus \varphi$$

Proof. As we have shown in Lemma 16 that $K \perp_{\mathcal{L}_h} \varphi = K \perp_{\mathcal{L}_0} \varphi$, it suffices to see that, by definition, $K - \varphi = \bigcap \gamma(K \perp_{\mathcal{L}_h} \varphi) = \bigcap \gamma(K \perp_{\mathcal{L}_0} \varphi)$. As $-$ satisfies *(enforced closure)*, $K - \varphi = C_h(K - \varphi) = C_h(\bigcap \gamma(K \perp_{\mathcal{L}_0} \varphi)) = K \ominus \varphi$. ☐

Lemma 17 shows that while our general notions of hyperintensional belief contraction may differ, in well-behaved logics such as Horn Logic, these notions may coincide. This fact indicates that our notions are more general and may apply to a wide range of logics while maintaining appropriate rationality criteria for belief change.

The connection of our hyperintensional contractions to belief change in Horn Logic is not limited to Horn Belief Set Contractions but can be extended to Delgrande and Wassermann's (2013) belief base contractions for Horn Logic as well.

Definition 18. *(Delgrande and Wassermann 2013) Let $\mathcal{L}_h = \langle L_h, C_h \rangle$ be the Horn Logic. We say an operation $- : 2^{L_h} \times L_h \rightarrow 2^{L_h}$ is a Horn Contraction operation on a set of Horn Formulas $K \subseteq L_h$ iff there is a selection function γ s.t. for any Hor formula $\varphi \in L_h$, it holds that:*

$$K - \varphi = \bigcap \gamma(K \perp_{\mathcal{L}_h} \varphi)$$

To define our hyperintensional base contractions, which we will use to interpret Delgrande and Wassermann's (2013) Horn Contractions, we will employ the following $\mathcal{L}_0$-sound consequence operator.

Definition 19. *We define the Horn restriction operator $C_h : 2^{L_0} \rightarrow 2^{L_0}$ as*

$$C_{\downarrow h}(X) = X \cap L_h$$

It is not difficult to see that the postulates above imply the postulates of C-base contractions, considering the Horn restriction operator C_h. To show that this connection holds, let us redefine our C-base contractions to the particular case of Horn Logic.

Definition 20. *Let $\mathcal{L}_0$ be the classical propositional logic. We say an operation $- : 2^{L_0} \times L_0 \to 2^{L_0}$ is a Horn Base Contraction operation on a set $K \subseteq L_0$ iff there is a selection function γ s.t. for any formula $\varphi \in L_0$ it holds that:*

$$K - \varphi = C_{\downarrow h}(K) \cap \bigcap \gamma(Cn(K)\bot_{\mathcal{L}_0}\varphi)$$

Finally, it is easy to see that Delgrande and Wassermann's Horn Contractions coincide with our Horn Base Contractions.

Lemma 21. *Let $\mathcal{L}_h = \langle L_h, C_h \rangle$ be the Horn Logic and $K \subseteq L_h$ be a set of Horn Formulas. An operation $- : 2^{L_h} \times L_h \to 2^{L_h}$ is a Horn Contraction operation on K iff there is a Horn Base C_h-sensitive contraction operation on K, $\ominus : 2^{L_0} \times L_0 \to 2^{L_0}$, s.t. for any $\varphi \in L_h$, it holds that*

$$K - \varphi = K \ominus \varphi$$

Proof. It suffices to see that, as $K \subseteq L_h$ and $\varphi \in L_h$, $K' \in K\bot_{\mathcal{L}_h}\varphi$ iff there is some $K'' \in Cn(K)\bot_{\mathcal{L}_0}\varphi$ s.t. $K' = K \cap K''$.

Take $K' \in K\bot_{\mathcal{L}_h}\varphi$, then K' is a maximal subset s.t. $K' \subseteq K$ and $\varphi \notin C_h(K')$. Well, $C_h(K') = Cn(K') \cap L_h = C_{\downarrow h}(Cn(K'))$, thus $\varphi \notin Cn(K')$. As Cn is compact, there is a $K'' \in Cn(K)\bot_{\mathcal{L}_h}\varphi$ s.t. $K' \subseteq K''$. Clearly, $K' \subseteq K \cap K''$. To show the reverse inclusion, it suffices to see that if there is some $\beta \in (K \cap K'')\backslash K'$, then there is a set $K''' = K' \cup \{\beta\}$ s.t. $K' \subset K''' \subseteq K$ and $\varphi \notin C_h(K''') = Cn(K''') \cap L_h \subseteq Cn(K'')$, contradicting the maximality of K'. ☐

5.2 Belief Contraction in Description Logics

Description Logics (DLs) are a wide family of logic-based languages used for knowledge representation and reasoning, differing in their expressive power and the computational properties of their reasoning tasks (Baader et al. 2003). These languages have been used as the foundation of knowledge representation languages such as the Ontology Web Language (Motik, Hayes, and Horricks 2004), the standard language for the descriptions of computational ontologies for the Semantic Web, among other applications, e.g. (Moreira et al. 2005; Souza et al. 2015). While there is a significant number of DLs proposed in the literature, in this work, we will focus on the basic logic $\mathcal{ALC}$ and its sublanguages (Baader et al. 2003).

A terminological signature (or simply signature) of a description language is a tuple $\langle N_C, N_R, N_I \rangle$ such that N_C is a non-empty set of atomic concept names, N_R is a non-empty set of atomic role names, and N_I is a non-empty set of individuals. Given a signature, we construct the language of $\mathcal{ALC}$ by the following grammar, where $A \in N_C$ and $R \in N_R$:

$$C ::= A \mid \top \mid \bot \mid (\neg C) \mid (C \sqcap C) \mid (C \sqcup C) \mid (\forall R.C) \mid (\exists R.C)$$

The formal semantics of concepts may be given by interpretations $\mathcal{I}$, consisting of a non-empty set (Δ) and an interpretation function (I). In such an interpretation, a primitive atomic concept A is assigned to a set $A^{\mathcal{I}} \subseteq \Delta$ and each primitive atomic role R to a binary relation $R^{\mathcal{I}} \subseteq \Delta \times \Delta$ and each individual name $a \in N_I$ is assigned to an element of the domain of discourse $a^{\mathcal{I}} \in \Delta$. The semantics of the formulas of $\mathcal{ALC}$ is, then, defined on Table 1.

C	$C^{\mathcal{I}}$
$\top$	Δ
$\bot$	$\varnothing$
$\neg C$	$\Delta \backslash C^{\mathcal{I}}$
$(C \sqcap D)$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
$(C \sqcup D)$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
$\forall R.C$	$\{x \in \Delta \mid \forall y \in \Delta : \langle x, y \rangle \in R^{\mathcal{I}} \to y \in C^{\mathcal{I}}\}$
$\exists R.C$	$\{x \in \Delta \mid \exists y \in \Delta : \langle x, y \rangle \in R^{\mathcal{I}}$ and $y \in C^{\mathcal{I}}\}$

Table 1: Interpretation of $\mathcal{ALC}$ concepts

The $\mathcal{ALC}$ terminological language ($\mathcal{ALC}_{\mathcal{T}}$) is defined as the set of all formulas

$$\varphi ::= C \sqsubseteq D \mid C(a) \mid R(a, b) \mid a = b$$

where, C, D are formulas of $\mathcal{ALC}$, $R \in N_R$ is a role name and $a, b \in N_I$ are individuals.

We say an interpretation $\mathcal{I}$ satisfy a formula φ, denoted $\mathcal{I} \models \varphi$, if:

$$\begin{array}{lll} \mathcal{I} \models C \sqsubseteq D & \text{if} & C^{\mathcal{I}} \subseteq D^{\mathcal{I}} \\ \mathcal{I} \models C(a) & \text{if} & a^{\mathcal{I}} \in C^{\mathcal{I}} \\ \mathcal{I} \models R(a, b) & \text{if} & \langle a, b \rangle \in R^{\mathcal{I}} \\ \mathcal{I} \models a = b & \text{if} & a^{\mathcal{I}} = b^{\mathcal{I}} \end{array}$$

We say that a terminological formula $\varphi \in \mathcal{ALC}_{\mathcal{T}}$ is satisfiable if there is some interpretation that satisfies it, and that it is valid, if for any interpretation it is satisfied. In this context, a knowledge base is a set $K \subseteq \mathcal{ALC}_{\mathcal{T}}$ of terminological formulas, and we say that an interpretation $\mathcal{I}$ satisfies K, denoted $\mathcal{I} \models K$, if for any $\varphi \in K$ it holds that $\mathcal{I} \models \varphi$. We say a knowledge base K implies formula $\varphi \in \mathcal{ALC}_{\mathcal{T}}$, denoted $K \models \varphi$, if any interpretation that satisfies K also satisfies φ. Thus, we define $Cn(K) = \{\varphi \in \mathcal{ALC}_{\mathcal{T}} \mid K \models \varphi\}$.

As Cn is a monotonic and compact operator, based on Hansson and Wassermann's (2002) characterisation of partial meet contraction, we know that there we can define partial meet contraction operators for $\mathcal{ALC}_{\mathcal{T}}$. We will employ these operators to construct belief contraction operators for sublanguages of $\mathcal{ALC}_{\mathcal{T}}$ based on our hyperintensional contractions. With that, we can show that belief change in logics of restricted reasoning can arise from their relationship with more expressive logics, for which we can construct belief change operators.

In Knowledge Representation, different DLs are defined as syntactic restrictions on the kind of formulas that can appear in a knowledge base. For example, the well-known DL $\mathcal{EL}$ (Baader 2003) is defined by limiting the concepts C, D occurring in a terminological formula $C \sqsubseteq D$. For any such logic $\mathcal{L} = \langle L, C \rangle$ with $L \subseteq \mathcal{ALC}_{\mathcal{T}}$ sufficiently strong, i.e. any logic monotonic and compact such that for $K \subseteq L$, it holds that $C(K) = Cn(K) \cap L$, we can easily see, as for Horn Logic, that any partial meet contraction $- : 2^L \times L \to 2^L$ can be defined as a hyperintensional base contraction on $\mathcal{ALC}_{\mathcal{T}}$. Let us show this formally.

Definition 22. *We define the L restriction operator $C_L : 2^{\mathcal{ALC}_\mathcal{T}} \rightarrow 2^{\mathcal{ALC}_\mathcal{T}}$ as*

$$C_L(X) = X \cap L$$

With that, we can redefine our C_L-base contractions to the particular case of subsets of $\mathcal{ALC}$.

Definition 23. *Let $\mathcal{L} = \langle L, C \rangle$ a monotonic and compact logic, s.t. $L \subseteq \mathcal{ALC}_\mathcal{T}$ and $C(X) = Cn(X) \cap L$. We say an operation $- : 2^{\mathcal{ALC}_\mathcal{T}} \times \mathcal{ALC}_\mathcal{T} \rightarrow 2^{\mathcal{ALC}_\mathcal{T}}$ is an $\mathcal{ALC}$-Base Contraction operation on a set $K \subseteq \mathcal{ALC}_\mathcal{T}$ iff there is a selection function γ s.t. for any $\varphi \in \mathcal{ALC}_\mathcal{T}$, it holds that:*

$$K - \varphi = C_L(K) \cap \bigcap \gamma(Cn(K) \bot_{\mathcal{ALC}_\mathcal{T}} \varphi)$$

Clearly, as $\mathcal{ALC}_\mathcal{T}$ satisfies inclusion, $\mathcal{ALC}$-Base Contractions are C_L-Base Contractions. Finally, it is easy to see that partial meet contractions on $\mathcal{L}$ coincide with $\mathcal{ALC}$ Base Contractions.

Lemma 24. *Let $\mathcal{L} = \langle L, C \rangle$ a monotonic and compact logic, s.t. $L \subseteq \mathcal{ALC}_\mathcal{T}$ and $C(X) = Cn(X) \cap L$, $K \subseteq L$ be a set of L-formulas. An operation $- : 2^L \times L \rightarrow 2^L$ is a partial meet contraction operation on K iff there is a $\mathcal{ALC}$-Base Contraction $\ominus : 2^{\mathcal{ALC}_\mathcal{T}} \times \mathcal{ALC}_\mathcal{T} \rightarrow 2^{\mathcal{ALC}_\mathcal{T}}$ s.t. for any $\varphi \in L$, it holds that*

$$K - \varphi = K \ominus \varphi$$

The proof of the above result is similar to that of Horn Logic. As such, we can see that for any sufficiently strong monotonic and compact sublogics of $\mathcal{ALC}$, we can construct belief contraction operators based on $\mathcal{ALC}$. Notice that our assumption that the sublogic $\mathcal{L}$ is *sufficiently strong*, i.e. $C(X) = Cn(X) \cap L$, is not an unreasonable one, since this property holds for several sublanguages of $\mathcal{ALC}$, such as the logics in the $\mathcal{EL}$ family (Baader 2003), and in the DL-Lite family (Calvanese et al. 2005), etc.

More importantly, based on our hyperintensional belief contractions, we can propose belief change operations for non-monotonic Description Logics (Governatori and Rotolo 2004), which is not possible in the framework of partial-meet contraction.

With the connection of hyperintensional belief contraction, horn contractions and belief contractions in description logics, we point out our framework's potential to study belief change in non-classical logics.

6 Final Considerations

In this work, we investigated the application of belief change operations arising from hyperintensional treatments of beliefs to define belief contraction operations on non-classical logics. Firstly, we defined and characterised three different notions of hyperintensional belief contraction arising from generalisations of operations in the literature. In doing so, we generalise Santos et al.'s (2018) previous results on the characterisation of Cn*-pseudo contractions (here called C-dependent contractions) and their applications of hyperintensional belief change to construct belief change operations for non-classical logics.

Notice that, by encoding hyperintensional reasoning by means of the relation among two different consequence operators C and Cn, our approach is general enough to be connected to different foundational theories of hyperintensionality - such as the structural perspective underlying the structured propositions tradition (Cresswell 1975) and the informational perspective underlying Berto's mereological treatment (Berto 2019). As such, even if sentences such as p :"*3 is a prime number*" and q :"*3068 is divisible by 13*" are necessarily equivalent, i.e $Cn(p) = Cn(q)$, we can abstractly represent the hyperintensional semantic difference between the formulas by imposing that $C(p) \neq C(q)$.

The connection between Horn contractions and hyperintensional contractions indicates that hyperintensional belief change can be used to unify different work on non-classical belief change, particularly for sublogics of classical logic (Gabbay, Rodrigues, and Russo 2008). That is the case, for example, of belief change in Description Logics (Flouris, Plexousakis, and Antoniou 2005; Ribeiro and Wassermann 2009), as explored in this work. Our results explain how we construct and characterise variations of belief base change to these subclassical logics and point to a general framework for studying belief change in non-classical logics.

This connection between contractions in subclassic logics and hyperintensional logics, while theoretically valuable, is not surprising in the sense that subclassic logics necessarily impose hyperintensional differentiations in the underlying logic. Our work highlights that, as a subclassical logic can be understood as a hyperintensional interpretation within the foundational logic, belief change in the former is intrinssicaly defined by belief change in the latter. This fact indicates that the approach proposed by Gabbay (2008) for defining belief change operators for non-classical logics is semantically well-behaved, in the sense that it preserves desirable properties of the resulting belief change operator for a great variety of underlying logics.

While we focus on an abstract interpretation of hyperintensionality, we believe that the semantic treatment of such notions can provide insightful intuitions for the construction of different belief change operations. Particularly, since competing approaches in the literature represent hyperintensional contexts in different ways, we believe that general frameworks such as impossible-world semantics (Rantala 1982) or Sedlar's (2019) Montague-Scott semantics can be used to explain diverse proposals of belief change operations in the literature and their logical characterisations. More yet, we believe semantic-based operations can be more easily connected to other recent dynamic logic-based theories of belief and mental change (Girard and Rott 2014; Souza, Vieira, and Moreira 2020) and to the work on iterated belief change (Jin and Thielscher 2007; Fermé and Wassermann 2018).

In future work, we aim to investigate the use of these semantic frameworks to characterise belief change in non-classical logics in a unified manner. This will help, in our opinion, to provide belief change operators for a great range of logics, such as non-monotonic and non-compact logics, which are of interest to both Artificial Intelligence and Philosophy.

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