On the Progression of Belief

Daxin Liu *, Qihui Feng
RWTH Aachen University
liu@kbsg.rwth-aachen.de, qihui.feng@rwth-aachen.de

Abstract
Based on weighted possible-world semantics, Belle and Lakemeyer recently proposed the logic DS, a probabilistic extension of a modal variant of the situation calculus with a model of belief. The logic has many desirable properties like full introspection and it is able to precisely capture the beliefs of a probabilistic knowledge base in terms of the notion of only-believing. While the proposal is intuitively appealing, it is unclear how to do planning with such logic. The reason behind this is that the logic lacks projection reasoning mechanisms. Projection reasoning, in general, is to decide what holds after actions. Two main solutions to projection exist: regression and progression. Roughly, regression reduces a query about the future to a query about the initial state while progression, on the other hand, changes the initial state according to the effects of actions and then checks whether the formula holds in the updated state. In this paper, we study projection by progression in the logic DS. It is known that the progression of a categorical knowledge base wrt a noise-free action corresponds to what is only known after that action. We show how to progress a type of probabilistic knowledge base wrt noisy actions by the notion of only-believing after actions. Our notion of only-believing is closely related to Lin and Reiter’s notion of progression.

1 Introduction
Rich representation of knowledge and actions has been a goal that many AI researchers pursue. Among all proposals, perhaps, the situation calculus by McCarthy (1963) is the most widely-studied, where actions are treated as logical terms and agent’s knowledge is represented by logical formulas. The language has been extended to incorporate many features like time, concurrency, procedures, etc. Later, combining it with probability, Bacchus, Halpern, and Levesque (1999) (BHL) provided a rich account of dealing with degrees of belief and noisy sensing. The main advantage of a logical account like BHL is that it allows partial or incomplete specifications of beliefs depending on what information is actually available in a particular domain.

Alternatively, Belle and Lakemeyer (BL) (2017) proposed a formulation of BHL’s ideas based on a modal variant of the situation calculus (Lakemeyer and Levesque 2004), extending earlier work on static probabilistic beliefs (Belle, Lakemeyer, and Levesque 2016). Unlike the axiomatic BHL, BL’s logic DS is based on possible-world semantics with distributions over possible-worlds. More concretely, a distribution is just an assignment of non-negative weights to the possible worlds. An epistemic state is then defined as a set of such distributions and a sentence \( \phi \) is believed with degree \( r \) if and only if the normalized sum of the weight of worlds that satisfies \( \phi \) equals \( r \) in all distributions of the epistemic state. Later, beliefs after a sequence of actions are defined by the notions of action likelihood and observational-indistinguishability which captures the idea that the agent might not be able to distinguish between certain actions.

The logic has many interesting properties such as full introspection of beliefs. Besides, it is possible to express all the agent’s beliefs of a probabilistic knowledge base (KB) by appealing to a notion of only-believing. Nevertheless, the problem of how to plan with such logic is still open. The reason behind this is the lack of projection reasoning mechanisms. Projection reasoning, in general, is to decide what holds after actions. There are two main solutions to the projection problem: regression and progression. Roughly, regression reduces a query about the future to a query about the initial state while progression, on the other hand, changes the initial state according to the effects of actions and then checks whether the formula holds in the updated state. Compared with regression, progression is more challenging as Lin and Reiter (1997) proved that progression in general requires second-order logic.

Progression has been developed since then, mainly by appealing to the notion of forgetting. Later, Lakemeyer and Levesque (2009) showed that the progression of a categorical knowledge base specified by only-knowing wrt to a noise-free action amounts to what is only known by the agent after that action. In the setting of quantitative beliefs and noisy actions, the progression would correspond to what is only believed after actions. However, the current semantics of the only-believing \( O \) in DS is problematic to reflect this correctly. To see a concrete example, consider a robot moving toward a wall as in Fig. 1. Suppose a fluent \( h \) indicates the robot’s distance to the wall and the robot is equipped with an accurate sonar (specified by the action model \( \Sigma \)). In Lakemeyer and Levesque’s work, the following holds:

\[
| = O((h = 1 \lor h = 2) \land \Sigma) \supset [\text{sonar}(2)]O(h = 2 \land \Sigma)
\]

In English, only-knowing the distance is 1 or 2 and the act-
The rest of the paper is organized as follows. In section 2, we introduce the syntax and semantics of logic $\mathcal{DS}_p$. The semantics of progression is presented in section 3, where we address our solution of progression wrt noisy sensing and stochastic actions. In section 4 and 5, we discuss related work and conclude the paper, respectively.

\section{The Logic $\mathcal{DS}_p$}

$\mathcal{DS}_p$ is a modal language with equality and sorts of type \textit{object} and \textit{action}. Implicitly, we assume that \textit{number} is a sub-sort of object and refers to the computable numbers $\mathbb{C}$.\footnote{We use the computable numbers as they are still enumerable and allow representing distributions mentioning real numbers such as Euler’s number $e$ (Turing 1937).}

Before presenting the formal definitions, here are the main features:

- \textbf{standard names}: The language includes (countably many) standard names $\mathcal{N}$ for both objects $\mathcal{N}_O$ and actions $\mathcal{N}_A$ ($\mathcal{N} = \mathcal{N}_O \cup \mathcal{N}_A$). This can be viewed as a fixed infinite domain closure with the unique name assumption, which further allows first-order (FO) quantification to be understood substitutionally. Moreover, equality can also be treated in a simpler way: every ground term will have a coreferring standard name, and two terms are equal if their coreferring standard names are identical.
- \textbf{rigid and fluent functions}: The language contains both fluent and rigid function symbols. For simplicity, all action functions are rigid and we do not include predicate symbols in the language. Fluents vary as the result of actions, yet meaning of rigid functions is fixed.
- \textbf{belief and truth}: The language includes modal operators $\mathcal{B}$ and $\mathcal{O}$ for degrees of belief and only-believing respectively. Such operators allow us to distinguish between sentences that are true and sentences that are believed to be true with positive degrees.
- \textbf{observational-indistinguishability}: Finally, unlike deterministic domains, the effects of action could be non-deterministic. This is characterized by stochastic actions. Instead of saying an action might have non-deterministic effects, we say the action is stochastic and has non-deterministic alternatives which are not observationally distinguishable to the agent (as indicated by a special function $\mathcal{O}$).

\subsection{The Language}

\textbf{Definition 1.} The symbols of $\mathcal{DS}_p$ are taken from the following vocabulary:

- \textbf{first-order variables}: $u, v, x, y \ldots a, a' \ldots$;
- \textbf{second-order (SO) function variables}: $F, F' \ldots$;
- \textbf{rigid function symbols of every arity}, such as $\text{sonar}(x)$, including arithmetical functions like $+, \times, \text{etc.}$;
- \textbf{fluent function symbols of every arity, such as distanceTo(x), heightOf(y)}, including an unary special symbols $l$ and a special binary symbols $oi$. Roughly, $l$ returns the likelihood of an action and $oi$ describes the observational-indistinguishability (alternative choices) among actions $\mathcal{O}$\footnote{We don’t include the usual $\text{poss}$ function (action precondition), as in stochastic setting, that an action is impossible can always be specified by saying that action has 0 likelihood.};
- \textbf{connectives and other symbols}: $=, \wedge, \neg, \forall, \mathcal{B}, \mathcal{O}, [\cdot], \square$, round and square parentheses, period, colon, comma. $\mathcal{B}$ and $\mathcal{O}$ are called epistemic operators.

\textbf{Definition 2.} The \textbf{terms} of the language are the least set of expressions such that:

- \textbf{every standard name and FO variable is a term};
- if $t_1, \ldots t_k$ are terms, $f$ a $k$-ary function symbol, then $f(t_1, \ldots t_k)$ is a term;
- if $t_1, \ldots t_k$ are terms, $F$ a $k$-ary SO variable, then $F(t_1, \ldots t_k)$ is a term.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{robot-wall.png}
\caption{robot moving toward a wall}
\end{figure}
A term is said to be rigid, if and only if it does not contain fluents. Ground terms are terms without variables while SO ground terms are terms without FO variables. Primitive terms are terms of the form \( f(n_1, \ldots, n_k) \), where \( f \) is a function symbol and \( n_i \) are object standard names. SO primitive terms are defined likewise by replacing \( f \) with \( F \), a second-order variable. We denote the sets of primitive terms of sort object and action as \( \mathcal{P}_O \) and \( \mathcal{P}_A \), respectively, and the set of all SO primitive terms as \( \mathcal{P}_{SO} \). While object standard names are syntactically like constants, we require that action standard names are all the primitive action terms, i.e. \( \mathcal{N}_A = \mathcal{P}_A \). For example, the action \( \text{sonar}(5) \), where a sonar returns the number 5, is considered a standard action name. Furthermore \( \mathcal{Z} \) refers to the set of all finite sequences of action standard names, including the empty sequence \( \emptyset \). We reserve standard names \( T \), \( \bot \) in \( \mathcal{N}_O \) for truth values (to simulate predicates).

**Definition 3.** The well-formed formulas of the language are the least set of expressions such that:

- If \( t_1, t_2 \) are terms, then \( t_1 = t_2 \) is a formula;
- If \( \alpha \) is a term of sort action and \( \alpha \) a formula, then \( [t_\alpha] \alpha \) is a formula;
- If \( \alpha \) and \( \beta \) are formulas, \( v \) a FO variable, \( F \) a SO variable, \( r, r_1, r_2 \) rigid terms, then \( \alpha \land \beta \), \( \alpha \lor \beta \), \( \alpha \land \beta \), \( \alpha \lor \beta \) are also formulas.

\[ t_\alpha \alpha \text{ should be read as } \alpha \text{ holds after action } t_\alpha \text{ and } \alpha \alpha \text{ as } \alpha \text{ holds after any sequence of actions.} \]

The epistemic expression \( B(\alpha : r) \) should be read as \( \alpha \) is believed with a degree \( r \). \( K \alpha \text{ means } \alpha \text{ is known and is an abbreviation for } B(\alpha : 1) \). \( O(\alpha_1, \ldots, \alpha_k : r_k) \) may be read as “the \( \alpha_i \) with a probability \( r_i \) are all that is believed”. Similarly, \( O \alpha \text{ means } \alpha \text{ is only known} \text{ and is an abbreviation for } O(\alpha : 1) \). For action sequence \( z = a_1 \cdots a_k \), we write \( [z] \alpha \text{ to mean } [a_1] \cdots [a_k] \alpha \), \( \alpha \alpha \) is the formula obtained by substituting all free occurrences of \( x \) in \( \alpha \) by \( t \). As usual, we treat \( \alpha \lor \beta \), \( \alpha \lor \beta \), \( \alpha \beta \), \( \alpha \beta \) and \( \exists v. \alpha \) as abbreviations.

A sentence is a formula without free variables. We use **true** as an abbreviation for \( \forall x(x = x) \), and **false** for its negation. A formula with no \( \Box \) is called **bounded**; a formula with no \( \Box \) or \( [t_\alpha] \alpha \) is called **static**; a formula with no \( B \) or \( O \) is called **object**; a formula with no fluent, \( \Box \) or \( [t_\alpha] \alpha \) outside \( B \) or \( O \) is called **subjective**; a formula with no \( B \), \( O \), \( \Box \), \( [t_\alpha] \alpha \), \( I \) is called a fluent formula; a fluent formula without fluent functions is called a rigid formula.

### 2.2 The Semantics

The semantics is given in terms of **possible worlds**, which define what is true initially and after any sequence of actions. Compared to non-probabilistic accounts with deterministic actions (Lakemeyer and Levesque 2004), a number of challenges need to be addressed, including how to specify probabilities over uncountably many possible worlds, how to deal with multiple probability distributions entertained by the agent, and how to deal with probabilistic action effects, which may be unobservable by the agent.

A world \( w \) is a mapping from the primitive terms (\( \mathcal{P}_O \cup \mathcal{P}_A \)) and \( \mathcal{Z} \) to \( \mathcal{N} \) of the right sort, satisfying:

1. **Rigidity:** If \( t \) is a rigid primitive term, then for all \((w, z), (w', z') \), \( w[t, z] = w'[t, z'] \);
2. **Arithmetical Correctness:** If \( f \) is an arithmetical expression and \( val \) is its value in the usual sense, then for all \((w, z), w[f, z] = val \). For example, \( w[1 + 1, z] = 2 \).

Let \( \mathcal{W} \) be the set of all such worlds. FO free variables are handled substitutionally by using standard names. To interpret free SO variables, we need variable maps. A variable map \( \lambda \) is a mapping from \( \mathcal{P}_{SO} \) to \( \mathcal{N}_O \). We write \( \lambda \sim F \lambda' \) to mean \( \lambda \) and \( \lambda' \) agree except perhaps on SO primitives involving \( F \). We now define the co-referring standard names for SO ground terms (essentially, the denotation of terms). Given a SO ground term \( t \), a world \( w \), and action sequence \( z \), a variable map \( \lambda \), we define \([t]_{\lambda w}^z \) (read: the co-referring standard name for \( t \) given \( w, z, \lambda \)) recursively by:

1. If \( t \in \mathcal{N}_A \), then \([t]_{\lambda w}^z = t \);
2. \( f(t_1, \ldots, t_k)[\lambda w] = w[f(t_1)_{\lambda w}, \ldots, t_k]_{\lambda w}, z \);
3. \( F(t_1, \ldots, t_k)[\lambda w] = \lambda(F(t_1)_{\lambda w}, \ldots, t_k)_{\lambda w}, z \).

For a rigid SO ground term \( t \), we use \([t]_{\lambda} \) instead of \([t]_{\lambda w}^z \) for its denotation, moreover, if \( t \) is first-order, we write \([t]_{\lambda} \). We will require that \( l(a) \) is of sort number, and \( o(a, a') \) only takes values \( \top \) or \( \bot \), and \( o(a, a') \) behaves like an equivalence relation (reflexive, symmetric, and transitive).

\( o(a, a') \) means \( a \) and \( a' \) are observationally indistinguishable actions. In the example of Fig 1, the robot might perform a stochastic action \( f\text{wd}(x, y) \), where \( x \) is its intended forward distance and \( y \) is the actual outcome selected by nature. \( x \) is observable to the robot while \( y \) is not. Then, \( o\text{fwd}(1, 1, 1), f\text{wd}(1, 0, 9) \) says that nature can non-deterministically select 1.1 or 0.9 as a result for the intended value 1.

By a distribution \( d \) we mean a mapping from \( \mathcal{W} \) to \( \mathbb{R}_{\geq 0} \) (non-negative real) and an **epistemic state** \( e \) is any set of distributions. By a model, we mean a 4-tuple \((e, w, z, \lambda) \). In order to prepare for the semantics, we need to extend \( l(a), o(a, a') \) from actions to action sequences:

**Definition 4.** We define

1. \( l(\bullet) : \mathcal{W} \times \mathcal{Z} \rightarrow \mathbb{R}_{\geq 0} \) as
   \( l(\bullet)(\emptyset) = 1 \) for every \( w \in \mathcal{W} \);
   \( l(\bullet)(w, z \cdot a) = l(\bullet)(w, z) \times n \) where \( w[l(a)], z) = n \);
2. \( z \sim_w z' \) as
   \( z \sim_w z' \) if \( z' = \emptyset \);
   \( z \cdot a \sim_w z' \) if \( z' = z \ast a \ast z \sim_w z \ast w[o(a, a'), z) = \top \);

Obviously, \( \sim_w \) is an equivalence relation. We require that \( w[l(a)], z) \leq 1 \) and \( \sum_{a, a':o(a, a')} w[l(a)], z) = 1 \), for all \( w \in \mathcal{W}, z \in \mathcal{Z} \), and \( a \in \mathcal{N}_A \). Since \( \mathcal{W} \) is uncountable, to obtain a well-defined sum over uncountably many worlds, the following three conditions are used for evaluating beliefs:

**Definition 5.** We define **BND**, **EQN**. Normalize for any distribution \( d \) and any set \( \mathcal{V} \)

1. \( \text{BND}(d, V, n) \) iff \( \exists k, (w_1, z_1), \ldots, (w_k, z_k) \in V \text{ such that } \sum_{i=1}^k d(w_i) \times l^\bullet(w_i, z_i) > n \);
2. \( \text{EQN}(d, V, n) \) iff \( \text{BND}(d, V, n) \) and there is no \( n' < n \) such that \( \text{BND}(d, V, n') \) holds.
3. for any $U \subseteq V$, $\text{Norm}(d, U, V, n)$ iff $\exists b \neq 0$ such that $\text{Eq}(d, U, b \times n)$ and $\text{Eq}(d, V, b)$.

Intuitively, given $\text{Norm}(d, U, V, n)$, $n$ can be viewed as the normalized sum of the weights of worlds in $U$ wrt $d$ in relation to $V$. Here $\text{Eq}(d, V, b)$ expresses that the weight of the worlds wrt $d$ in $V$ is $b$, and finally $\text{Bnd}(d, V, b)$ ensures the weights of worlds in $V$ is bounded by $b$. In essence, even if $W$ is uncountable, the condition $\text{Eq}$ and $\text{Bnd}$ on $d$ admit a well-defined summation of the weights on worlds, i.e. only countably many worlds have non-zero weight wrt $d$ (Belle, Lakemeyer, and Levesque 2016).

The truth of sentences in $DS_p$ is defined as:

- $e, w, z, \lambda = \tau_1 \iff |\tau_1|_{w, \lambda}$ and $|\tau_2|_{w, \lambda}$ are identical;
- $e, w, z, \lambda = \neg \alpha$ iff $e, w, z, \lambda \neq \alpha$;
- $e, w, z, \lambda = \alpha \land \beta$ iff $e, w, z, \lambda = \alpha$ and $e, w, z, \lambda = \beta$;
- $e, w, z, \lambda = \forall v, \alpha$ iff $e, w, z, \lambda = \alpha^n$ for every standard name $n$ of the right sort;
- $e, w, z, \lambda = \forall F, \alpha$ iff $e, w, z, \lambda' = \alpha$ for all $\lambda' \sim_F \lambda$;
- $e, w, z, \lambda = \neg \alpha$ iff $e, w, z, n. \lambda = \alpha$ and $n = |\tau|_{w, \lambda}$;
- $e, w, z, \lambda = \Box \alpha$ iff $e, w, z, z' \cdot \lambda = \alpha$ for all $z' \in \mathbb{Z}$.

To prepare for the semantics of epistemic operators, let $W_{\alpha}^{\langle \rangle} = \{(w', z') \mid z' \sim_w z \text{ and } e, w', \{\}, \lambda = |z'\rangle \alpha\}$. If $z = \langle \rangle$, we ignore $z$ and write $W_{\alpha}^{\langle \rangle}$. If $\alpha$ is FO, we ignore $\lambda$ and write $W_{\alpha}^{\langle \rangle}$. If the context is clear, we write $W_\alpha$. Intuitively, $W_\alpha$ is the set of alternatives (world and action sequence pairs) of $z$ that might result in $\alpha$. A distribution $d$ is regular iff $\text{Eq}(d, W^{\langle\rangle}_{\text{TRUE}})$, for some $n \in \mathbb{R} \geq 0$. We denote the set of all regular distributions as $\mathcal{D}$.

Definition 6. Given $w \in W, d \in D, z \in \mathbb{Z}$, we define

- $w$ as a world such that for all primitive terms $t$ and $z' \in \mathbb{Z}, w[t, z'] = w[t, z \cdot z']$;
- $d_z$ a mapping such that for all $w \in W$, $d_z(w) = \sum_{\{w' : d(w') > 0\}} \sum_{z' : z' \sim_w z} d(w') \times \text{fst}(w', z')$. $d_z$ is called the progressed world of $w$ wrt $z$. A remark is that the $d_z$ might not be regular for a regular $d$. For example, if the likelihood of $z$'s alternatives is all zero in worlds with non-zero weights, then $\text{Eq}(d, W_{\text{TRUE}}^{\langle\rangle}, 0)$. Hence we define:

Definition 7. A distribution $d$ is compatible with action sequence $z$, $d \sim_{\text{comp}} z$ iff $d_z \in D$; given an epistemic state $e$, the set $e_z = \{d_z \mid d \in D, d \sim_{\text{comp}} z\}$ is called the progressed epistemic state of $e$ wrt $z$.

As a consequence, $d \sim_{\text{comp}} \{\}$ iff $d \in D$. Note that the progressed epistemic state of $e$ are only about its regular subset $e \cap D$ and $d_z \subseteq D$, therefore $e \notin \{\}$ in general.

Proposition 1. Given $d \in D$ and $z, z' \in \mathbb{Z}, d_z \sim_{\text{comp}} (d_z)_{z'}$.

The truth of $B$ and $O$ are given by:

- $e, w, z, \lambda = B(\alpha; r)$ iff $\forall d \in e_z, \text{Norm}(d, W^{\langle\rangle}_{\alpha} \lambda, W^{\langle\rangle}_{\text{TRUE}}; n)$ for $n \in \mathbb{R}$ and $n = |\alpha|_{\lambda}$;
- $e, w, z, \lambda = O(\alpha; r_1, \ldots, r_k)$ iff $\forall d \in e_z$ iff for all $1 \leq i \leq k, \text{Norm}(d, W^{\langle\rangle}_{\alpha} \lambda, W^{\langle\rangle}_{\text{TRUE}}; n_i)$ for $n_i \in \mathbb{R}$ and $n_i = |\alpha|_{\lambda}$;

Intuitively, an epistemic state $e$ only believes something after action sequence $z$, if and only if the progressed epistemic state $e_z$ is the maximal epistemic state consisting of all distributions that believe it. However, this definition of $O$ is problematic to some extents. To see why we need the following notation.

Let $S(d)$ be the support set of $d$, i.e. $S(d) = \{w | d(w) > 0\}$, we define a distance function $\rho$ for regular distributions $D$ as $\rho(d, d') = \sum_{w \in S(d) \cup S(d')} |d(w) - d'(w)|$. Clearly $(D, \rho)$ forms a metric space. Given an infinite sequence of regular distributions $\{d_i, d_2, \ldots, d_n, \ldots\}$, we say it converges to a regular distribution $d$ if for any $\epsilon > 0$, there exists a natural number $N \in \mathbb{N}$ s.t. for any $n > N$, $\rho(d_n, d) < \epsilon$. If such a $d$ exists, we write $\lim_{n \to \infty} d_n = d$ and call $d$ the limit of $\{d_1, d_2, \ldots, d_n, \ldots\}$.

Definition 8 (Closure). For all $e \subseteq D$, the closure of $e$ is defined as $\text{cl}(e) = \{d \in D, \exists\{d_1, d_2, \ldots, d_n, \ldots\}, \forall i \in \mathbb{N}, d_i \in e, \lim_{n \to \infty} d_n = d\}$.

We call an $e \subseteq D$ closed if and only if $e = \text{cl}(e)$.

Proposition 2. Given regular $d$ and $\{d_1, d_2, \ldots, d_n, \ldots\}$ s.t. $d = \lim_{n \to \infty} d_n$, for any $w \in W$, the limit of $d_n(w)$ exists and $d(w) = \lim_{n \to \infty} d_n(w)$

Theorem 1. Let $e$ be an epistemic state, $\lambda$ a variable map, $\alpha$ a sentence $\alpha$, and $r$ a rigid term, if for all $d \in e$, $\text{Norm}(d, W^{\langle\rangle}_{\alpha} \lambda, W^{\langle\rangle}_{\text{TRUE}}; |\lambda|)$, then for all $d' \in \text{cl}(e)$, $\text{Norm}(d', W^{\langle\rangle}_{\alpha} \lambda, W^{\langle\rangle}_{\text{TRUE}}; |\lambda|)$.

The proof require the definition of limit and Prop. 2. The theorem suggests that $\{d | \text{Norm}(d, W^{\langle\rangle}_{\alpha} \lambda, W^{\langle\rangle}_{\text{TRUE}}; |\lambda|)\}$ is a closed set for any $\alpha, \lambda$, and $r$. Therefore, $e_z$ should be closed to enable the maximal semantics for only-believing. Hence, we have the following definition:

Definition 9 (Progressed Epistemic State). Given $e, z \in Z$, we defined $e_z \sim_{\text{comp}} \{d \mid d \in D, d \sim_{\text{comp}} z\}$, i.e. $e_z = \{d \in D \mid \exists\{d_1, d_2, ..., d_n, \ldots\}, \forall i \in \mathbb{N}, d_i \in e \cap D, d_i \sim_{\text{comp}} z, d = \lim_{n \to \infty}(d_i)\}$

A remark is that Definition 9 ensures that $e_z$ is closed, yet whether the $e_z$ by Definition 7 already ensures $e_z$ is closed remains open. In case the answer is positive, the two definitions are exactly the same. In case negative, the maximal semantics for only-believing only works for Definition 9. Now, the truth of $B$ and $O$ are given exactly the same as before except $e_z$ with the new definition.

For sentence $\alpha$, we write $e, w, \models \alpha$ to mean $e, w, \{\}, \lambda = \alpha$ for all variable maps $\lambda$. When $\Sigma$ is a set of sentences and $\alpha$ is a sentence, we write $\models \alpha = \alpha(\Sigma)$; $\alpha(\Sigma)$, which basically assigns $\alpha$ to mean that for every set of regular distributions $e$ and $w$, if $e, w, \models \alpha'$ for every $\alpha' \in \Sigma$, then $e, w, \models \alpha$. We say that $\alpha$ is valid $(\models \alpha)$ if $\{\} \models \alpha$. Satisfiability is then defined in the usual way. If $\alpha$ is an objective formula, we write $w \models \alpha$ instead of $e, w, \models \alpha$. Similarly, we write $e, w, \models \alpha$ instead of $e, w, \models \alpha$ if $\alpha$ is subjective.

2.3 Comparison with $DS$ and Some Properties

The language of $DS$ and $DS_p$ are rather similar except that $DS$ considers fluent predicates while $DS_p$ only has fluent
functions. While in $D_S$ every closed term is a standard name, we follow (Lakemeyer and Levesque 2011), who use special standard names for objects and actions. The semantic structures in both logics are essentially the same, consisting of worlds, sets of distributions over worlds serving as epistemic states, and action sequences. Our use of rigid mathematical functions, which are not considered in $D_S$, is similar to the $R$-interpretation in (Belle and Levesque 2018). Among other things, this allows us later to express degrees of belief specified by arbitrary rigid terms.

The main difference lies in the semantics of beliefs and only-believing. To appreciate the difference, it is instructive to review the semantics in $D_S$. While many notions are exactly the same, things diverge from the definition of $W_{\alpha}$. First, $D_S$ keeps the traditional special $poss$ predicate to specify action preconditions and a notion of $exec(z)$, defined recursively by $poss$, to express action sequence $z$ is executable. Second, $D_S$ defines a compatibility between worlds wrt $oi$: $w' \sim_{oi} w$ iff for all $a, a'$, and $z, w'\{oi(a, a'), z\} = w\{oi(a, a'), z\}$, that is $w'$ is compatible with $w$ if they agree on $oi$. Finally, $W_{\alpha}$ in $D_S$ is then defined wrt triples $e, w, z$: $W_{\alpha, w, z}^e = \{w'\sim_{oi} w, w'\sim_{oi} w\}$, and $e, w', (\{z\} \land exec(z'))$. Note that worlds in $W_{\alpha, w, z}^e$ agree with $w$ wrt observational-indistinguishability.

Truth of beliefs ($B'$) and only-believing ($O'$) is given by

- $e, w, z \models B'(\alpha: n) \iff \forall d \in C(e, w, z)$,
- $\text{Norm}(W_{\alpha, w, z}^e, \text{true})$ for constant $n$;

One consequence of only allowing constants in degrees of belief is that formula like $B(p: 0.1 + 0.2)$ is not well-defined. The new logic overcomes this by a special treatment of rigid terms that is terms like 0.1 + 0.2 have the same denotations in all worlds, i.e. 0.3. This, among other things, enables us to include formulas like $\forall w.B(h = w : G(\{z\}, u))$ as well-formed formula, where $h$ is a fluent and $G(\{z\}, u)$ stands for geometry distribution with expectation 2.

Another observation is that, while $W_{\alpha}$ in $B'$ and $O'$ involves $e$ and $w$, $W_{\alpha}$ in $B$ and $O$, however, only involves individual $d \in e$. Such a change has major impacts on properties of the logic like introspection and meta-beliefs. Discussing them would go beyond the scope of this paper.

We comment that the idea of special treatment of rigid terms and using individual $d \in e$ in $W_{\alpha}$ for $B$ and $O$ is from (Liu and Lakemeyer 2021). For the reason of being self-contained, we reiterate it here and it’s not the main contribution of this work. Yet, we want to emphasize that:

**Theorem 2.** For any sentence $\alpha$ and rigid ground term $r$, $O(\alpha: r)$ is satisfactory.

For any $\lambda$, let $e = \{d|\text{Norm}(W_{\alpha, d}^e, \text{true}), r|\lambda\}$, clearly $e, \lambda \models O(\alpha: r)$. This is not the case when replacing $O$ with $O'$, even if $r$ is FO, due to the $e$ in $W_{\alpha, w, z}^e$ in $O'$.

**Theorem 3.** Given an objective sentence $\alpha$ and constant $n$,
- $\models B(\alpha: n) \equiv B'(\alpha: n)$;
- $\models O'(\alpha: n) \supset O(\alpha: n)$;

Intuitively, in static case, when $\alpha$ is objective, $poss$, $e$, and $\{d\}$ in $W_{\alpha}$ play no role, therefore $W_{\alpha}$ coincides in the two logics. Hence $B$ and $B'$ are equivalent. As for only-believing, suppose an $e \subseteq D_S.t.e \models O'(\alpha: n)$, we have $e = \{d|\text{Norm}(d, W_{\alpha}, \text{true})\}$. By Theorem 1, $e$ is closed, therefore, $e = e_0$. By semantics, $e \models O(\alpha: n)$. The converse does not hold, since there might exists an open set $e.t.e \models O(\alpha: n)$, yet $e \notin \{d|\text{Norm}(d, W_{\alpha}, \text{true})\}$ = $c(e)$ and therefore $e \neq O'(\alpha: n)$.

Although our notion of only-believing is weaker than its counterpart of $D_S$, we still retain the properties of only-believing:

**Proposition 3.** Let $\alpha$ and $\alpha_i$ be arbitrary sentences, we have
1. $\models O(\alpha_1: r_1, \ldots, \alpha_k: r_k) \supset \bigwedge B(\alpha_i: r_i)$;
2. $\models O(\alpha: r) \supset \neg B(h(n) = m: r')$ for all $r', r$, and $\alpha$, where $n$ and $m$ are std. names and $h$ is a fluent not in $\alpha$;
3. For any $e$, $e \models O(\alpha_1: r_1, \ldots, \alpha_k: r_k) \iff c(e) \models O'(\alpha_1: r_1, \ldots, \alpha_k: r_k)$

The second part says that the agent has no beliefs about things not mentioned in the KB. Note that this is not true if $O$ is replaced by $B$. In the rest of paper, whenever we write $e \models O(\alpha_1: r_1, \ldots, \alpha_k: r_k)$, we mean a closed $e$, unless stated otherwise.

Before we move on, let us turn back to the motivational example in the introduction:

**Example 1.** Suppose $\Sigma$ is the conjunction of the following
1. $\square \text{pos}(a) \equiv \text{true}$
2. $\square \text{oi}(a, a') = \exists y. a = \text{sonar}(y) \land a = a'$
3. $\square (a) = x \equiv x = 1 \land \exists y.h = y \land a = \text{sonar}(y)$, then $\not \models O'(h = 1: 0, h = 2: 0.5, \Sigma: 1) \supset [\text{sonar}(2)]O'(h = 2 \land \Sigma: 1)$

Suppose the opposite holds and $e, w \models O'(h = 1: 0.5, h = 2: 0.5, \Sigma: 1)$. Consider a distribution $d$ s.t. $\{d, w \models K'(h = 1 \land \Sigma: 1), clearly, d \not \models e \iff e, w \models B(h = 1: 0.5)$. In addition, one can check $\{d, w \models [\text{sonar}(2)]K'(h = 1 \land \Sigma: 1)$. However, by hypothesis, $e, w \models [\text{sonar}(2)]O'(h = 2 \land \Sigma: 1)$. By semantics of $O'$, $d \in e$, contradiction. The reason is that truth of $O'$ refers to $e$ but not the progressed epistemic state $e_2$.

**Lemma 1.** $(w_2)z' = w_2, z'$ and $(e_2)z' = e_2, z'$.

The proof of the first is trivial by definition of $w_2$ while the proof of the later requires not just the definition of $e_2$ but also the triangle property of distance $\rho$.

**Theorem 4.** For any sentence $\psi, e, w, z \models \psi$ iff $e, w, z \models \psi$.

The proof is by induction on the structure of $\psi$. The base case is by definition of $w_2$ while the induction on $\neg, \lor, \exists, B$, and $O$ are simply by semantics. The induction on $[\cdot]$ and $\square$ requires Lemma 1.
3 The Semantics of Progression

Sometimes, it would be desirable to include usual mathematical functions as logical terms not just +, \times like the uniform distribution in the introduction \( U_{1,2} \). Specially, we might use summation \( \sum \) as a logical term. One way is to assume infinite rigid function symbols, one for each such function, and semantically ensure all worlds in \( W \) interpret them identically, another way is to specify them syntactically by axioms. Here, we take the later approach and call these axioms definitional axioms and these functions definitional functions, terms constructed by definitional functions definitional terms. E.g. the following axiom specifies \( U_{1,2} \).

\[
\forall v, \forall u. U_{1,2}(u) = v \equiv (u = 1 \lor u = 2) \land v = 0.5 \\
\lor -(u = 1 \lor u = 2) \land v = 0 \tag{1}
\]

3.1 Basic Action Theories

BATs were first introduced by Reiter (2001) to encode the effects and preconditions of actions. Given a finite set of fluents \( H \), a BAT \( \Sigma \) over \( H \) consists of the union of the following sets:

- \( \Sigma_{ss} \): A set of successor state of axioms, one for each fluent in \( H \), of the form \( \square a(h(\bar{p})) = u \equiv \gamma_h \) to characterize action effects, also providing a solution to the frame problem (Reiter 2001). Here \( \gamma_h \) is a fluent formula with free variables \( \bar{p} \) and it is functional in \( u \).
- \( \Sigma_{oi} \): A single axiom of the form \( \square \text{oi}(a, a') = \top \equiv \psi \), where \( \psi \) is a rigid formula, to represent the observational-indistinguishability relation among actions.
- \( \Sigma_l \): A set of axioms of the form \( \square l(a) = \mathcal{L}(a) \) to capture action likelihoods. Here \( \mathcal{L}(a) \) is a definitional term with free variable \( a \).

We require \( \Sigma_{ss} \), \( \Sigma_{oi} \), and \( \Sigma_l \) to be first-order. The condition that \( \psi \) is rigid ensures that the observational-indistinguishability among actions is fixed. Besides BATs, to infer future state, we need to specify what holds initially. This is achieved by a set of fluent sentences \( \Sigma_0 \) (might be second-order). By belief distribution, we mean the joint distribution of a finite set of random variables. Formally, assuming all fluents in \( H \) are nullary, \( H = \{ h_1, \ldots, h_k \} \), a belief distribution \( B^f \) of \( H \) is a formula of the form

\[
\forall \bar{u}. B(\bar{h} = \bar{u}; f(\bar{u})), \text{ where } \bar{u} \text{ is a set of variables, } \bar{h} = \bar{u} \text{ stands for } \bigwedge h_i = u_i, f \text{ is a definitional function of sort number with free variables } \bar{u}. \text{ Finally, by a knowledge base, we mean a sentence of the form } O(B^f \land \Sigma). \text{ Although the logic allows KB specified by multiple distributions, for example, } O(B^f_1 \lor B^f_2), \text{ we only consider KB with single distribution in this paper.}
\]

Example 2. The following is a possible BAT \( \Sigma \) for our robot moving example:

\[
\square a(h = u \equiv \exists x, y, a = \text{wd}(x, y) \land u = \max\{0, h - y\}) \\
\lor \forall x, y, a \neq \text{wd}(x, y) \land h = u \\
\square \text{oi}(a, a') = \top \equiv \exists x, y, z, a = \text{wd}(x, y) \land a' = \text{wd}(x, z) \\
\lor a = \text{sonar}(z) \land a' = a \\
\square l(a) = \mathcal{L}(a) \text{ with } \\
\mathcal{L}(a) = \begin{cases} 
\theta(x, y, 0, 2, 0.6) & \exists x, \exists y, a = \text{wd}(x, y) \\
\theta(z, h, 0.1, 0.8) & \exists z, a = \text{sonar}(z) 
\end{cases} \text{ and further more, } \theta \text{ is given by } \\
\theta(x, y, m, n) = \begin{cases} 
m & x = y \\
0 & \text{o.w.}
\end{cases}
\]

In English, distance \( h \) can only be affected by \( \text{wd}(x, y) \) and the value is determined by value \( u \), not the intended value \( x \); the robot cannot get across the wall \( (u = \max\{0, h - y\}) \); two actions are observationally indistinguishable if and only if they are both forward actions with the same intended value or they are identical sensing action; likelihood of stochastic action \( \text{wd}(x, y) \) and noisy sensing \( \text{sonar}(z) \) is specified by \( \theta \). The agent considers a uniform distribution among \( \{1, 2, 3\} \) possible initially.

For the objective fragment, our definition of progression is similar to (Claßen 2013):

**Definition 10.** Given \( \Sigma_0, \Sigma, \) and an rigid ground action term \( t \), a set of fluent formula \( \Sigma'_0 \) is called the progression of \( \Sigma_0 \) wrt \( t \), \( \Sigma'_0 \) iff for all \( w \models \Sigma_0 \cup \Sigma \) iff there exists \( w \) s.t. \( w \models \Sigma'_0 \land \Sigma \land w(t) = w' \).

**Theorem 5** (Lin and Reiter’s progression). The following is a progression of \( \Sigma_0 \) wrt \( \Sigma, t \):

\[
\exists F_\Sigma(\Sigma_0)_{\Sigma, t} \land \forall \bar{p}. \forall u. h(\bar{p}) = u \equiv (\gamma_h)_{\Sigma, t}^{\Sigma_0}
\]

, where \( F_\Sigma \) are SO variables s.t. \( F_i, h_i \) are of the same arity.

Since LR progression was proposed, efforts have been made to find fragments where \( \Sigma'_0 \) is first-order. For example, if the successor state of axioms is local-effect (Liu and Levesque 2005), the progression is first-order definable. Intuitively, local-effect means actions can only affect locally, like the block world example, the action \( \text{move}(x, y, z) \), i.e. moving object \( x \) from \( y \) to \( z \), only affects locally on the object \( x \) and location \( y \) and \( z \) but nothing globally. Here we show that the progression of the nullary fluents fragment is first-order definable.

Here, “\( \in \)” should be understood as a finite disjunction. For readability, we write the definition functions in this form, they should be understood as logical formula as Equation 1.
Theorem 6. Given $\Sigma_0$, $\Sigma$, and an ground action term $t$ where every fluents in $H$ are nullary, then the following is a progression of $\Sigma_0$ wrt $\Sigma$, t: $\exists \psi. (\Sigma_0)^{\psi} \wedge \forall \nu. h = u \equiv (\gamma_t)_{\nu, \bar{v}}$

The proof is based on the fact that $\exists V. \alpha$ is logically equivalent to $\exists V. (\alpha)^\psi$ if $V$ is nullary. A formal proof is by an induction on the size of $\alpha$. We omit it here for space reasons. We mark the FO progression given by Theorem 6 as Pro$(\Sigma_0, \Sigma, t)$. For example, let $\Sigma_0 = \{ h = 1 \}$, then $\Sigma_0(t) = \{ h = 2 \}$.

Definition 11. We call a formula $O(\Psi \wedge \Sigma)$ the progression of the knowledge base $O(B^f \wedge \Sigma)$ wrt ground action $t$, if and only if $O(B^f \wedge \Sigma) \models [t]O(\Psi \wedge \Sigma)$.

Essentially, this means that when the sonar reads some value, the agent knows it that value but no others.

3.2 Progression after Sensing Actions

To begin with, it’s necessary to define what are sensing actions. Our view of sensing actions is the same as (Lakemeyer and Levesque 2009), namely these are actions that provide information about the world but do not change it. In a formal way, they are actions that appear in the likelihood axioms but not in the successor state axioms, like the sonar in Example 2. Additionally, sensing actions have no alternatives. Intuitively, this means that when the sonar reads some value, the agent knows it that value but no others.

Theorem 7. Given a KB $O(B^f \wedge \Sigma)$ and a sensing action $t_{\text{sen}}$ s.t. $O(B^f \wedge \Sigma) \models [t_{\text{sen}}]O(B^f \wedge \Sigma)$ where $f^*$ is a definitional function in term of $f$ as:

$$f^*(u) = \frac{1}{\eta} f(u) \times L(t_{\text{sen}})^{\eta}_u$$

and $\eta = \sum_{w \in (X_0)^k} f(u) \times L(t_{\text{sen}})^{\eta}_u$.

Namely, the progression of a KB is another KB with belief distribution $B^f$; the relation between $B^f$ and $B^f$ is such that the new degrees of belief of $(h = i)$ is just the normalized product of the old degrees of belief and likelihoods of the sensing action. While intuitively the result might be straightforward, the proof is non-trivial. Suppose $e \models O(B^f \wedge \Sigma)$ and $e' \models O(B^f \wedge \Sigma)$, the central task of the proof is to show that $e_{t_{\text{sen}}} = e'$. The direction $e_{t_{\text{sen}}} \subseteq e'$ is straightforward, whereas the other one is sophisticated. In fact, for the direction $e_{t_{\text{sen}}} \subseteq e'$, we have

Lemma 2. $B^f \wedge K \Sigma = [t_{\text{sen}}]B^f$, where $B^f$, $\Sigma$, $B'$, and $t_{\text{sen}}$ are the same as Theorem 7.

Definition 12. Given BAT $\Sigma$ wrt fluents $H$, let $P \Sigma$ be the set of all primitive terms of fluents not in $H$, we define a relation $\sim_H$ over $W$ as $w \sim_H w'$ iff for all $t \in P \Sigma$ and all $z \in Z$, $w[t, z] = w'[t, z]$.

Namely, $w \sim_H w'$ iff $w$ and $w'$ assigns the same denotation for terms without fluent in $H$. Clearly, $\sim_H$ is an equivalence relation. We denote the set of all equivalence classes wrt BAT $\Sigma$ as $W_H$. Note that $W_H$ is countable.

Proposition 4. Given BAT $\Sigma$ and $C \in W_H$, for all standard names $\bar{n}$, there is a unique world $w$ s.t. $w \in C$ and $w \models \bar{n} \wedge \Sigma$, we mark this world as $w_{\bar{n}, \bar{n}}$.

Lemma 3. Let $B^f$, $B'^f$, $\Sigma$, and $t_{\text{sen}}$, be as in Theorem 7. For all $d' \in D$ such that $\{ d' \} \models B^f \wedge K \Sigma$, there exists a distribution $d \in D$ s.t. $\{ d \} \models B^f \wedge K \Sigma$ and $d_{\text{sen}} = d'$.

The construction of such $d$ is based on three main steps:

1. By virtue of Prop. 4 and the fact that $S(d')$ is countable, there exists a minimal countable set $W_{\text{con}}(d') \subseteq W_H$ such that $\forall w \in S(d')$, $\exists C' \in W_{\text{con}}(d')$, $w \in C'$, namely $W_{\text{con}}(d')$ covers all worlds in $S(d')$.
2. For each $C' \in W_{\text{con}}(d')$, we select a $C \in W_H$ such that for every world $w' \in C'$, there exists a unique world $w \in C$ which can progress to $w'$ after $t_{\text{sen}}$. The selected $C$ forms $W_{\text{con}}(d)$.
3. The last step is to assign weights to $w$ which is exactly the weight of $w'$ under $d'$ divided by the likelihood of $t_{\text{sen}}$.

Essentially, $f(i) = f'(i) / L(t_{\text{sen}})^{\eta}$. if we ignore the $\eta$ in Theorem 7. Our construction of $d$ is to reconstruct such relation at the semantical level: If $d'$ assigns some weights to a world $w'$, then $d$ assigns a world $w$, which progresses to $w'$ after $t_{\text{sen}}$, with the same weights but divided by the likelihood of $t_{\text{sen}}$. Moreover, the feature of sensing actions ensures this semantical property can be reflected correctly at the syntactical level.

Proof of Theorem 7. Suppose two closed $e, e'$ s.t. $e \models O(B^f \wedge \Sigma)$ and $e' \models O(B^f \wedge \Sigma)$, by Theorem 4, it suffices to show $e_{t_{\text{sen}}} = e'$

"$e_{t_{\text{sen}}} \subseteq e'$": Since $O(B^f \wedge \Sigma) \models B^f \wedge K \Sigma$, $e \models B^f$. By semantics of $O$ and hypothesis, $e_{t_{\text{sen}}} \subseteq e'$

"$e' \subseteq e_{t_{\text{sen}}}$": Since $e' \models O(B^f \wedge K \Sigma)$, for all $d' \in e'$, $\{ d' \} \models B^f \wedge K \Sigma$. By Lemma 3, $d_{\text{sen}} = d'$. Hence $d' \in \{ d \in D | d \models \Sigma \wedge d \sim_{\text{comp}} t_{\text{sen}} \}$. Therefore $d' \in \{ d \in D | d \models e \wedge d \sim_{\text{comp}} t_{\text{sen}} \}$, hence $d' \models e$. That is $e' \subseteq e_{t_{\text{sen}}}$.}

Example 3. Given KB $O(B^f \wedge \Sigma)$ as Example 2, then $O(B^f \wedge \Sigma) \models [\text{sonar}(2)]O(B^f \wedge \Sigma)$, where $f'$ is a definitional function as:

$$f'(u) = \frac{1}{\eta} f(u) \times [\text{sonar}(2)]_{\eta}$$

$$= \begin{cases} \frac{1}{\eta} \times 0.1 & u \in \{1, 3\} \\ \frac{1}{\eta} \times 0.8 & u = 2 \\ 0 & \text{o.w.} \end{cases}$$
The second equality is by the specification of \( \mathcal{L}(a) \), the third equal is due to \( f'(u) \) is non-zero only among \( \{1, 2, 3\} \). The last one is because \( \eta = \frac{1}{3} \).

### 3.3 Progression after Stochastic Actions

Unlike sensing, stochastic actions have observationally indistinguishable actions as alternatives, sometimes even infinite alternatives. Besides, stochastic actions do affect the real world. This makes the progression wrt stochastic actions more complicated than sensing actions

**Theorem 8.** Given a KB \( O(B^f \land \Sigma) \) and a stochastic action \( t_{sa} \), \( O(B^f \land \Sigma) \models [t_{sa}]O(B^f \land \Sigma) \), where \( f' \) is a definitional function in term of \( f \) as:

\[
f'(\vec{u}) = \sum_{\vec{u} \in \mathcal{N}_d} \sum_{a \in \mathcal{A}_d} f(\vec{u}) \times L(a)_{\vec{u}}^0 \times \mathcal{I}(\vec{u}, \vec{a}, a, t_{sa}),
\]

where \( \mathcal{I} \) is a definitional function given by

\[
\mathcal{I}(\vec{u}, \vec{u}', a, t_{sa}) = \begin{cases} 1 & \text{Proto}(\vec{h} = \vec{u}', a)_{\vec{a}}^0 \land \mathcal{L}(\vec{u}) \land (\psi)^{a'}_{t_{sa}} \land \alpha.w. \end{cases}
\]

Like sensing actions, the central task of the proof is to show \( \epsilon_{t_{sa}} = \epsilon' \) and the direction \( \epsilon_{t_{sa}} \leq \epsilon' \) is straightforward. Formally, we have

**Lemma 4.** \( B^f \land K \Sigma \models [t_{sa}]B^f \), where \( B^f, B'^f \), and \( t_{sa} \) are the same as Theorem 8.

The proof is rather similar to Lemma 2 but additionally requires the fact that for all world \( w \) and stochastic action \( a \), \( \sum_{\{a, \alpha : \alpha_w = a\}} f(a) = 1 \). Unfortunately, the techniques for sensing actions to construct a distribution as Lemma 3 does not apply to stochastic actions. This is because: 1) there is not an explicit inverse of \( f \) in terms of \( f' \); and 2) the correspondence between \( C \) and \( C' \) breaks. More concretely, assuming \( \{a_1, a_2, \ldots, a_m\} \) are mutual alternatives, given \( w' \in C' \), there might be a set of world \( \{w_1, w_2, \ldots\} \) s.t. \( w' = (w_i)_a \), and \( w_1 \in C \) for different \( C_i \). Conversely, given a world \( w \in C \), \( w_1 \) might belong to different \( C_i \). To solve this problem, we first consider the case where action alternatives are finite.

**Lemma 5 (Finite Action Alternatives).** Let \( B^f, B'^f \), and \( \Sigma \) be as in Theorem 8, \( t_{sa} \) a stochastic actions with finitely many alternatives under \( \Sigma \). For all \( \vec{d} \in D \) such that \( \{\vec{d}\} \models B^f \land K \Sigma \), there exists a distribution \( d \in D \) s.t. \( \{\vec{d}\} \models B'^f \land K \Sigma \) and \( d_{t_{sa}} = d' \).

In the following, we only consider the case that \( Eq(d', \forall \mathcal{V}_{\mathcal{R}_{\text{TRUE}}}). \) If \( Eq(d', \forall \mathcal{V}_{\mathcal{R}_{\text{TRUE}}}, c) \) for \( c \neq 1 \), a distribution can be constructed in the same way except that the weight of worlds is proportionally increased by \( c \).

We first observe that given a world \( w \) s.t. \( w \models \vec{h} = \vec{n} \land \Sigma \) for some \( \vec{n} \), due to the finite alternatives hypothesis, there are only finite alternatives \( \{a_{\vec{n},1}, a_{\vec{n},2}, \ldots, a_{\vec{n},m}\} \) whose likelihoods are positive. Moreover, \( w \) might progress to \( m \) worlds \( w_{a_{\vec{n},1}}, w_{a_{\vec{n},2}}, \ldots, w_{a_{\vec{n},m}} \), with equivalence class \( C_{a_{\vec{n},1}, C_{a_{\vec{n},2}, \ldots, C_{a_{\vec{n},m}}} \}. \) Conversely, given an equivalence class \( C_{a_{\vec{n},1}, C_{a_{\vec{n},2}, \ldots, C_{a_{\vec{n},m}}} \} \), there exists a world \( w \) s.t. \( w \models \vec{h} = \vec{n} \land \Sigma \) and \( w_{a_{\vec{n},i}} \in C_{a_{\vec{n},i}}, \) for all \( 1 \leq i \leq m \). In fact, there are infinitely many such worlds: the condition \( w_{a_{\vec{n},i}} \in C_{a_{\vec{n},i}}, \) only restricts how \( w \) interprets terms \( t \in P_{\vec{H}} \) after actions.
Lemma 6. Let $B^f$, $B^{\prime}$, and $\Sigma$ be as in Theorem 8, $t_{sa}$ a stochastic actions. For all $d' \in D$ such that $\{d'\} \models B^{\prime} \land K\Sigma$ and any $\epsilon > 0$, there exists a distribution $d \in D$ s.t. $\{d\} \models B^f \land K\Sigma$ and $\rho(d_{t_{sa}}, d') < \epsilon$.

The idea is to only consider a finite subset of $t_{sa}$'s alternatives and construct a distribution $d$ using the above procedure w.r.t the finite set of alternatives. It can be shown that $d$ satisfies $B^f \land K\Sigma$. Additionally, by increasing the size of the finite set of alternatives, the distance $\rho(d_{t_{sa}}, d')$ decreases accordingly and is eventually less than $\epsilon$.

Proof of Theorem 8. Suppose $e \models O(B^f \land \Sigma)$ and $e' \models O(B^{\prime} \land \Sigma)$, by Theorem 4, it suffices to show $e_{t_{sa}} = e'$ "$e_{t_{sa}} \subseteq e'"$ : the proof is exactly the same as its counterpart of noisy sensing.

"$e' \subseteq e_{t_{sa}}":$ Since $e' \models O(B^{\prime} \land K\Sigma)$, for all $d' \in e'$, $\{d'\} \models B^{\prime} \land K\Sigma$. By Lemma 6, $\exists d \models B^f \land K\Sigma$ and $\rho(d_{t_{sa}}, d') < \epsilon$ for any $\epsilon$. Therefore, $\exists d_1, \ldots, d_n \ldots$ s.t. $\{d_1\} \models B^{\prime} \land K\Sigma$ and there exists $N$ for $n > N$, $\rho(d_{t_{sa}}, d_n') < \epsilon$ for any $\epsilon$. That is, $\exists d_1, \ldots, d_n \ldots$ s.t. $\{d_n\} \models B^{\prime} \land K\Sigma$ and $\lim_{n \to \infty}(d_n)_{t_{sa}} = d'$. By definition of $e_{t_{sa}}$, $d' \in e_{t_{sa}}$.

Example 4. Let $B^{f}$ and $\Sigma$ be as Example 2, then $O(B^f \land \Sigma) \models [fwd(2, 2)]O(B^{f} \land \Sigma)$ where $f'$ is given by and $f'(u) = \begin{cases} 1 & u = 2 \\ 4/15 & u = 1 \\ 1/3 & u = 0 \\ 0 & o.w. \end{cases}$

By definition, we have:

$I(u, u', a, fwd(2, 2)) = \begin{cases} 1 & \text{Pro}(h = u', a)^{h}, u, \psi_{fwd(2,2)} \\ 0 & o.w. \end{cases}$

where $\psi_{fwd(2,2)} = (\exists x, y. a = fwd(x, y) \land u = \max(0, u' - y) \\
\forall x, y. a = fwd(x, y) \land u = u')$

$L(a)_u^{h} = \begin{cases} \theta(x, y, 0.2, 0.6) & \exists x, y. a = fwd(x, y) \\ \theta(z, u', 0.1, 0.8) & \exists z. a = sonar(z) \end{cases}$

Therefore,

$f'(u) = \sum_a \sum_a f(u')L(a)_u^{h}I(u, u', a, fwd(2, 2)) = \sum_a \sum_a f(u')  \begin{cases} \theta(2, y, 0.2, 0.6) & \exists y. a = fwd(2, y) \land u = \max(0, u' - y) \\ 0 & o.w. \end{cases}$

$= \begin{cases} 1/2 \times 0.2 & u = 2 \\ 1/2 \times 0.2 + 1/2 \times 0.6 & u = 1 \\ \left(1/2 \times 0.2 + 1/2 \times 0.6 + 1/3 \times 0.2\right) & u = 0 \\ 0 & o.w. \end{cases}$

The second line is by $L(a)_u^{h} \times I(u, u', a, fwd(2, 2))$. The third is because $\theta$ and $\theta$ are zero when $u'$ and $y$ are not among $\{1, 2, 3\}$, respectively, therefore $f'$ is non-zero only ony if $u \in \{0, 1, 2\}$ and the degrees of belief for each value of $u$ equals to the sum of products between $f(u')$ and $\theta(2, y, 0.2, 0.6)$ of all combinations of $u'$ and $y$ that result that value (according to $u = \max\{0, u' - y\}$).

4 Related Work

We revisit related work from two aspects: knowledge representation and projection by progression.

In terms of knowledge representation, our logic builds on the logic $DS$, a probabilistic extension of a modal variant of the situation calculus with a model of belief and only-believing. What distinguishes us is that our proposed logic has richer expressiveness that allows us to express a probabilistic knowledge base with arbitrary belief distributions and the utility of our notion of only-believing does not constrain to the static case. The logic $DS$ is based on the first-order logic $OBL$ (Belle, Lakemeyer, and Levesque 2016), a probabilistic logic of only-knowing. It is shown that $OBL$ fully captures the features of the logic $OL$, the pioneering work on only-knowing by Levesque (1990). In a game theory context, Halpern and Pass (2009) have considered a (propositional) version of only knowing with probability distributions.

For the aspect of degrees of belief, $DS_p$ is inspired by the work BHL (Bacchus, Halpern, and Levesque 1999), an axiomatic proposal with a conceptually attractive definition of belief in a first-order dynamic setting. In a less restricted setting, reasoning about knowledge and probability was studied prior to BHL (Nilsson 1986; Fagin and Halpern 1994; Monderer and Samet 1989). Notably, the work of Fagin and Halpern (1994) can be seen to be at the heart of BHL.

On progression, Lin and Reiter (1997) proposed the most general account of progression and showed that progression is second-order definable. Restricted forms of LR-progression, which are first-order definable, are discussed there as well and later in (Liu and Levesque 2005; Claßen et al. 2007; Vassos and Levesque 2007). Based on the notion of progressed worlds, Lakemeyer and Levesque (2009) show that the progression of categorical knowledge against noise-free actions amounts to only-knowing after actions. For a limited type of theory, the progression of discrete degrees of belief wrt context-completeness is considered in (Belle and Lakemeyer 2011). Belle and Levesque (2020) studied the progression of continuous degrees of belief for the so called invertible BATs which exclude our BATs in Example 2. As a result, we gap the general account of progression in discrete degrees of belief.

5 Conclusion

In this work, we lift the expressiveness of the logic $DS$. As a result, we are able to express a probabilistic knowledge base with arbitrary belief distribution. For an interesting fragment, we show that classical progression is first-order definable. Besides, based on our notion of progressed distributions, we show how the progression of discrete degrees of belief is related to only-believing after actions. In terms of future work, it would be interesting to see how the idea of progression can be used in verification of belief programs (Belle and Levesque 2015).

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