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Approximation Fixpoint Theory for Non-Deterministic Operators and Its Application in Disjunctive Logic Programming

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Abstract

Approximation fixpoint theory (AFT) constitutes an abstract and general algebraic framework for studying the semantics of nonmonotonic logics. It provides a unifying study of the semantics of different formalisms for nonmonotonic reasoning, such as logic programming, default logic and autoepistemic logic. In this paper we extend AFT to non-deterministic constructs such as disjunctive information. This is done by generalizing the main constructions and corresponding results to non-deterministic operators, whose ranges are sets of elements rather than single elements. The applicability and usefulness of this generalization is illustrated in the context of disjunctive logic programming.

1 Introduction

Disjunctive information has a central role in knowledge representation and reasoning, and disjunctive reasoning capabilities provide an additional way of expressing uncertainty and indeterminism to many formalisms for non-monotonic reasoning. It is not surprising that the introduction of disjunctive reasoning often increases the computational complexity of formalisms and extends their modeling capabilities (Eiter and Gottlob 1993).

The integration of non-deterministic reasoning with non-monotonic reasoning (NMR) has often proven non-trivial, as witnessed e.g. by the large body of literature on disjunctive logic programming (Lobo, Minker, and Rajasekar 1992; Minker and Seipel 2002). The implementation of non-deterministic reasoning in NMR yielded the formulation of some (open) problems that are related to the combination of non-monotonic and disjunctive reasoning (Beirlaen, Heyninck, and Straßer 2017; Beirlaen, Heyninck, and Straßer 2018; Bonevac 2018), or was restricted to limited cases (Gelfond et al. 1991).

The goal of this work is to provide an adequate framework for modeling disjunctive reasoning in NMR. Our framework is based on *approximation fixpoint theory* (AFT), a general technique for constructively characterizing a variety of non-monotonic operators. This approach underlies many non-monotonic formalisms, including all the major semantics for autoepistemic and default logic (Denecker, Marek, and Truszczyński 2000; Denecker, Marek, and Truszczyński 2003), a variety of logic programs including first order logic programs, and formal argumentation (Strass 2013). AFT also allows to define attractive semantics for non-monotonic formalisms, such as extensions of logic programs (Pelov, Denecker, and Bruynooghe 2007; Antić, Eiter, and Fink 2013; Charalambidis, Rondogiannis, and Symeonidou 2018) and weighted abstract dialectical frameworks (ADFs, (Bogaerts 2019)).

The extension of AFT to disjunctive reasoning is made here by the incorporation of *non-deterministic operators*. The idea of looking at non-deterministic operators was introduced in (Pelov and Truszczynski 2004), together with some results on two-valued semantics for DLP. In this paper, we develop an AFT for non-deterministic operators, which, among others, allows a generalization of the results of (Pelov and Truszczynski 2004) to the three-valued case. In more detail, we define several interesting classes of approximating fixpoints and show the existence and consistency of some of them. An application of this theory is demonstrated in the context of disjunctive logic programming.

Our theory is a conservative extension of AFT for deterministic operators in the sense that all the concepts introduced in this paper coincide with the deterministic counterparts when the theory is restricted to deterministic operators.

Paper Outline: In Section 2 we review some preliminaries, in particular those of disjunctive logic programming (Section 2.1) and approximation fixpoint theory (Section 2.2). In Section 3 we introduce the primary concepts of the paper: non-deterministic operators and their approximations. Section 4 is a study of the theory of non-deterministic AFT, including the consistency of non-deterministic operators (Section 4.1), properties of Kripke-Kleene fixpoints (Section 4.2) and constructions of stable fixpoints (Section 4.3). In Section 5 we illustrate the application of this framework to disjunctive logic programming, and in Section 6 we conclude.

2 Background and Preliminaries

In this section, we recall the basics of approximation fixpoint theory (AFT) for deterministic operators. We start with a brief survey on disjunctive logic programming (DLP), which will serve to illustrate concepts and results of the general theory of non-deterministic AFT.

2.1 Disjunctive Logic Programming

A (propositional) disjunctive logic program $\mathcal{P}$ (a dlp, for short) is a finite set of rules of the form $\bigvee_{i=1}^{n} p_i \leftarrow \psi$, where $\bigvee_{i=1}^{n} p_i$ (the rule's head) is a disjunction of atoms, and ψ (the rule's body) is a (propositional[1]) formula that may include the propositional constants T (representing truth), F (falsity), U (unknown), and C (contradictory information). A rule is called *normal* if its body is a conjunction of literals (i.e., atomic formulas or a negated atoms), and its head is atomic. A logic program is called *normal*, if it consists only of normal rules; It is *positive*, if there are no negations in the rules' bodies. The set of atoms occurring in $\mathcal{P}$ is denoted $\mathcal{A}_\mathcal{P}$.

Given a four-valued lattice $\mathsf{F} \leq_t \mathsf{U}, \mathsf{C} \leq_t \mathsf{T}$ [2] and a $\leq_t$-involution $-$ on it (i.e, $-\mathsf{F} = \mathsf{T}$, $-\mathsf{T} = \mathsf{F}$, $-\mathsf{U} = \mathsf{U}$ and $-\mathsf{C} = \mathsf{C}$), a *four-valued interpretation* of a program $\mathcal{P}$ is a pair (x, y), where $x \subseteq \mathcal{A}_\mathcal{P}$ is the set of the atoms that are assigned a value in $\{\mathsf{T}, \mathsf{C}\}$ and $y \subseteq \mathcal{A}_\mathcal{P}$ is the set of atoms assigned a value in $\{\mathsf{T}, \mathsf{U}\}$.[3] Interpretations are compared by *information order* $\leq_i$, where $(x, y) \leq_i (w, z)$ iff $x \subseteq w$ and $z \subseteq y$ (different 'precision'), and by *truth order* $\leq_t$, where $(x, y) \leq_t (w, z)$ iff $x \subseteq w$ and $y \subseteq z$ (increased 'positive' evaluations). Truth assignments to complex formulas are then recursively defined as follows:

- $(x,y)(\phi) = \begin{cases} \mathsf{T} & \text{if } \phi \in x \text{ and } \phi \in y, \\ \mathsf{U} & \text{if } \phi \notin x \text{ and } \phi \in y, \\ \mathsf{F} & \text{if } \phi \notin x \text{ and } \phi \notin y, \\ \mathsf{C} & \text{if } \phi \in x \text{ and } \phi \notin y. \end{cases}$
- $(x, y)(\neg\phi) = -(x, y)(\phi)$,
- $(x, y)(\psi \wedge \phi) = \min_{\leq_t}\{(x, y)(\phi), (x, y)(\psi)\}$,
- $(x, y)(\psi \vee \phi) = \max_{\leq_t}\{(x, y)(\phi), (x, y)(\psi)\}$.

A four-valued interpretation of the form (x, x) may be associated with a *two-valued* (or *total*) interpretation x, in which for an atom p, $x(p) = \mathsf{T}$ if $p \in x$ and $x(p) = \mathsf{F}$ otherwise. We say that (x, y) is a *three-value* (or *consistent*) interpretation, if $x \subseteq y$. Note that in consistent interpretations there are no C-assignments.

We now consider semantics for dlps. First, given a two-valued interpretation, an extension to dlps of the immediate consequence operator for normal programs (van Emden and Kowalski 1976) is defined as follows:

Definition 1. Given a dlp $\mathcal{P}$ and a two-valued interpretation x, we define:

- $HR_\mathcal{P}(x) = \{\Delta \mid \bigvee\Delta \leftarrow \psi \in \mathcal{P} \text{ and } (x, x)(\psi) = \mathsf{T}\}$.
- $IC_\mathcal{P}(x) = \min_\subseteq\{y \mid \forall\Delta \in HR_\mathcal{P}(x),\ y \cap \Delta \neq \emptyset\}$.

Thus, denoting by $\wp(\mathcal{L})$ the powerset of $\mathcal{L}$, $IC_\mathcal{P}$ is an operator on the lattice $\langle \wp(\mathcal{A}_\mathcal{P}), \subseteq\rangle$. $IC_\mathcal{P}(x)$ are the minimal two-valued interpretations that validate all disjunctions which are derivable from $\mathcal{P}$ given x.

Other semantics for dlps, this time based on three-valued interpretations, are defined next:

Definition 2. Given a dlp $\mathcal{P}$ and a consistent interpretation (x, y). We say that (x, y) is:

- *(three–valued) model* of $\mathcal{P}$, iff for every $\phi \leftarrow \psi \in \mathcal{P}$, $(x, y)(\phi) \geq_t (x, y)(\psi)$. We denote by $mod(\mathcal{P})$ the set of the three-valued models of $\mathcal{P}$.
- *weakly supported model* of $\mathcal{P}$, iff for every $p \in \mathcal{A}_\mathcal{P}$ such that $(x, y)(p) = \mathsf{T}$ $[(x, y)(p) = \mathsf{U}]$, there is $\bigvee\Delta \leftarrow \phi \in \mathcal{P}$ such that $p \in \Delta$ and $(x, y)(\phi) = \mathsf{T}$ $[(x, y)(p) = \mathsf{U}]$.
- *supported model* of $\mathcal{P}$, iff for every $p \in \mathcal{A}_\mathcal{P}$ such that $(x, y)(p) = \mathsf{T}$ $[(x, y)(p) = \mathsf{U}]$, there is $\bigvee\Delta \leftarrow \phi \in \mathcal{P}$ such that $p \in \Delta$ and $(x, y)(\phi) = \mathsf{T}$ $[(x, y)(p) = \mathsf{U}]$ and $\Delta \cap x = \{p\}$ $[\Delta \cap y = \{p\}]$.[4]

Another common way of providing semantics to dlps is by the following reduct (Gelfond and Lifschitz 1991):

Definition 3. The GL-transformation $\frac{\mathcal{P}}{(x,y)}$ of a normal dlp $\mathcal{P}$ w.r.t. a consistent interpretation (x, y), is the positive program obtained by replacing, in every rule $p_1 \vee \ldots \vee p_n \leftarrow q_1, \ldots, q_m, \neg r_1, \ldots, \neg r_k \in \mathcal{P}$, any negated literal $\neg r_i$ $(1 \leq i \leq k)$ by: (1) F if $(x, y)(r_i) = \mathsf{T}$, (2) T if $(x, y)(r_i) = \mathsf{F}$, and (3) U if $(x, y)(r_i) = \mathsf{U}$.

An interpretation (x, y) is a *three-valued stable model* of $\mathcal{P}$ iff it is in $\min_{\leq_t}(mod(\frac{\mathcal{P}}{(x,y)}))$.[5]

2.2 Approximation Fixpoint Theory

We now recall basic notions from approximation fixpoint theory (AFT), as described in (Denecker, Marek, and Truszczyński 2000). AFT introduces constructive techniques for approximating the fixpoints of an operator $O_\mathcal{L}$ over a lattice $L = \langle \mathcal{L}, \leq\rangle$. The motivation for this is the observation that while fixpoints of a (possibly non-monotonic) operator might not always exist, the existence of *approximations* of such fixpoints on a *bilattice* (Ginsberg 1988; Fitting 2006), constructed on the basis of L, is always guaranteed.

Definition 4. Given a lattice $L = \langle \mathcal{L}, \leq\rangle$, we let $L^2 = \langle \mathcal{L}^2, \leq_i, \leq_t\rangle$ be a structure (called *bilattice*), in which $\mathcal{L}^2 = \mathcal{L} \times \mathcal{L}$, and for every $x_1, y_1, x_2, y_2 \in \mathcal{L}$,

- $(x_1, y_1) \leq_i (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \geq y_2$,
- $(x_1, y_1) \leq_t (x_2, y_2)$ iff $x_1 \leq x_2$ and $y_1 \leq y_2$.

An approximating operator $\mathcal{O} : \mathcal{L}^2 \to \mathcal{L}^2$ of an operator $O_\mathcal{L} : \mathcal{L} \to \mathcal{L}$ is defined by specifying two operators $\mathcal{O}_l$ and $\mathcal{O}_u$ which calculate, respectively, a *lower* and an *upper bound* for the value of $O_\mathcal{L}$. It is observed in (Denecker, Marek, and Truszczyński 2000) that many formalisms can be characterized by a symmetric operator where the upper bound can be calculated by "inversing" the lower bound (and vice versa),

Definition 5. Let $O_\mathcal{L} : \mathcal{L} \to \mathcal{L}$ and $\mathcal{O} : \mathcal{L}^2 \to \mathcal{L}^2$.

- $\mathcal{O}$ is called an *approximation* of $O_\mathcal{L}$, if $\forall x, y, \in\ \mathcal{L}$, $\mathcal{O}(x, y) = (\mathcal{O}_l(x, y), \mathcal{O}_u(x, y))$, where $\mathcal{O}_l : \mathcal{L}^2 \to \mathcal{L}$ and

[1] For simplicity and due to lack of space, we restrict ourselves to the propositional case.

[2] In the sequel, we use the same notation for a truth value and the corresponding propositional constant.

[3] Somewhat skipping ahead, the intuition here is that x (y) is a lower (upper) approximation of the true atoms.

[4] Two-valued supported and weakly supported models are defined in (Brass and Dix 1995a). Their generalization to the 3-valued case is, to the best of our knowledge, novel.

[5] If $x = y$, (x, y) is called a *two-valued* stable model of $\mathcal{P}$.

$\mathcal{O}_u : \mathcal{L}^2 \to \mathcal{L}$ are a lower and upper bound, respectively, of $O_\mathcal{L}$, i.e.: $\mathcal{O}_l(x,y) \leq O_\mathcal{L}(x)$ and $\mathcal{O}_u(x,y) \geq O_\mathcal{L}(y)$.

- $\mathcal{O}$ is *symmetric*, if $\mathcal{O}(x,y) = (\mathcal{O}_l(x,y), \mathcal{O}_l(y,x))$ for some $\mathcal{O}_l : \mathcal{L}^2 \to \mathcal{L}$; $\mathcal{O}$ is *$\leq_i$-monotonic*, if when $(x_1,y_1) \leq_i (x_2,y_2)$, also $\mathcal{O}(x_1,y_1) \leq_i \mathcal{O}(x_2,y_2)$; $\mathcal{O}$ is *approximating*, if it is symmetric and $\leq_i$-monotonic.

Remark 1. One can define an approximating operator $\mathcal{O}$ without having to specify which operator $O_\mathcal{L}$ it approximates, and indeed it will often be convenient to study approximating operators without having to refer to the approximated operator. However, one can easily obtain all the operators $O_\mathcal{L}$ that $\mathcal{O}$ approximates by simply taking operators for which $\mathcal{O}_l(x,y) \leq O_\mathcal{L}(x)$ and $\mathcal{O}_u(x,y) \geq O_\mathcal{L}(x)$.

The *stable operator*, defined next, is used for expressing the semantics of many non-monotonic formalisms.

Definition 6. Given a lattice $L = \langle \mathcal{L}, \leq \rangle$, let $\mathcal{O} : \mathcal{L}^2 \to \mathcal{L}^2$ be an approximating operator.

- $\mathcal{O}_l(\cdot, y) = \lambda x. \mathcal{O}_l(x,y)$, i.e.: $\mathcal{O}_l(\cdot,y)(x) = \mathcal{O}_l(x,y)$.[6]
- The *complete stable operator for $\mathcal{O}$* is: $C(\mathcal{O})(z) = \mathit{lfp}(\mathcal{O}_l(\cdot,z)) = \min_\leq \{x \in \mathcal{L} \mid x = \mathcal{O}_l(x,z)\}$.
- The *stable operator for $\mathcal{O}$* is: $S(\mathcal{O})(x,y) = (C(\mathcal{O})(y), C(\mathcal{O})(x))$.

Stable operators capture the idea of minimizing truth, since for any $\leq_i$-monotonic operator $\mathcal{O}$ on $\mathcal{L}^2$, fixpoints of the stable operator $S(\mathcal{O})$ are $\leq_t$-minimal fixpoints of $\mathcal{O}$ (Denecker, Marek, and Truszczyński 2000, Theorem 4). This motivates the following notions:

- Kripke-Kleene fixpoint of $\mathcal{O}$: $\{(x,y) \in \mathcal{L}^2 \mid (x,y) = \mathit{lfp}_{\leq_i}(\mathcal{O}(x,y))\}$.
- three-valued stable models of $\mathcal{O}$: $\{(x,y) \in \mathcal{L}^2 \mid S(\mathcal{O})(x,y) = (x,y)\}$.
- two-valued stable models of $\mathcal{O}$: $\{(x,x) \in \mathcal{L}^2 \mid S(\mathcal{O})(x,x) = (x,x)\}$.
- the well-founded model of $\mathcal{O}$: the $\leq_i$-minimal (three-valued) stable model of $\mathcal{O}$.

In (Denecker, Marek, and Truszczyński 2000) it is shown that a unique well-founded model exists for any approximating operator. In (Pelov, Denecker, and Bruynooghe 2007) it is shown that for normal logic programs, the fixpoints based on the immediate consequence operator for a logic program give rise to the following correspondences: the three-valued stable models coincides with the three-valued semantics as defined by (Przymusinski 1990), the well-founded model coincides with the homonymous semantics as defined by (Przymusinski 1990; Van Gelder, Ross, and Schlipf 1991), and the two-valued stable models coincide with the two-valued (or total) stable models of a logic program.

3 Non-Deterministic Operators and Approximations

In order to characterize (two-valued) semantics for DLP, Pelov and Truszczynski (2004) introduced the notion of non-deterministic operators and accordingly extended AFT to non-deterministic AFT.

Definition 7. A *non-deterministic operator on $\mathcal{L}$* is a function $O_\mathcal{L} : \mathcal{L} \to \wp(\mathcal{L}) \setminus \{\emptyset\}$.

Example 1. The operator $IC_\mathcal{P}$ from Definition 1 is a non-deterministic operator on the lattice $\langle \wp(\mathcal{A_P}), \subseteq \rangle$.

Like deterministic AFT, non-deterministic AFT aims at approximating operators, this time non-deterministic ones, by *non-deterministic approximating operators* (ndao's, in short) $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$, producing a set of pairs of lower- and upper-bound approximations. These approximation are expressed by means of two operators $\mathcal{O}_l : \mathcal{L}^2 \to \wp(\mathcal{L})$ and $\mathcal{O}_u : \mathcal{L}^2 \to \wp(\mathcal{L})$. Unlike the deterministic case, however, the transition between the operator and its lower/upper approximations $\mathcal{O}_l$ and $\mathcal{O}_u$ is not entirely straightforward. In particular, one cannot always expect to represent an ndao $\mathcal{O}$ by $\mathcal{O}(x,y) = \{(l,u) \mid l \in \mathcal{O}_l(x,y), u \in \mathcal{O}_u(x,y)\}$.[7] This can be overcome by using a selection function $f : \wp(\mathcal{L}^2) \to \wp(\mathcal{L}^2)$ for $\mathcal{O}$, such that $f(\{(l,u) \mid l \in \mathcal{O}_l(x,y), u \in \mathcal{O}_u(x,y)\}) = \mathcal{O}(x,y)$ for any $x,y \in \mathcal{L}$.

Remark 2. For any operator $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$ one can define the functions $\mathcal{O}_l$ and $\mathcal{O}_u$ by letting $\mathcal{O}_l(w,z) = \{x \mid (x,y) \in \mathcal{O}(w,z)\}$ and $\mathcal{O}_u(w,z) = \{y \mid (x,y) \in \mathcal{O}(w,z)\}$. Clearly, there exists a selection function f s.t. $\mathcal{O}(w,z) = f(\{(l,u) \mid l \in \mathcal{O}_l(x,y), u \in \mathcal{O}_u(x,y)\})$. We will sometimes refer to f, $\mathcal{O}_l$, $\mathcal{O}_u$ without explicitly defining them.

As in the deterministic case, a bilattice structure is defined for the approximating operators. This is done by the following relation, known as *Smyth order* (Smyth 1976), which is used in the context of DLP in several works (e.g., (Fernández and Minker 1995; Alcântara, Damásio, and Pereira 2005)).

Definition 8. Let $L = \langle \mathcal{L}, \leq \rangle$ be a lattice.

- We denote: $\wp_\leq(\mathcal{L}) = \{X \subseteq \mathcal{L} \mid \min_\leq(X) = X\}$.
- Let $X, Y \in \wp(\mathcal{L})$. Then $X \preceq_L^S Y$ iff for every $y \in Y$ there is an $x \in X$ such that $x \leq y$.

Remark 3. It is easy to verify that $\preceq_L^S$ is reflexive and transitive on $\wp(\mathcal{L})$, and a partial order on $\wp_\leq(\mathcal{L})$.

The Smyth order is extended to $\wp_\leq(\mathcal{L}^2)$, yielding on it a bilattice structure (cf. Definition 4).

Definition 9. Given a lattice $L = \langle \mathcal{L}, \leq \rangle$,

- $\wp(\mathcal{L}^2)$ consists of all sets of pairs of elements of $\mathcal{L}$.
- $\wp_\leq(\mathcal{L}^2) = \{\mathbf{X} \in \wp(\mathcal{L}^2) \mid \{x \mid (x,y) \in \mathbf{X}\}, \{y \mid (x,y) \in \mathbf{X}\} \in \wp_\leq(\mathcal{L})\}$
- $\wp_{\leq_i}(\mathcal{L}^2) = \{\mathbf{X} \in \wp(\mathcal{L}^2) \mid \min_{\leq_i}(\mathbf{X}) = \mathbf{X}\}$.

Let $\mathbf{X}, \mathbf{Y} \in \wp(\mathcal{L}^2)$:

- $\mathbf{X} \preceq_i^S \mathbf{Y}$ iff for every $(y_1,y_2) \in \mathbf{Y}$ there is $(x_1,x_2) \in \mathbf{X}$ such that $(x_1,x_2) \leq_i (y_1,y_2)$.

[6] The choice of the lower bound is arbitrary here since, in view of the symmetry of $\mathcal{O}$, $\mathcal{O}_u(z,.) = \mathcal{O}_l(.,z)$.

[7] To see that not every ndao can be naively reconstructed by combining its lower and upper bounds, let $\mathcal{L} = \{1,2,3\}$ and suppose that $\mathcal{O}(1,2) = \{(1,3),(1,4),(2,3)\}$. If there were $\mathcal{O}_l, \mathcal{O}_u : \mathcal{L}^2 \to \wp(\mathcal{L})$ such that $\mathcal{O}(1,2) = \{(l,u) \mid l \in \mathcal{O}_l(1,2), u \in \mathcal{O}_u(1,2)\}$, then since $1,2 \in \mathcal{O}_l(1,2)$ and $3,4 \in \mathcal{O}_u(1,2)$, we would get that $(2,4) \in \mathcal{O}(1,2)$, contradicting the assumption on $\mathcal{O}$.

- $\mathbf{X} \preceq_t^S \mathbf{Y}$ iff for every $(y_1, y_2) \in \mathbf{Y}$ there is $(x_1, x_2) \in \mathbf{X}$ such that $(x_1, x_2) \leq_t (y_1, y_2)$.

The orders $\preceq_i^S$ and $\preceq_t^S$ are reflexive and transitive on $\wp(\mathcal{L}^2)$, while $\preceq_i^S$ is also anti-symmetric on $\wp_{\leq_i}(\mathcal{L}^2)$. Both $\preceq_i^S$ and $\preceq_t^S$ are anti-symmetric on $\wp(\mathcal{L}^2)$.

We can now define what non-deterministic approximating operators are.

Definition 10. Given a lattice $L = \langle \mathcal{L}, \leq \rangle$ and an operator $\mathcal{O} : \mathcal{L}^2 \to \wp_{\leq_i}(\mathcal{L}^2)$. Then:

- $\mathcal{O}$ is *symmetric*, if there is an $\mathcal{O}_l : \mathcal{L}^2 \to \wp_{\leq}(\mathcal{L})$ and a selection function f, such that for every $x, y \in \mathcal{L}$, $\mathcal{O}(x, y) = f(\{(l, u) \mid l \in \mathcal{O}_l(x, y), u \in \mathcal{O}_l(y, x)\})$.
- $\mathcal{O}$ is a *non-deterministic approximating operator* (ndao) iff it is symmetric and $\preceq_i^S$-monotonic in the sense of Definition 5.

A ndao $\mathcal{O}$ is *an approximation of* $O_{\mathcal{L}} : \mathcal{L} \to \wp(\mathcal{L})$ iff $\mathcal{O}(x, x) = f(O_{\mathcal{L}}(x), O_{\mathcal{L}}(x))$ for every $x \in \mathcal{L}$. In that case we say that $\mathcal{O}$ *approximates*[8] $O_{\mathcal{L}}$. A pair $(x, y) \in \mathcal{L}^2$ is *consistent* if $x \leq y$. A ndao $\mathcal{O}$ is *consistent*, if for every consistent $(x, y) \in \mathcal{L}^2$, all the elements in $\mathcal{O}(x, y)$ are consistent.

Example 2. For a dlp $\mathcal{P}$ and an interpretation (x, y), we define the following operators:

$\mathcal{HR}^l_{\mathcal{P}}(x, y) = \{\Delta \mid \bigvee\Delta \leftarrow \phi \in \mathcal{P}, (x, y)(\phi) \in \{\mathsf{T}, \mathsf{C}\}\}$,
$\mathcal{IC}^l_{\mathcal{P}}(x, y) = \min_{\subseteq}(\{v \mid \forall \Delta \in \mathcal{HR}^l_{\mathcal{P}}(x, y),\ v \cap \Delta \neq \emptyset\})$,
$\mathcal{IC}_{\mathcal{P}}(x, y) = (\mathcal{IC}^l_{\mathcal{P}}(x, y), \mathcal{IC}^l_{\mathcal{P}}(y, x))$.
$\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}(x, y) = \{(x, y) \in \mathcal{IC}_{\mathcal{P}}(x, y) \mid x \subseteq y\}$.

It can be verified that the operator $\mathcal{IC}_{\mathcal{P}}$ defined above is an approximation of the non-deterministic operator $IC_{\mathcal{P}}$ in Example 1 (and Definition 1), and that $\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}$ is consistent.

The following lemma shows that an ndao is composed of a $\preceq_L^S$-monotonic lower-bound operator and a $\preceq_L^S$-anti-monotonic upper-bound operator:

Lemma 1. *A symmetric operator $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$ is $\preceq_i^S$-monotonic iff for every $z \in \mathcal{L}$, $\mathcal{O}_l(\cdot, z)$ is $\preceq_L^S$-monotonic and $\mathcal{O}_l(z, \cdot)$ is anti-$\preceq_L^S$ monotonic.*

4 Theory of Non-Deterministic AFT

We now develop a general theory of approximation of non-deterministic operators. Given a non-deterministic operator on the lattice $L = \langle \mathcal{L}, \leq \rangle$, we are interested in approximating fixpoints of this operator by looking at fixpoints of a corresponding approximating operator on the bilattice $L^2 = \langle \mathcal{L}^2, \preceq_i^S, \preceq_t^S \rangle$. By introducing the information order $\preceq_i^S$, formulating an $\preceq_i^S$-monotonic approximating operator guarantees the existence of such fixpoints. The $\preceq_i^S$-least such fixpoint is the *Kripke-Kleene-fixpoint*, considered in Section 4.2. More precise approximations can often be obtained based on complete operators, including the well-founded fixpoint. We study such fixpoints in Section 4.3. First, in the next section, we consider the consistency property of the above-mentioned fixpoints.

4.1 Consistency of Approximations

For deterministic operators, consistency of any approximating operator is guaranteed by Proposition 9 in (Denecker, Marek, and Truszczyński 2000). However, this is no longer the case when non-determinism is introduced.[9] To avoid the inclusion of inconsistent pairs, we define a consistent version $\mathcal{O}^{\mathsf{cons}}$ of a non-deterministic operator $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$ as follows: for $\mathbf{X} \subseteq \mathcal{L}^2$, we let $f^{\mathsf{cons}}(\mathbf{X}) = \{(x, y) \in \mathbf{X} \mid x \leq y\}$, and:

$$\mathcal{O}^{\mathsf{cons}}(x, y) = f^{\mathsf{cons}}(\mathcal{O}(x, y)).$$

Clearly, $\mathcal{O}^{\mathsf{cons}}$ is nonempty and consistent for any ndao $\mathcal{O}$. We observe that $\mathcal{O}^{\mathsf{cons}}$ is itself an approximating operator, and it approximates an operator $O_{\mathcal{L}}$ if so does $\mathcal{O}$. It follows, then, that (even though ndaos are not guaranteed to be consistent) one can always construct, for any ndao $\mathcal{O}$, an ndao $\mathcal{O}^{\mathsf{cons}}$ that is consistent. See, for instance, the operator $\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}$ in Example 2.

Proposition 1. *Given a lattice $L = \langle \mathcal{L}, \leq \rangle$ and an ndao $\mathcal{O}$, then:*

1. *$\mathcal{O}^{\mathsf{cons}}$ is an ndao.*
2. *$\mathcal{O}^{\mathsf{cons}}$ approximates an operator $O_{\mathcal{L}} : \mathcal{L} \to \wp(\mathcal{L})$ if $\mathcal{O}$ approximates $O_{\mathcal{L}}$.*

Proof. Item 1: For $\preceq_i^S$-monotonicity, let $(x, y) \leq_i (w, z)$ and $(w', z') \in \mathcal{O}^{\mathsf{cons}}(w, z)$. Then, since $(w', z') \in \mathcal{O}(w, z)$ and $\mathcal{O}$ is $\preceq_i^S$-monotonic, there is some $(x', y') \in \mathcal{O}(x, y)$ s.t. $(x', y') \leq_i (w', z')$, i.e., $x' \leq w'$ and $z' \leq y'$. Since (w', z') is consistent, $w' \leq z'$, thus by transitivity $x' \leq y'$, and so $(x', y') \in \mathcal{O}^{\mathsf{cons}}(x, y)$. For symmetry, suppose that $(w, z) \in \mathcal{O}^{\mathsf{cons}}(x, y)$, i.e., $w \leq z$, $w \in \mathcal{O}_l(x, y)$ and $z \in \mathcal{O}_u(x, y)$. By the symmetry of $\mathcal{O}$, $z \in \mathcal{O}_l(y, x)$.
Item 2: If $\mathcal{O}$ approximates $O_{\mathcal{L}} : \mathcal{L} \to \wp(\mathcal{L})$, i.e., for every $x \in \mathcal{L}$, $\mathcal{O}(x, x) = f(O_{\mathcal{L}}(x), O_{\mathcal{L}}(x))$, then $\mathcal{O}^{\mathsf{cons}}(x, x) = \{(w, z) \mid (w, z) \in f(O_{\mathcal{L}}(x), O_{\mathcal{L}}(x))$ and $w \leq z\}$. □

4.2 Kripke-Kleene Semantics

Recall that the Kripke-Kleene fixpoint of a *deterministic* approximating operator is the $\leq_i$-least fixpoint of $\mathcal{O}$, which, as shown in (Denecker, Marek, and Truszczyński 2000), is unique, consistent and always exists. Now, while consistency is assured also when moving to non-deterministic operators, the next example shows that the other properties are no longer guaranteed.

Example 3. Consider the dlp $\mathcal{P} = \{p \vee q \leftarrow\}$. There are *two* $\leq_i$-minimal consistent fixpoints of $\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}$ (Example 2): $(\{p\}, \{p\})$ and $(\{q\}, \{q\})$.

To see that $\leq_i$-minimal consistent fixpoints of $\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}$ may not exist, consider the following dlp (taken from (Przymusinski 1991)):

$$\mathcal{P} = \{p \vee q \vee r;\ p \leftarrow \neg q;\ r \leftarrow \neg p;\ q \leftarrow \neg r\}.$$

[8] Sometimes "extend" is used instead of "approximate". Notice that in the deterministic case, approximation of an operator according to our definition is weaker than approximation according to Definition 5.

[9] For instance, if $\mathcal{O} = \mathcal{O}_l \times \mathcal{O}_u$ and $\mathcal{O}_l(1, 2) = \mathcal{O}_u(1, 2) = \{1, 2\}$, then $(2, 1) \in \mathcal{O}(1, 2)$, but this is not a consistent pair.

We show that there is no (x, y) with $x \subseteq y \subseteq \{p, q, r\}$ s.t. $(x, y) \in \mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}(x, y)$. We consider three cases (the other cases are analogous or follow by symmetry):

- $(\emptyset, \{p, q, r\})$. Recall that this pair encodes the interpretation where p, q, r are assigned the value U. Note that $\{p, q, r\} \in \mathcal{HR}^{l}_{\mathcal{P}}(\emptyset, \{p, q, r\})$, and so $\emptyset \notin \mathcal{IC}^{l}_{\mathcal{P}}(\emptyset, \{p, q, r\})$.
- $(\{p\}, \{p, q, r\})$. $\mathcal{HR}^{u}_{\mathcal{P}}(\{p\}, \{p, q, r\}) = \{p \vee q \vee r, p, q\}$ and $\{p, q, r\} \supset \{p, q\} \in \mathcal{IC}^{u}_{\mathcal{P}}(\{p\}, \{p, q, r\})$, thus by the minimality of the models in $\mathcal{IC}^{u}_{\mathcal{P}}$, $\{p, q, r\} \notin \mathcal{IC}^{u}_{\mathcal{P}}(\{p\}, \{p, q, r\})$.
- $(\{p, q\}, \{p, q\})$. $\mathcal{HR}^{l}_{\mathcal{P}}(\{p, q\}, \{p, q\}) = \{p \vee q \vee r, q\}$ and thus $\{q\} \in \mathcal{IC}^{l}_{\mathcal{P}}(\{p, q\}, \{p, q\})$, which prohibits $\{p, q\} \in \mathcal{IC}^{l}_{\mathcal{P}}(\{p, q\}, \{p, q\})$.

To obtain an analogue to the Kripke-Kleene fixpoint for ndaos, which is guaranteed to be the unique $\preceq^S_i$-least one, we observe that $(\wp_{\leq_i}(\mathcal{L}^2), \preceq^S_i)$ is a partial order, and so we can generalize the idea behind the usual proof that shows there is a unique fixpoint, if we have a $\preceq^S_i$-monotonic operator $\mathcal{O}' : \wp(\mathcal{L}^2) \to \wp_{\leq_i}(\mathcal{L}^2)$. Such an operator aims at correlating the arguments and the values of an ndao $\mathcal{O} : \mathcal{L}^2 \to \wp_{\leq_i}(\mathcal{L}^2)$. More specifically, we define an operator $\mathcal{O}'$ on the basis of $\mathcal{O}$ as follows:

Definition 11. Given an operator $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$, the operator $\mathcal{O}' : \wp(\mathcal{L}^2) \to \wp_{\leq_i}(\mathcal{L}^2)$ is defined, for every $\mathbf{X} \in \wp(\mathcal{L}^2)$, by:[10]

$$\mathcal{O}'(\mathbf{X}) = \min_{\leq_i} \bigcup_{(x_1, x_2) \in \mathbf{X}} \mathcal{O}(x_1, x_2).$$

Remark 4. Sets of three-valued interpretations as in the construction of the operator $\mathcal{O}'$ are a direct generalization of the idea of *Herbrand states* (Lobo, Minker, and Rajasekar 1992), which are sets of disjunctions. Sets of three-valued interpretations are thus a semantical, three-valued counterpart of Herbrand states. States are a central concept in formulating many generalizations of the well-founded semantics for disjunctive logic programming, e.g., as in (Baral, Lobo, and Minker 1992) or (Brass and Dix 1995b).

As we show in Proposition 2 below, the operator $\mathcal{O}'$ admits a unique $\preceq^S_i$-minimal fixpoint for any ndao $\mathcal{O}$. For this, we first need some preliminaries:

Definition 12. A set $\mathbf{X} \in \wp(\mathcal{L}^2)$ is a *pre-fixpoint* of an operator $\mathcal{O}' : \wp(\mathcal{L}^2) \to \wp(\mathcal{L}^2)$ iff $\mathcal{O}'(\mathbf{X}) \preceq^S_i \mathbf{X}$.

Lemma 2. *Let $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$ be $\preceq^S_i$-monotonic operator and let $\mathcal{O}' : \wp(\mathcal{L}^2) \to \wp_{\leq_i}(\mathcal{L}^2)$ be the operator from from Definition 11. Then (1) $\mathcal{O}'$ is $\preceq^S_i$-monotonic, and (2) if $\mathbf{X} \in \wp(\mathcal{L}^2)$ is a $\preceq^S_i$-minimal pre-fixpoint of $\mathcal{O}'$ then $\mathbf{X}$ is a $\preceq^S_i$-minimal fixpoint of $\mathcal{O}'$.*

Lemma 3. *If $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$ is a $\preceq^S_i$-monotonic operator, then $\mathcal{O}'$ has a $\preceq^S_i$-minimal pre-fixpoint.*

Proof outline: Denote by $\bot$ and $\top$ the $\leq$-minimal and the $\leq$-maximal elements (respectively) of $\mathcal{L}$. Then, by Lemma 2, and since $\{(\bot, \top)\} \preceq^S_i \mathcal{O}'(\{(\bot, \top)\})$, we can construct a $\preceq^S_i$-chain, starting at $\{(\bot, \top)\}$, by iteratively applying $\mathcal{O}'$. By (Smyth 1976, Theorem 4) this chain has a supremum $(\mathcal{O}')^{\beta}(\{(\bot, \top)\})$, (for some ordinal β) which is pre-fixpoint of $\mathcal{O}'$.

To show $\preceq^S_i$-minimality of $(\mathcal{O}')^{\beta}(\{(\bot, \top)\})$, suppose that there is $\mathbf{Y} \in \wp(\mathcal{L}^2)$ such that $\mathbf{Y} \prec^S_i (\mathcal{O}')^{\beta}(\{(\bot, \top)\})$ and $\mathcal{O}'(\mathbf{Y}) \prec^S_i \mathbf{Y}$. Since $\{(\bot, \top)\} \preceq^S_i \mathbf{Y}$, by the $\preceq^S_i$-monotonicity of $\mathcal{O}'$ (Lemma 2), $\mathcal{O}'(\{(\bot, \top)\}) \preceq^S_i \mathcal{O}'(\mathbf{Y})$ we get: $(\mathcal{O}')^{\beta}(\{(\bot, \top)\}) \preceq^S_i (\mathcal{O}')^{\beta}(\mathbf{Y})$. Since Y is a prefixpoint of the operator $\mathcal{O}'$, this implies that $(\mathcal{O}')^{\beta}(\{(\bot, \top)\}) \preceq^S_i \mathbf{Y}$, contradicting the assumption that $\mathbf{Y} \prec^S_i (\mathcal{O}')^{\beta}(\{(\bot, \top)\})$. □

Definition 13. The $\preceq^S_i$-least fixpoint of the operator $\mathcal{O}' : \wp(\mathcal{L}^2) \to \wp_{\leq_i}(\mathcal{L}^2)$ is called the *Kripke-Kleene-state of* $\mathcal{O}$, and is denoted $\mathsf{K}(\mathcal{O})$. We say that $\mathsf{K}(\mathcal{O})$ (or, in general, a state) is *consistent*, if for every $(x, y) \in \mathsf{K}(\mathcal{O})$, $x \leq y$.

Example 4. Consider the following dlp:

$$\mathcal{P} = \{a \vee b \leftarrow \neg c;\ c \leftarrow d;\ d \leftarrow c\}.$$

Construction of the Kripke-Kleene state for $\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}$:

$(\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}})'^{1}(\emptyset, \mathcal{A}_{\mathcal{P}}) = \{(\emptyset, \{a, c, d\}), (\emptyset, \{b, c, d\})\}$.
$(\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}})'^{2}(\emptyset, \mathcal{A}_{\mathcal{P}}) = (\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}})'^{1}(\emptyset, \mathcal{A}_{\mathcal{P}})$.

Proposition 2. *Let $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$ be a consistent ndao for an operator $O_{\mathcal{L}} : \mathcal{L} \to \wp(\mathcal{L})$. Then the Kripke-Kleene-state $\mathsf{K}(\mathcal{O})$ of $\mathcal{O}$ exists, is consistent, and approximates every fixpoint of $O_{\mathcal{L}}$ (i.e., for every $x \in \mathcal{L}$ such that $x \in O_{\mathcal{L}}(x)$ there is a $(z, y) \in \mathsf{K}(\mathcal{O})$ such that $z \leq x \leq y$, or equivalently, $\mathsf{K}(\mathcal{O}) \preceq^S_i \{(x, x)\}$).*

Proof. Let $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$ be an ndao for a non-deterministic operator $O_{\mathcal{L}} : \mathcal{L} \to \wp(\mathcal{L})$. By Lemmas 3 and 2, $\mathsf{K}(\mathcal{O})$ is the $\preceq^S_i$-least fixpoint of $\mathcal{O}'$. We now show $\mathsf{K}(\mathcal{O})$ approximates every fixpoint of $O_{\mathcal{L}}$. Indeed, take some $x \in \mathcal{L}$ s.t. $x \in O_{\mathcal{L}}(x)$. Clearly, $(\bot, \top) \leq_i (x, x)$. By the $\preceq^S_i$-monotonicity of $\mathcal{O}$, we get $\mathcal{O}(\bot, \top) \preceq^S_i \mathcal{O}(x, x)$. Since $x \in O_{\mathcal{L}}(x)$ and $\mathcal{O}$ approximates $O_{\mathcal{L}}$, $\mathcal{O}(\bot, \top) \preceq^S_i \{(x, x)\}$. Keeping this iteration, we have that for any ordinal β, $\mathsf{K}(\mathcal{O}) \preceq^S_i \{(x, x)\}$. Consistency follows from the fact that $\mathcal{O}$ is consistent. □

Remark 5. Consider the following dlp:

$$\mathcal{P}^* = \{p \vee s \leftarrow;\ p \leftarrow s;\ s \leftarrow p\}.$$

Here, $\mathsf{K}(\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}) = \{(\{p\}, \{p, s\}), (\{s\}, \{p, s\})\}$, but the unique stable model of $\mathcal{P}$ is $(\{p, s\}, \{p, s\})$. Intuitively, this is so since $\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}(\{s\}, \{p, s\}) = \{(\{p\}, \{p, s\})\}$ and $\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}(\{p\}, \{p, s\}) = \{(\{s\}, \{p, s\})\}$, and thus a fixed point of $\mathcal{O}'$ is reached even though $\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}$ oscillates between the different members of this fixed point. This can be solved by defining the following cumulative operator for an approximating operator $\mathcal{O}$ (where $\sqcup$ is the join operator of L and $\mathbf{X} \in \wp(\mathcal{L}^2)$):

$$\mathcal{O}_c(x, y) = \bigcup_{(w,z) \in \mathcal{O}(x,y)} (x \sqcup w, z), \quad \mathcal{O}'_c(\mathbf{X}) = \min_{\leq_i} \bigcup_{(x,y) \in \mathbf{X}} \mathcal{O}_c(x, y).$$

[10] This is equivalent to defining $\mathcal{O}'(\mathbf{X}) = \min_{\leq}(f(\{(x, y) \mid x \in \mathcal{O}_l(w, z), y \in \mathcal{O}_u(w, z), (w, z) \in \mathbf{X}\}))$ for some selection function f.

Intuitively, we now make cumulative approximations in the sense that the previous approximation (x) is incorporated in the new approximation $(w,z) \in \mathcal{O}(x,y)$.[11] One can verify that:

1. $\mathcal{O}'_c$ is a $\preceq_i^S$-monotonic operator if so is $\mathcal{O}$ (e.g., since $\mathcal{O}$ is an ndao). Consequently, by Lemmas 2 and 3, $\mathcal{O}'_c$ admits a $\preceq_i^S$-minimal fixpoint, denoted $\mathsf{K}_c(\mathcal{O})$.
2. $\mathsf{K}_c(\mathcal{O})$ is more informative than $\mathsf{K}(\mathcal{O})$ but less informative than any fixpoint of $\mathcal{O}$: For any $\preceq_i^S$-monotonic ndao $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$, $\mathsf{K}(\mathcal{O}) \preceq_i^S \mathsf{K}_c(\mathcal{O}) \preceq_i^S \{(x,y)\}$ for any $(x,y) \in \mathcal{O}(x,y)$.
3. If $\mathcal{O}$ is an approximation of $O_\mathcal{L}$, then any fixpoint of $O_\mathcal{L}$ will be approximated by $\mathcal{O}_c$. That is, if $x \in O_\mathcal{L}(x)$ then $(x,x) \in \mathcal{O}_c(x,x)$.

Consider now again the dlp $\mathcal{P}^*$ defined above. Then $(\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}^*})'^{1}_c(\emptyset, \mathcal{A}_{\mathcal{P}^*}) = \{(\{p\},\{p,s\}),(\{s\},\{p,s\})\}$, and after the next iteration the model of $\mathcal{P}^*$ is reached: $(\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}^*})'^{2}_c(\emptyset, \mathcal{A}_{\mathcal{P}^*}) = \{(\{p,s\},\{p,s\})\}$.

Another possible generalization of Kripke-Kleene fixpoints to the non-deterministic setting is to simply say that (x,y) is a *Kripke-Kleene interpretation* of $\mathcal{O}$ iff (x,y) is a $\leq_i$-minimal fixpoint of $\mathcal{O}$. As noted above, existence and uniqueness of Kripke-Kleene interpretations is not guaranteed. Yet, as we will see in Section 5, such operators do play a role in applications to logic programming. The following proposition expresses the relation between the Kripke-Kleene state, Kripke-Kleene interpretations and other fixpoints of $\mathcal{O}$.

Proposition 3. *Given an ndao $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$, if there is a Kripke-Kleene (KK) interpretation for $\mathcal{O}$, then we have:*

1. $\mathsf{K}(\mathcal{O}) \preceq_i^S \{(x,y)\}$ *for any KK-interpretation (x,y).*
2. *for every fixpoint (x,y) of $\mathcal{O}$ there is a KK-interpretation (w,z) of $\mathcal{O}$ s.t. $(w,z) \leq_i (x,y)$.*
3. *any fixpoint of $\mathcal{O}$ (and so also any KK-interpretation) is consistent, if $\mathcal{O}$ is consistent.*

Proof. For Item 1, let β be the minimal ordinal s.t. $(\mathcal{O}')^\beta(\{(\bot,\top)\}) = \mathsf{K}(\mathcal{O})$. Since $(\bot,\top) \leq_i (x,y)$, by the $\preceq_i^S$-monotonicity of $\mathcal{O}$ (and thus of $\mathcal{O}'$, by Lemma 2), $(\mathcal{O}')^\beta(\{(\bot,\top)\}) \preceq_i^S (\mathcal{O}')^\beta(\{(x,y)\})$. Since (x,y) is a fixpoint of $\mathcal{O}$, $(x,y) \in (\mathcal{O}')^\beta(\{(x,y)\})$ (by (a transfinite) induction on β). Thus, $(\mathcal{O}')^\beta(\{(\bot,\top)\}) = \mathsf{K}(\mathcal{O}) \preceq_i^S \{(x,y)\}$.

Item 2 is immediate since a Kripke-Kleene interpretation is $\leq_i$-minimal, and Item 3 follows from the fact that $\mathcal{O}$ is assumed to be consistent. $\square$

Remark 6. When $\mathcal{O}$ is a deterministic operator, $\mathsf{K}(\mathcal{O})$ coincides with the unique Kripke-Kleene interpretation, which is guaranteed to exist, and which also coincides with the Kripke-Kleene fixpoint of $\mathcal{O}$ according to (Denecker, Marek, and Truszczyński 2000).

[11] In general, $\mathcal{O}'_c$ may give rise to self-supporting conclusions (e.g., $(\mathcal{IC}^{\mathsf{cons}}_{\emptyset})_c(\{p\},\{p\}) = (\{p\},\emptyset)$) and thus it should not be seen as useful for obtaining *any* fixpoint of $\mathcal{O}$. However, when iteratively applying it to $(\emptyset, \mathcal{A}_\mathcal{P})$, such self-supporting behaviour will not occur.

4.3 Stable Non-Deterministic Operators

We now turn to stable operators in non-deterministic AFT (cf. Definition 6). The motivation for the definition of this operator resembles the one in the deterministic case (recall Definition 6). Given an ndao $\mathcal{O}$, instead of refining an approximation (x,y) by $\mathcal{O}(x,y) = f(\{(l,u) \mid l \in \mathcal{O}_l(x,y), u \in \mathcal{O}_l(y,x)\})$, we consider more precise bounds: Since $\mathcal{O}_l(.,z)$ is a $\preceq_L^S$-monotonic operator for any $z \in \mathcal{L}$, one may improve the lower bound estimation by computing the least-fixed point of $\mathcal{O}_l(.,y)$, if it exists, and similarly x can be used to obtain an improved estimation of the upper bound by taking the $\preceq_L^S$-minimal fixpoint of $\mathcal{O}_u(.,x)$. By symmetry, we can equivalently consider $\mathcal{O}_l(x,.)$. The complete stable operator $C(\mathcal{O})$ and the stable operator $S(\mathcal{O})$ are then constructed as in the next definition:

Definition 14. Let $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$ be an ndao on $\mathcal{L}^2$ such that $\mathcal{O}(x,y) = f(\mathcal{O}_l(x,y), \mathcal{O}_l(y,x))$.

- The *complete stable operator for* $\mathcal{O}$ is:
  $C(\mathcal{O})(y) = \mathit{lfp}(\mathcal{O}_l(\cdot,y)) = \min_\leq\{x \in \mathcal{L} \mid x \in \mathcal{O}_l(x,y)\}$.
- The *stable operator for $\mathcal{O}$:*
  $S(\mathcal{O})(x,y) = f(C(\mathcal{O})(y), C(\mathcal{O})(x))$.
- A *stable fixpoint of $\mathcal{O}$:*
  a fixpoint of $S(\mathcal{O})$, i.e., (x,y) s.t. $(x,y) \in S(\mathcal{O})(x,y)$.

Example 5. Let $\mathcal{P} = \{p \vee q \leftarrow;\ r \leftarrow \neg p;\ r \leftarrow \neg q\}$. Then: $C(\mathcal{IC}^{\mathsf{cons}}_\mathcal{P})(\{p,r\}) = \mathit{lfp}(\mathcal{IC}^l_\mathcal{P}(.,\{p,r\}) = \{\{p,r\},\{q,r\}\}$, $S(\mathcal{IC}^{\mathsf{cons}}_\mathcal{P})(\{p,r\},\{p,r\}) = \{(x,y) \mid x,y \in \{\{p,r\}, \{q,r\}\}, x \subseteq y\} = \{(\{p,r\},\{p,r\}),(\{q,r\},\{q,r\})\}$. Thus, $(\{p,r\},\{p,r\})$ is a stable fixpoint of $\mathcal{IC}^{\mathsf{cons}}_\mathcal{P}$. We note that it is also a stable model of $\mathcal{P}$. As we shall see in Section 5, this is not a coincidence.

Stable fixpoints of $\mathcal{O}$ give rise to fixpoints of $\mathcal{O}$ that are minimal with respect to the truth-ordering:

Proposition 4. *Let $L = \langle \mathcal{L}, \leq \rangle$ be a complete lattice and let $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$ be a $\preceq_i^S$-monotonic operator. Every fixpoint of $S(\mathcal{O})$ is a $\preceq_t^S$-minimal fixpoint of $\mathcal{O}$.*[12]

Proof. Suppose that $(x,y) \in S(\mathcal{O})$, i.e., $x \in C(\mathcal{O})(y)$ and $y \in C(\mathcal{O})(x)$. Thus $x \in \mathit{lfp}(\mathcal{O}_l(\cdot,y))$, so $x \in \mathcal{O}_l(x,y)$. Likewise, $y \in \mathcal{O}_u(x,y)$, and so $(x,y) \in \mathcal{O}(x,y)$.

To see that (x,y) is a $\preceq_t^S$-minimal fixpoint of $\mathcal{O}$, suppose that there is some $(x',y') \in \mathcal{O}(x',y')$ s.t. $(x',y') \leq_t (x,y)$, i.e., $x' \leq x$ and $y' \leq y$. By Lemma 1, $\mathcal{O}_u(\cdot,y')$ (or equivalently, $\mathcal{O}_l(y',.)$ by symmetry) is $\preceq_L^S$-monotonic, and so $\mathcal{O}_u(x,y') \preceq_L^S \mathcal{O}_u(x',y')$. Thus (since $y' \in \mathcal{O}_u(x',y')$ in view of $(x',y') \in \mathcal{O}(x',y')$) there is a $w \in \mathcal{O}_u(x,y')$ s.t. $w \leq y'$. Since $y' \leq y$, by transitivity, $w \leq y$. Since $y \in \mathit{lfp}(\mathcal{O}_u(x,.)) = \mathit{lfp}(\mathcal{O}_l(.,x))$, $y \leq w$ and thus $y = y'$. Similarly, we can show that $x = x'$. $\square$

Unfortunately, the complete stable operator might not be available, since there might be operators $\mathcal{O}$ for which no $\preceq_L^S$-least fixpoint of $\mathcal{O}_l(.,x)$ exists for some $x \in \mathcal{L}$. A case in point is Example 3. Nevertheless, we will see below that the

[12] This is a generalization to ndao's of a similar result from (Denecker, Marek, and Truszczyński 2000).

stable operator is useful in applications to knowledge representation. In Definition 16 we overcome the possible non-existence of the $\preceq_L^S$-least fixpoint of $\mathcal{O}_l(.,x)$. For this, we first need to trade the Smyth order $\preceq_L^S$ (recall Definition 8) by (the stronger) Plotkin order $\preceq_L^P$ (Plotkin 1976):

Definition 15. Given a lattice $L = \langle \mathcal{L}, \leq \rangle$, $\mathbf{X}, \mathbf{Y} \in \wp_{\leq}(\mathcal{L}^2)$,[13] and $X, Y \in \wp_{\leq}(\mathcal{L})$. We define:

- $X \preceq_L^H Y$ iff for every $x \in X$, there is $y \in Y$ s.t. $x \leq y$.
- $X \preceq_L^P Y$ iff $X \preceq_L^S Y$ and $X \preceq_L^H Y$.
- $\mathbf{X} \preceq_i^H \mathbf{Y}$ iff for every $(x, y) \in \mathbf{X}$, there is $(w, z) \in \mathbf{Y}$ s.t. $(x, y) \leq_i (w, z)$.
- $\mathbf{X} \preceq_i^P \mathbf{Y}$ iff $\mathbf{X} \preceq_i^S \mathbf{Y}$ and $\mathbf{X} \preceq_i^H \mathbf{Y}$.

It is easy to verify that Plotkin order $\preceq_L^P$ is reflexive, transitive and anti-symmetric over $\wp_{\leq}(\mathcal{L}^2)$.

Definition 16. A *non-deterministic state approximating operator* (ndsao, in short) is a $\preceq_i^P$-monotonic operator $\mathcal{O}^s : \wp_{\leq}(\mathcal{L}^2) \to \wp_{\leq}(\mathcal{L}^2)$ that can be decomposed by $\mathcal{O}^s(\mathbf{X}) = f(\mathcal{O}_l^s(\mathbf{X}), \mathcal{O}_u^s(\mathbf{X}))$ for some $\mathcal{O}_l^s, \mathcal{O}_u^s : \wp_{\leq}(\mathcal{L}^2) \to \wp_{\leq}(\mathcal{L})$ and a selection function f, such that $\mathcal{O}_u^s(\mathbf{X}) = \mathcal{O}_l^s(\{y \mid (x, y) \in \mathbf{X}\}, \{x \mid (x, y) \in \mathbf{X}\})$ for every $\mathbf{X} \in \wp_{\leq}(\mathcal{L}^2)$.

We say that an ndsao $\mathcal{O}^s$ *approximates* an ndao $\mathcal{O}$ iff $\mathcal{O}^s(\{(x, y)\}) = \mathcal{O}(x, y)$ for every $x, y \in \mathcal{L}$, and that $\mathcal{O}^s$ approximates an operator $O_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}$ iff $\mathcal{O}^s(\{(x, x)\}) = \{(O_{\mathcal{L}}(x), O_{\mathcal{L}}(x))\}$ for every $x \in \mathcal{L}$.

It is clear that if an ndsao $\mathcal{O}^s$ approximates an ndao $\mathcal{O}$ that on its turn approximates $O_{\mathcal{L}}$, then $\mathcal{O}^s$ approximates $O_{\mathcal{L}}$.

Example 6. $\mathcal{IC}_{\mathcal{P}}^s(X, Y) = (\mathcal{IC}_{\mathcal{P}}^{s,l}(X, Y), \mathcal{IC}_{\mathcal{P}}^{s,l}(Y, X))$, where $\mathcal{IC}_{\mathcal{P}}^{s,l}(X, Y) = \min_{\subseteq}(\mathcal{IC}_{\mathcal{P}}^l(\bigcap X, \bigcup Y))$ (for every $X, Y \subseteq \mathcal{L}$) is a $\preceq_i^P$-monotonic operator, and approximates $\mathcal{IC}_{\mathcal{P}}$.[14] We observe, however, that $\mathcal{IC}'_{\mathcal{P}}$ (Definition 11) is *not* an ndsao, since it is not $\preceq_i^P$-monotonic over $\wp_{\leq}(\mathcal{L}^2)$, and neither is $\min_{\leq} \mathcal{IC}'_{\mathcal{P}}$ for that matter.

Notions like $\preceq_i^P$-monotonicity of an ndsao are completely analogous to the $\preceq^S$-based notions.

We now study stable operators based on ndsao's. First, we characterize ndsao's in terms of their component operators $\mathcal{O}_l$ and $\mathcal{O}_u$ (cf. Lemma 1).

Lemma 4. *A symmetric operator $\mathcal{O}^s : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$ is $\preceq_i^P$-monotonic iff for every $Z \subseteq \mathcal{L}$, $\mathcal{O}_l^s(\cdot, Z)$ is $\preceq_L^P$-monotonic and $\mathcal{O}_l^s(Z, \cdot)$ is anti-$\preceq_L^P$-monotonic.*

By Lemma 4, if $\mathcal{O}^s$ is $\preceq_i^P$-monotonic then $\mathcal{O}_l^s(., Y)$ is a $\preceq_L^P$-monotonic operator for any $Y \subseteq \mathcal{L}$. From this and the fact that $(\wp_{\leq}(\mathcal{L}), \preceq_L^P)$ forms a lattice, it then follows with the Tarski-Knaster theorem that $\mathcal{O}_l^s$ *does* admit a $\preceq_L^P$-least fixpoint. We can then define the *complete stable state operator* $C(\mathcal{O}^s)$ (for any $X \in \wp_{\leq}(\mathcal{L})$) as follows:

$$\begin{aligned} C(\mathcal{O}^s)(X) &= \mathit{lfp}_{\preceq_L^P}(\mathcal{O}_l^s(., X)) \\ &= \min_{\preceq_L^P}\{Y \in \wp_{\leq}(\mathcal{L}) \mid Y = \mathcal{O}_l^s(Y, X)\}. \end{aligned}$$

Thus, just like the stable complete operator, the stable complete state operator calculates a new lower bound based on the upper bound X, only that now this is done on the basis of a set of upper bounds X instead of a single upper bound x. Likewise, the *stable state operator* $S(\mathcal{O}^s)$ is defined in analogy to the stable operator (cf. Definition 14) as follows: for any $\mathbf{X} \in \wp_{\leq}(\mathcal{L}^2)$,[15]

$$\begin{aligned} S(\mathcal{O}^s)(\mathbf{X}) = f(&C(\mathcal{O}^s)(\{z \mid (w, z) \in \mathbf{X}\}), \\ &C(\mathcal{O}^s)(\{w \mid (w, z) \in \mathbf{X}\})) \end{aligned}$$

The following are important results on the complete stable and stable state operators of ndsao's:

Lemma 5. *Given a complete lattice $L = \langle \mathcal{L}, \leq \rangle$ and an ndsao $\mathcal{O}^s : \wp_{\leq}(\mathcal{L}^2) \to \wp_{\leq}(\mathcal{L}^2)$. Then:*

- *the complete stable state operator $C(\mathcal{O}^s)$ is $\preceq_L^P$-anti-monotonic.*
- *the stable state operator $S(\mathcal{O}^s)$ is $\preceq_i^P$-monotonic.*
- *a fixpoint of $S(\mathcal{O}^s)$ is a fixpoint of $\mathcal{O}^s$.*

The next approximation is now defined as follows.

Definition 17. Given a lattice $L = \langle \mathcal{L}, \leq \rangle$ and an ndsao $\mathcal{O}^s$. The *well-founded state* $\mathsf{W}(\mathcal{O}^s)$ of $\mathcal{O}^s$ is the $\preceq_i^P$-least fixpoint of $S(\mathcal{O}^s)$.

We show that the well-founded state always exists:

Proposition 5. *For a lattice $L = \langle \mathcal{L}, \leq \rangle$ and an ndsao $\mathcal{O}^s$ that approximates $O_{\mathcal{L}}$, the well-founded state $\mathsf{W}(\mathcal{O}^s)$ of $\mathcal{O}^s$ exists, is consistent, and approximates every fixpoint of $O_{\mathcal{L}}$.*

Proof. Existence and minimality are shown as in the proof of Proposition 2, since by Lemma 5, $S(\mathcal{O}^s)$ is $\preceq_i^P$-monotonic over the partial order $\langle \preceq_i^P, \wp_{\leq}(\mathcal{L}^2) \rangle$. Approximation of the fixpoints of $O_{\mathcal{L}}$ follows from Lemma 5. $\square$

Example 7. Consider again the disjunctive logic program $\mathcal{P} = \{a \vee b \leftarrow \neg c;\ c \leftarrow d;\ d \leftarrow c\}$ from Example 4. The well-founded state $\mathsf{W}(\mathcal{IC}_{\mathcal{P}}^s)$ of $\mathcal{IC}_{\mathcal{P}}^s$ is obtained as follows:

$(S(\mathcal{IC}_{\mathcal{P}}^s)(\emptyset, \mathcal{A}_{\mathcal{P}}))^1 =$
$(\mathit{lfp}(\mathcal{IC}_{\mathcal{P}}^{s,l}(., \mathcal{A}_{\mathcal{P}})\ ,\ \mathit{lfp}(\mathcal{IC}_{\mathcal{P}}^{s,l}(\emptyset, .)) = \{(\emptyset, \{a\}), (\emptyset, \{b\})\}$[16]
$(S(\mathcal{IC}_{\mathcal{P}}^s)(\emptyset, \mathcal{A}_{\mathcal{P}}))^2 =$
$(\mathit{lfp}(\mathcal{IC}_{\mathcal{P}}^{s,l}(., \{\{a\}, \{b\}\})\ ,\ \mathit{lfp}(\mathcal{IC}_{\mathcal{P}}^{s,l}(\emptyset, .)) =$
$\{(\{a\}, \{a\}), (\{b\}, \{b\})\}$
$(S(\mathcal{IC}_{\mathcal{P}}^s)(\emptyset, \mathcal{A}_{\mathcal{P}}))^3 =$
$((S(\mathcal{IC}_{\mathcal{P}}^s))'(\emptyset, \mathcal{A}_{\mathcal{P}}))^2 = \{(\{a\}, \{a\}), (\{b\}, \{b\})\}$

- At Stage 3, we have reached a fixpoint.
- The well-founded state of $\mathcal{IC}_{\mathcal{P}}$ is in this case more precise than the Kripke-Kleene state of $\mathcal{IC}_{\mathcal{P}}$ (Example 4), since $\{(\emptyset, \{a, c, d\}), (\emptyset, \{b, c, d\})\} \preceq_i^S \{(\{a\}, \{a\}), (\{b\}, \{b\})\}$.

The next proposition lists some desirable properties of the fixpoints that are related to $\mathcal{O}$. To show a relation between the Kripke-Kleene state and the well-founded state, we introduce an additional notion:

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[13] Notice that $(\wp_{\leq}(\mathcal{L}^2)) \neq \wp_{\leq_i}(\mathcal{L}^2)$ (recall Definition 9). In fact, it holds that $(\wp_{\leq}(\mathcal{L}^2)) \subseteq \wp_{\leq_i}(\mathcal{L}^2)$).

[14] This operator is somewhat similar to the ultimate approximation (Denecker, Marek, and Truszczynski 2002).

[15] Notice that we slightly abuse notation and write (X, Y) for $\{(x, y) \mid x \in X, y \in Y\}$.

[16] $\mathit{lfp}(\mathcal{IC}_{\mathcal{P}}^{s,l}(., \mathcal{A}_{\mathcal{P}})) = \{\emptyset\}$ and $\mathit{lfp}(\mathcal{IC}_{\mathcal{P}}^{s,l}(., \emptyset)) = \{\{a\}, \{b\}\}$.

Definition 18. We say an ndsao $\mathcal{O}^s$ is *refining, if it holds that* $\bigcup_{(x,y)\in\mathbf{X}} \mathcal{O}^s(x,y) \preceq_i^S \mathcal{O}^s(\mathbf{X})$.

Proposition 6. *Let L be a lattice, $\mathcal{O} : \mathcal{L}^2 \to \wp(\mathcal{L}^2)$ a $\preceq_i^P$-monotonic ndao, and $\mathcal{O}^s$ be an ndsao that approximates $\mathcal{O}$. We denote by $\mathsf{W}(\mathcal{O}^s)$ the well-founded state of $\mathcal{O}^s$ and by $\mathsf{S}(\mathcal{O})$ the stable fixpoints of $\mathcal{O}$. Then:*

1. $\mathsf{W}(\mathcal{O}^s) \preceq_i^S \mathsf{S}(\mathcal{O})$.
2. *If for every $(x,y) \in \mathsf{W}(\mathcal{O}^s)$ it holds that $x = y$, then for every consistent $(w,z) \in \mathsf{S}(\mathcal{O})$, $w = z$.*
3. *If $\mathcal{O}^s$ is refining and approximates $\mathcal{O}$, then $\mathsf{K}(\mathcal{O}) \preceq_i^S \mathsf{W}(\mathcal{O}^s)$.*

Proof. Item 1: Observe that $(\emptyset, \mathcal{L}) \preceq_i^P (x,y)$ for every $(x,y) \in \mathsf{S}(\mathcal{O})$. Furthermore, by the $\preceq_i^P$-monotonicity of $\mathcal{O}^s$, $\mathcal{O}^s(\emptyset,\mathcal{L}) \preceq_i^P \mathcal{O}^s(\{x\},\{y\})$. Since $\mathcal{O}^s$ approximates $\mathcal{O}$, $\mathcal{O}^s(\{x\},\{y\}) = \mathcal{O}(x,y)$, and so $\mathcal{O}^s(\emptyset,\mathcal{L}) \preceq_i^P \mathcal{O}(x,y)$. Since $(x,y) \in \mathcal{O}(x,y)$, $\mathcal{O}^s(\emptyset,\mathcal{L}) \preceq_i^P (x,y)$. This process can be iterated until $\mathsf{W}(\mathcal{O}^s)$ is obtained on the left hand side of the last $\preceq_i^P$-comparison.

For Item 2, consider some $(w,z) \in \mathsf{S}(\mathcal{O})$. By Item 1 we know that there is some $(x,y) \in \mathsf{W}(\mathcal{O}^s)$ s.t. $(x,y) \leq_i (w,z)$, i.e., $x \leq w$ and $z \leq y$. Since $x = y$, and since by consistency $w \leq z$, we obtain $w = z$.

For Item 3, let $\mathbf{X} \in \mathsf{K}(\mathcal{O})$ (hence, in particular, $\mathbf{X} \in \wp_{\leq_i}(\mathcal{L}^2)$). Since $\mathcal{O}^s$ approximates $\mathcal{O}$, we have that $\mathcal{O}(\bot,\top) = \mathcal{O}^s(\bot,\top)$, and so $\mathcal{O}(\bot,\top) \preceq_i^S \mathcal{O}^s(\bot,\top)$. Then, by the definition of $\mathcal{O}'$, for any $\mathbf{Y} \in \wp_{\leq_i}(\mathcal{L}^2)$, $\mathcal{O}'(\mathbf{Y}) \preceq_i^S \bigcup_{(x,y)\in\mathbf{Y}} \mathcal{O}(x,y)$, and this holds in particular for $\mathbf{X}$. Since $\mathcal{O}^s$ approximates $\mathcal{O}$, $\bigcup_{(x,y)\in\mathbf{X}} \mathcal{O}(x,y) = \bigcup_{(x,y)\in\mathbf{X}} \mathcal{O}^s(x,y)$. Moreover, since $\mathcal{O}^s$ is refining, this implies that $\bigcup_{(x,y)\in\mathbf{X}} \mathcal{O}(x,y) \preceq_i^S \mathcal{O}^s(\mathbf{X})$. Thus, $\mathcal{O}'(\mathbf{X}) \preceq_i^S \mathcal{O}^s(\mathbf{X})$. The claim then follows by an induction on the applications of $\mathcal{O}'$ respectively $S(\mathcal{O}^s)$. □

A summary of the relations between the central fixpoint operators studied in this paper is presented in Figure 1. An arrow from node N1 to N2 means that the set described in N1 is in $\preceq_i^S$-relation to the set described in N2. A dotted arrow means that the relation only holds under certain conditions (see Proposition 6).

Table 1 summarizes the main properties of the approximating operators discussed in this section.

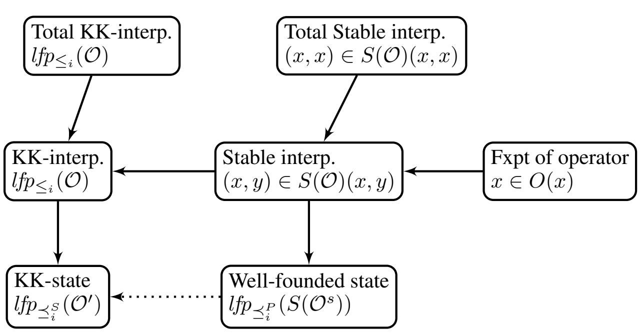

Figure 1: Figure summarizing the main fixpoints defined in this paper and their $\preceq_i^S$-relations.

Name	Definition	Exists?	Unique?	Prop.
KK state	$lfp_{\preceq_i^S}(\mathcal{O}')$	✓	✓	2,3
KK interp.	$lfp_{\preceq_i}(\mathcal{O})$	×	×	3
WF state	$lfp_{\preceq_i^P}(S(\mathcal{O}^s))$	✓($\preceq_i^P$)	✓($\preceq_i^P$)	5,6
Stable interp.	$fp(S(\mathcal{O}))$	×	×	6

Table 1: Approximating operators and their properties. ✓($\preceq_i^P$) means that the property is guaranteed only for $\preceq_i^P$-monotonic ndsaos.

Remark 7. Clearly, for a deterministic operator, the well-founded state coincides with the well-founded model of $\mathcal{O}^s$ according to (Denecker, Marek, and Truszczyński 2000) (since the $\preceq_i^S = \preceq_i^P = \leq_i$). In this sense, then, our theory of non-deterministic approximating operators can be said to be a generalization of the AFT from (Denecker, Marek, and Truszczyński 2000).

5 Applications to DLP

We now show how the approximating operators considered in the previous section may be used for computing the semantics of dlps. First, we show that the supported models of a dlp $\mathcal{P}$ (Definition 2) coincide with the fixpoints of $\mathcal{IC}_\mathcal{P}$. This is a generalization of Theorem 2 in (Pelov and Truszczynski 2004), where it is shown that total fixpoints $(x,x) \in \mathcal{IC}_\mathcal{P}(x,y)$ coincide with the two-valued supported interpretations of $\mathcal{P}$.

Theorem 1. *Given a dlp $\mathcal{P}$ and a consistent interpretation $(x,y) \in \wp(\mathcal{A}_\mathcal{P}^2)$, it holds that (x,y) is a supported model of $\mathcal{P}$ iff $(x,y) \in \mathcal{IC}_\mathcal{P}^{\mathsf{cons}}(x,y)$.*

Proof. [⇐] Suppose that $(x,y) \in \mathcal{IC}_\mathcal{P}^{\mathsf{cons}}(x,y)$. We first show that (x,y) is a model of $\mathcal{P}$. Indeed, suppose that for $\bigvee\Delta \leftarrow \phi \in \mathcal{P}$, $(x,y)(\phi) = \mathsf{T}$. Then $\Delta \in \mathcal{HR}_\mathcal{P}^l(x,y)$ and thus (since $(x,y) \in \mathcal{IC}_\mathcal{P}^{\mathsf{cons}}(x,y)$), $\Delta \cap x \neq \emptyset$, i.e., $(x,y)(\bigvee\Delta) = \mathsf{T}$. The case for $(x,y)(\phi) = \mathsf{U}$ is similar and the case for $(x,y)(\phi) = \mathsf{F}$ is trivial. We now show that (x,y) is supported. Indeed, let $p \in \Delta \cap x$. Suppose first that there is no $\bigvee\Delta \leftarrow \phi \in \mathcal{P}$ s.t. $(x,y)(\phi) = \mathsf{T}$. Then $x \notin min_\subseteq(\{v \mid v \cap \Delta \neq \emptyset \text{ for every } \Delta \in \mathcal{HR}_\mathcal{P}^l(x,y)\}$, since there is some $x' \subseteq x \setminus \{p\}$ such that $x' \in min_\subseteq(\{v \mid v \cap \Delta \neq \emptyset \text{ for every } \Delta \in \mathcal{HR}_\mathcal{P}^l(x,y)\}$. The proofs for $(x,y)(\phi) = \mathsf{U}$ and of the second condition in the definition of supported models are similar.
[⇒] We leave this to the full paper. □

Next, we show that stable fixpoints (Definition 14) of $\mathcal{IC}_\mathcal{P}^{\mathsf{cons}}$ coincide with the (three-valued) stable models of $\mathcal{P}$ (Definition 3). This is a generalization of Theorem 6 in (Pelov and Truszczynski 2004), where it is shown that total fixpoints of $S(\mathcal{IC}_\mathcal{P})(x,y)$ coincide with two-valued (total) stable models of $\mathcal{P}$.

Lemma 6. *The non-deterministic operator $\mathcal{HR}_\mathcal{P}^l(.,y)$ (Example 2) is $\leq_i$-anti-monotonic for any $y \subseteq \mathcal{A}_\mathcal{P}$.*

Theorem 2. *Given a dlp $\mathcal{P}$ and a consistent interpretation $(x,y) \in \wp(\mathcal{A}^2_{\mathcal{P}})$. Then (x,y) is a stable model of $\mathcal{P}$ iff $(x,y) \in S(\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}})(x,y)$.*

Proof. [$\Rightarrow$] Let (x,y) be a stable model of $\mathcal{P}$. We show that $x \in lfp(\mathcal{IC}^l_{\mathcal{P}}(.,y))$ (the proof that $y \in lfp(\mathcal{IC}^l_{\mathcal{P}}(x,.))$ is analogous). We first show that $x \in \mathcal{IC}^l_{\mathcal{P}}(x,y)$. Indeed, this immediately follows from the fact that any stable interpretation is supported and that any supported model is a fixpoint of $\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}$ (Theorem 1). It remains to show $\subseteq$-minimality. Suppose towards a contradiction that there is some $x' \subset x$ such that $x' \in \mathcal{IC}^l_{\mathcal{P}}(x',y)$. It is easy to check that $(x',y) \in mod(\frac{\mathcal{P}}{(x,y)})$, which contradicts $(x,y) \in \min_{\leq_t}(mod(\frac{\mathcal{P}}{(x,y)}))$ (the latter follows from the assumption that (x,y) is stable). Thus x is a $\subseteq$-minimal fixpoint of $\mathcal{O}_l(.,y)$, i.e., $x \in lfp(\mathcal{O}_l(.,y))$. Analogously $y \in lfp(\mathcal{O}_l(x,.))$, hence $(x,y) \in S(\mathcal{IC}_{\mathcal{P}})^{\mathsf{cons}}(x,y)$.
[$\Leftarrow$] We leave this to the full paper. $\square$

Remark 8. Since for stratified dlps, the perfect models (Przymusinski 1988) coincide with the stable models (Przymusinski 1991, Theorem 4.3), the results above also show that for stratified dlps the stable fixpoints of $\mathcal{IC}^{\mathsf{cons}}_{\mathcal{P}}$ coincide with the perfect models of the program.

We summarize our results for DLP as follows:

- We have shown that the fixpoints of $\mathcal{IC}_{\mathcal{P}}$ coincide with the (three-valued) supported models of $\mathcal{P}$.
- The stable fixpoints of $\mathcal{IC}_{\mathcal{P}}$ coincide with the (three-valued) stable models of $\mathcal{P}$.

Concerning the well-founded semantics $\mathsf{W}(\mathcal{IC}^s_{\mathcal{P}})$, we note that:

1. it uniquely exists for any disjunctive logic program (see Proposition 5),
2. it coincides with the well-founded semantics for non-disjunctive normal logic programs (Van Gelder, Ross, and Schlipf 1991) (see Remark 6), and
3. it is $\preceq^S_i$-related to the stable models of $\mathcal{P}$ (see Proposition 6).

Altogether, this shows that the general construction for a well-founded state satisfies some minimal desiderata for any well-founded semantics for disjunctive logic programming. In future work, we plan to compare $\mathsf{W}(\mathcal{IC}^s_{\mathcal{P}})$ with other well-founded semantics for DLP (see e.g. (Knorr and Hitzler 2007; Wang and Zhou 2005)), as well as look at well-founded states induced by other operators, such as $\mathcal{IC}^{\mathsf{mod}}_{\mathcal{P}}$ (Pelov and Truszczynski 2004) which is obtained by dropping the minimality constraint of $\mathcal{IC}_{\mathcal{P}}$ and the operators from (Antić, Eiter, and Fink 2013), which have close similarities to the operator $(\mathcal{IC}'_{\mathcal{P}})_c$. We conjecture that different well-founded semantics from the literature coincide with the well-founded state of variations of these operators.

6 Conclusion, in View of Related Work

Approximations of fixpoints of non-deterministic operators generalize standard AFT, as all the operators and fixpoints defined in this paper coincide with the respective counterparts for deterministic operators (see Remarks 6 and 7). This work also generalizes or allows to generalize the results in (Pelov and Truszczynski 2004; Antić, Eiter, and Fink 2013) to further semantics of disjunctive logic programs, thus answering an open question in these works. Furthermore, our framework allows to obtain additional semantics for formalisms such as disjunctive default logic (Gelfond et al. 1991; Bonevac 2018). The advantage of studying non-deterministic operators is thus at least twofold:

1. allowing to define a family of semantics for non-monotonic reasoning with disjunctive information,
2. clarifying similarities and differences between semantics stemming from the use of different operators.

To the best of our knowledge, the only setting with similar unifying potential that has been applied to non-deterministic or disjunctive reasoning is *equilibrium logic* (Pearce 2006). The similarities between equilibrium logic and AFT have been noted in (Denecker, Bruynooghe, and Vennekens 2012), where it was indicated that equilibrium semantics are defined for a larger class of logic programs than those that are represented by AFT, a limitation of AFT which we have overcome in this paper. Furthermore, defining three-valued stable and well-founded semantics is not possible in standard equilibrium logic, but requires an extension known as *partial equilibrium logic* (Cabalar et al. 2006; Cabalar et al. 2007), which can be seen as a six-valued semantics. In contrast, the well-founded semantics *is defined* in AFT using the same operator used to define the stable semantics. That being said, in future work we plan to compare in more detail the well-founded semantics for DLP by partial equilibrium logic and the well-founded semantics obtained in this work.

The introduction of disjunctive information in AFT points to a wealth of further research, such as defining three-valued and well-founded semantics for various disjunctive non-monotonic formalisms and studying on the basis of which operators various well-founded semantics for DLP can be represented in our framework. Moreover, this framework lays the ground for the generalization of various interesting concepts introduced (or adapted) to AFT, such as ultimate approximations (Denecker, Marek, and Truszczynski 2002), grounded fixpoints (Bogaerts, Vennekens, and Denecker 2015), strong equivalence (Truszczyński 2006), stratification (Vennekens, Gilis, and Denecker 2006) and argumentative representations (Heyninck and Arieli 2020) to a non-deterministic setting. Extensions to DLP with negations in the rules' heads and corresponding 4-valued semantics (Sakama and Inoue 1995) will also be considered in future work.

Another issue for future work is whether and how to construct an ndsao $\mathcal{O}^s$ from an ndao $\mathcal{O}$, just as (Denecker, Marek, and Truszczynski 2002) show how to construct an approximating operator $\mathcal{O}$ from $O_{\mathcal{L}}$.

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