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Main Track

Somebody Knows

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Abstract

Several different notions of group knowledge have been extensively studied in the epistemic and doxastic logic literature, including common knowledge, general knowledge (everybody-knows) and distributed knowledge. In this paper we study a natural notion of group knowledge between general and distributed knowledge: somebody-knows. While something is general knowledge if and only if it is known by *everyone*, this notion holds if and only if it is known by *someone*. This is stronger than distributed knowledge, which is the knowledge that follows from the total knowledge in the group. We introduce a modality for somebody-knows in the style of standard group knowledge modalities, and study its properties. Unlike the other mentioned group knowledge modalities, somebody-knows is not a normal modality; in particular it lacks the conjunctive closure property. We provide an equivalent neighbourhood semantics for the language with a single somebody-knows modality, together with a completeness result: the somebody-knows modalities are completely characterised by the modal logic EMN extended with a particular weak conjunctive closure axiom. We also show that the satisfiability problem for this logic is PSPACE-complete. The neighbourhood semantics and the completeness and complexity results also carry over to logics for so-called local reasoning (Fagin et al. 1995) with bounded "frames of mind", correcting an existing completeness result in the literature (Allen 2005).

1 Somebody Knows

It is well known that *group knowledge* (or *belief*[1]) has different meaning in different contexts. For example, from "the students know that the exam is today" it is natural to assume that *every* student knows that the exam is today, whereas "the burglars knows how to open the safe" seems to imply only that *together* the burglars know how to open the safe – maybe one of the burglers knows the location of the safe and another one knows the combination code. In epistemic logic (Fagin et al. 1995) the former variant of group knowledge is called *general knowledge* (everybody-knows) and expressed as $E_G\phi$ where ϕ is the fact that is known by the group G, while the latter is called *distributed knowledge* and is expressed by $D_G\phi$. Clearly, general knowledge is *stronger* than distributed knowledge; we have that the following is valid:

$$E_G\phi \to D_G\phi$$

However, there is a natural notion of group knowledge between the two. A sentence such as "the police know who the killer is" typically is intended to imply that *at least one* member of the police knows who the killer is. From a logical perspective this is similar to general knowledge, replacing the universal quantifier with an existential one. We call this *somebody knows*: $S_G\phi$. We can use individual knowledge modalities K_i, where i is an agent, to express both everybody-knows and somebody-knows:

$$E_G\phi \leftrightarrow \bigwedge_{i\in G} K_i\phi \qquad S_G\phi \leftrightarrow \bigvee_{i\in G} K_i\phi$$

Somebody-knows is indeed in-between, the following are valid:

$$E_G\phi \to S_G\phi \qquad S_G\phi \to D_G\phi$$

The objective of this paper is to study the somebody-knows modality. Natural questions include the following. General knowledge and distributed knowledge both have a standard relational modal logic semantics, given by the union and the intersection, respectively, of the individual accessibility relations in a Kripke model. Is there a corresponding standard relational semantics for somebody-knows, and if so what is it? If not, is there an alternative standard *non-normal* semantic characterisation of somebody-knows? Which properties commonly assumed of knowledge and belief hold for the somebody-knows operators, and what is the relationship between properties of individual knowledge and properties of somebody-knows? What is a *complete* characterisation of all properties of the somebody-knows modalities? What is the computational complexity of reasoning with the somebody-knows modalities?

But why study these operators when they are definable from individual knowledge operators? The most obvious reason is to get an understanding of their basic properties, like we have for individual and other group knowledge operators. Another is that while adding these operators to the standard epistemic language does not increase the expressive power of the language, it makes it *exponentially more succinct*. For example, the shortest formula in the standard epistemic language expressing S_GS_Gp with $G = \{1, 2, 3\}$ is

[1]Like in the standard reference (Fagin et al. 1995) we use the word "knowledge" liberally here, also for the case of non-veridical belief.

$K_1(K_1p \vee K_2p \vee K_3p) \vee K_2(K_1p \vee K_2p \vee K_3p) \vee K_3(K_1p \vee K_2p \vee K_3p)$ – and in general the size of the translation grows exponentially with the number of nested somebody-knows operators[2]. Thus, reasoning about somebody-knows from the first principles of epistemic logic is unrealistic in practice, at least if nesting of such operators like in "at least one attacker knows that somebody in the group knows the password" is relevant. Finding a complete set of basic principles of reasoning directly on the level of somebody-knows can thus have direct practical interest.

We start, in Section 2, by looking at some basic properties of the somebody-knows operators and the relationship between epistemic properties of individual knowledge and of everybody-knows. Of course, the properties of the somebody-knows operators are in a technical sense completely characterised by the formula $S_G\phi \leftrightarrow \bigvee_{i\in G} K_i\phi$. However, that does not shed much light on properties we are typically interested in for epistemic modalities, such as distribution over conjunction or positive introspection. In order to understand the core principles of somebody-knows, we will in most of the paper study a logical language with just a single somebody-knows modality for the grand coalition, introduced in Section 3. That will allow us to isolate the properties of the modality, and give a complete characterisation of its properties in isolation from other modalities.

Despite the intuitive similarity with the (normal) general knowledge operator, somebody-knows is *non-normal* – answering the first question above in the negative – it does satisfy the conjunctive closure principle. It does, however, satisfy all the principles of the weakest modal logic **E** with *neighbourhood semantics*. We identify a class of neighbourhood models that are modally equivalent to Kripke models with respect to somebody-knows (Section 4). We use this equivalent semantics to give a sound and complete axiomatisation, for the single-operator language (Section 5). In short, the properties of the modality are completely characterised by the sub-normal modal logic **EMN** extended with a "weak" conjunctive closure axiom B^n. We also show that the satisfiability problem for this logic is PSPACE-complete (Section 6).

It turns out that there is a strong connection between somebody-knows and another type of epistemic modalities: *local reasoning* modalities, developed to reason about inconsistent beliefs by allowing agents to have more than one "frame of mind" (Vardi 1986; Fagin et al. 1995). Under the restriction that these frames of mind are limited to at most n in number (Allen 2005), the (non-standard Kripke) semantics of the local reasoning modality and the somebody-knows modality in the case of n agents coincide. Thus, our three main results, viz. the characterisation using neighbourhood semantics and the completeness and complexity results, all carry over to n-bounded local reasoning as well.

Weak conjunctive closure has been studied before, most notably by (Schotch and Jennings 1980; Jennings and Schotch 1981) who developed a non-standard semantics for non-adjunctive logics using so-called n-ary relational models. These are characterised by adding weak conjunctive closure axioms K^n to **EMN**. (Allen 2005) claims that the same system is complete with respect to n-bounded local reasoning structures as well. We show, in Section 6, that that is actually not the case, and that our axiom B^n is stronger than K^n, correcting the completeness results and complexity proofs in (Allen 2005). We conclude in Section 7.

[2] In the case of S5 knowledge the translation in our particular example can be shortened, but the translation still grows exponentially.

2 Language, Semantics and Some Properties

The language $\mathcal{L}_S^n$ of multi-agent epistemic logic with somebody-knows is defined as follows, for $n \geq 1$ agents and a non-empty set of atomic propositions PROP. We write AG for $\{1, \dots, n\}$.

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid S_G\phi$$

where $p \in$ PROP, $i \in$ AG and $\emptyset \neq G \subseteq$ AG. We write $\hat{S}\phi$ for $\neg S \neg \phi$ (the dual). We write $\top$ for $p \vee \neg p$ for some (fixed but arbitrary) $p \in$ PROP. The purely epistemic language, without any S_G operators, is called $\mathcal{L}_{EL}^n$.

A Kripke model M over AG and PROP is a triple (S, R, V), where S is a nonempty set of states, $R : \text{AG} \to \wp(S \times S)$ assigns to every agent a a binary relation R_a on S, and $V : \text{PROP} \to S$ is a valuation which associates with every propositional variable a set of states where it is true. For any $s \in S$, the pair (M, s) is called a *pointed model*.

Definition 1 (Satisfaction)**.** *The truth in, or satisfaction by, a pointed model (M, s) with $M = (S, R, V)$ of a formula ϕ, denoted $(M, s) \models \phi$, is defined inductively as follows.*

$(M,s) \models p$	*iff*	$s \in V(p)$
$(M,s) \models \neg\phi$	*iff*	*not* $(M,s) \models \phi$
$(M,s) \models (\phi \wedge \psi)$	*iff*	$(M,s) \models \phi$ *and* $(M,s) \models \psi$
$(M,s) \models K_i\phi$	*iff*	*for all* $t \in S$, *if* sR_it *then* $(M,t) \models \phi$
$(M,s) \models S_G\phi$	*iff*	$\exists i \in G$ *such that for all* $t \in S$, *if* sR_it *then* $(M,t) \models \phi$

It is easy to see that $(M,s) \models S_G\phi$ iff $(M,s) \models \bigvee_{i\in G} K_i\phi$. A formula ϕ is valid, $\models \phi$, if it is true in all states in all models. An S5 model is a model where all the accessibility relations are equivalence relations.

Let's look at the properties of S_G, expressed as valid formulas. First, like distributed knowledge but unlike general knowledge it is *coalition monotonic*, the following is *valid*:

$$S_G\phi \to S_H\phi \qquad G \subseteq H \qquad (Cm)$$

Properties of S_G of course depend on the properties of individual knowledge. For example, the formula

$$S_G\phi \to S_GS_G\phi \qquad (4)$$

(positive introspection) is *valid* on Kripke models with *transitive* accessibility relations, but not valid on the class of all Kripke models. In the language of (Ågotnes and Wáng 2020), positive introspection is *preserved* when going from individual knowledge to somebody-knows. The same holds for the truth axiom:

$$S_G\phi \to \phi \qquad (T)$$

– it is *valid* on the class of reflexive Kripke models.

Not all S5 properties are preserved, however. In particular, neither negative introspection

$$\neg S_G\phi \rightarrow S_G\neg S_G\phi \tag{5}$$

nor the B axiom

$$\neg\phi \rightarrow S_G\neg S_G\phi \tag{B}$$

are preserved on S5 models. As a counter-example for both, let $G = \{a, b\}$ and take the S5 model[3] M

$$\bullet_t^{\phi} \overset{a}{\cdots\cdots} \bullet_s^{\neg\phi} \overset{b}{\cdots\cdots} \bullet_u^{\phi}$$

where we have that $(M, s) \models \neg\phi$, $(M, s) \models \neg S_G\phi$ since $(M, s) \models \neg K_a\phi \wedge \neg K_b\phi$, but $(M, t) \models S_G\phi$ since $(M, t) \models K_b\phi$ and $(M, u) \models S_G\phi$ since $(M, u) \models K_a\phi$. Thus, $(M, s) \models \neg S_G\neg S_G\phi$.

Thus, like for everybody-knows (Ågotnes and Wáng 2020) not all properties of individual knowledge are preserved for somebody-knows in the S5 case. While the former modality in that case is (KT)B, it might seem from the above that the latter is S4 since it satisfies both T and 4. However, it in fact is not: it does not satisfy the K axiom:

$$S_G(\phi \rightarrow \psi) \rightarrow (S_G\phi \rightarrow S_G\psi) \tag{K}$$

As a counter-example, take $G = \{a, b\}$ and take M to be the (S5) model

$$\bullet_t^{\neg\phi,\neg\psi} \overset{a}{\cdots\cdots} \bullet_s^{\phi,\psi} \overset{b}{\cdots\cdots} \bullet_u^{\phi,\neg\psi}$$

We have that $(M, s) \models K_a(\phi \rightarrow \psi) \wedge K_b\phi$ so $(M, s) \models S_G(\phi \rightarrow \psi) \wedge S_G\phi$, but $(M, s) \models \neg K_a\psi \wedge \neg K_b\psi$ so $(M, s) \not\models S_G\psi$.

In other words, S_G is not a *normal* modality. This despite the intuitive similarity with the E_G modality, which *is* normal. It follows immediately that S_G has no normal relational semantics, in the sense that the interpretation of S_G in a Kripke model defined above corresponds to interpreting it in the normal way (truth in all accessible worlds) in a Kripke model with an accessibility relation for S_G, like is the case for E_G (union) and D_G (intersection)[4] .

So, for example, on S5 models the S_G modality is not S4 – it is actually not even K. What is it then? In order to situate the modalities in the landscape of non-normal modal logics, let us look at some further properties commonly considered for non-normal modal logics (Pacuit 2017).

The monotonicity axiom

$$S_G(\phi \wedge \psi) \rightarrow (S_G\phi \wedge S_G\psi) \tag{M}$$

is *valid*. The converse,

$$(S_G\phi \wedge S_G\psi) \rightarrow S_G(\phi \wedge \psi) \tag{C}$$

however is *not valid*. A counter-example is M with $G = \{a, b\}$:

$$\bullet_t^{\phi,\neg\psi} \overset{a}{\cdots\cdots} \bullet_s^{\phi,\psi} \overset{b}{\cdots\cdots} \bullet_u^{\neg\phi,\psi}$$

We have that $(M, s) \models S_G\phi \wedge S_G\psi \wedge \neg S_G(\phi \wedge \psi)$.

The N axiom

$$S_G\top \tag{N}$$

is also *valid*.

Moving on to inference rules, the following all preserve validity:

$$\phi \leftrightarrow \psi \Rightarrow S_G\phi \leftrightarrow S_G\psi \tag{RE}$$

$$\phi \rightarrow \psi \Rightarrow S_G\phi \rightarrow S_G\psi \tag{RM}$$

$$\phi \Rightarrow S_G\phi \tag{Nec}$$

Thus, in the nomenclature of non-normal modal logics (Pacuit 2017), the logic of the S_G modality is not a *normal* modal logic, it is a *minimal* (or *classical*) modal logic, it is also a *monotonic* modal logic, but it is not a *regular* modal logic. Most importantly, the S_G modalities have all the properties described by the weakest system with *neighbourhood semantics*, commonly called **E**, essentially propositional logic plus the RE rule. This also means that uniform substitution is admissible. The logic of S_G also has the standard axioms M and N. It is known that RM is admissible in **EM** (i.e., **E** extended with axiom M) and that Nec is admissible in **EN**, so the closest standard non-normal modal logic seems (so far) to be **EMN** (or **EMNT4** in the case of S5 models).

The question is whether there are other validities, not derivable in **EMN**. The answer is in fact yes. Let $a, b \in$ AG, $a \neq b$. If all of $S_{\{a,b\}}p, S_{\{a,b\}}q, S_{\{a,b\}}r$ are true, then either a or b knows two of the three propositions: either $S_{\{a,b\}}(p \wedge q)$ or $S_{\{a,b\}}(p \wedge r)$ or $S_{\{a,b\}}(q \wedge r)$ must be true. Said in another way,

$$(S_{\{a,b\}}p \wedge S_{\{a,b\}}q \wedge S_{\{a,b\}}r) \rightarrow (S_{\{a,b\}}(p \wedge q) \vee S_{\{a,b\}}(p \wedge r) \vee S_{\{a,b\}}(q \wedge r))$$

is valid. More generally, we have that

$$S_G\phi_1 \wedge \cdots \wedge S_G\phi_{|G|+1} \rightarrow \bigvee_{i,j \leq |G|+1} S_G(\phi_i \wedge \phi_j) \tag{B^G}$$

is valid.

Are there yet other properties, or do the ones we have found so far, i.e., $\mathbf{EMNB^G}$, constitute a complete characterisation of all the properties of the S_G modalities? The problem with this question is that there are interaction properties between the S_G modalities for different groups G, and between S_G and K_i (the latter is actually an instance of the former, since $S_{\{i\}}$ is equivalent to K_i). Thus a complete characterisation of the properties of the S_G modalities is, strictly speaking, given by $S_G\phi \leftrightarrow \bigvee_{i \in G} K_i\phi$ plus axioms and rules for K_i. This characterisation is however not very informative when it comes to understanding the fundamental properties of somebody knowing, in the way that the axioms and rules studied above are, as discussed in the introduction.

[3]When drawing S5 models we omit reflexive loops and implicitly assume symmetry.

[4]Since the S_G modality cannot be interpreted using an accessibility relation in the standard way, we cannot talk about *preservation* of properties of knowledge in the same *semantic* way as for everybody-knows and distributed knowledge in (Ågotnes and Wáng 2020). We can, however, check it syntactically, e.g., whether the 4 axiom holds for S_G whenever individual knowledge has the corresponding semantic property (transitivity) – which is what we did above.

In order to understand the core principles of somebody-knows, we will in the following sections study the fragment of the language with just a single somebody-knows modality, for the grand coalition. That will allow us to isolate the properties of the modality, and give a complete characterisation of its properties in isolation from other modalities. We will be able to answer the question above in the negative: there are no more properties – $\mathbf{EMNB}^{\text{AG}}$ constitute a complete characterisation of all valid formulas in the minimal language.

We first collect and prove the statements about valid formulas and rules mentioned above.

Lemma 1. *Cm, M, N and B^G are valid on all models. 4 is valid on all transitive models, T is valid on all reflexive models. RE, RM and Nec preserve validity on all models.*

Proof. For B^G, let $(M, s) \models S_G\phi_1 \wedge \cdots \wedge S_G\phi_{|G|+1}$, i.e., for each k such that $1 \leq k \leq |G| + 1$ there is an agent $i_k \in G$ such that $M, s \models K_{i_k}\phi_k$. By the pigeonhole principle $i_k = i_l$ for some $k \neq l$. Then, $(M, s) \models K_{i_k}(\phi_k \wedge \phi_l)$, and thus $(M, s) \models S_G(\phi_k \wedge \phi_l)$.

The other cases are straightforward. □

3 A Minimal Language

In order to understand the properties of somebody-knows, we will use a minimal language without individual knowledge operators and with only a single somebody-knows operator for the grand coalition. This allows us to focus on the core principles of somebody-knowing, without having to consider the interaction properties of somebody-knows and individual knowledge or between somebody-knows of different groups.

The language $\mathcal{L}$ is the standard uni-modal language, defined as follows, given a non-empty set of atomic propositions PROP.

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid S\phi$$

where $p \in$ PROP. We write $\hat{S}\phi$ for $\neg S \neg \phi$ (the dual). Kripke models over PROP and n agents, and their interpretation of the language, are defined as in the previous section, with S being interpreted as S_{AG}. In particular:

Definition 2.

$$(M, s) \models S\phi \quad \textit{iff} \quad \textit{there exists } i \in \text{AG} \textit{ s.t. for all } t \in S, \textit{ if } sR_it \textit{ then } (M, t) \models \phi$$

It is easy to see that

$$(M, s) \models \hat{S}\phi \quad \text{iff} \quad \text{for all } i \in \text{AG there exists a } t \in S, \text{ such that } sR_it \text{ and } (M, t) \models \phi$$

In most of the rest of the paper we will assume the minimal language. As we have seen above, the S modality has all the properties of $\mathbf{E}$, which means that there is a possibility that it has a standard *neighbourhood semantics*. In the next section we show that that is indeed the case. We define a neighbourhood semantics for this language that is equivalent to the non-standard Kripke semantics given above: a class of neighbourhood models which are used to interpret the language in the standard way, and which has a one-to-one correspondence to the class of Kripke models. Then we prove that the system $\mathbf{EMNB}^{\text{AG}}$ is sound and complete with respect to these models, and hence to all Kripke models.

4 Neighbourhood Semantics

A neighbourhood model is a triple $\mathcal{M} = (S, N, V)$ where S and V are as in a Kripke model and $N : S \to \mathcal{P}(\mathcal{P}(S))$ is a *neighbourhood function*, assigning a set of subsets of S to every state in S.

Definition 3 (Satisfaction). *The truth in a pointed neighbourhood model (M, s) of a formula ϕ is defined inductively as follows.*

$(M, s) \models p$	*iff*	$s \in V(p)$
$(M, s) \models \neg\phi$	*iff*	*not* $(M, s) \models \phi$
$(M, s) \models (\phi \wedge \psi)$	*iff*	$(M, s) \models \phi$ *and* $(M, s) \models \psi$
$(M, s) \models S\phi$	*iff*	$[\![\phi]\!]_M \in N(s)$

where $[\![\phi]\!]_M = \{t \in S : (M, t) \models \phi\}$.

One of our main concerns are properties of the neighbourhood function. The following are some properties that will be of interest, some (the first five properties in Def. 4) of which are well known and some which are new. The *non-monotonic core* U_{nc} of a set U of subsets of S is defined as follows:

$$U_{nc} = \{X \in U : \forall Y \subseteq S \; Y \subset X \Rightarrow Y \notin U\}$$

Definition 4. *A set U of sets of subsets of a non-empty set S:*

- *is* monotonic *iff for all $X \in U$ and $Y \subseteq S$, if $X \subseteq Y$ then $Y \in U$;*
- contains the unit *iff $S \in U$;*
- *is* augmented *iff $\cap U \in U$ and U is monotonic;*
- *is* a filter *iff it is monotonic, contains the unit and is closed under binary intersections;*
- *is* core complete *iff for all $X \in U$ there exists $Y \in U_{nc}$ such that $Y \subseteq X$;*
- *is* n-bounded *iff $|U_{nc}| \leq n$;*
- *is n-augmented iff $U = \bigcup_{1 \leq i \leq n} U_i$ for some augmented sets $U_1, \ldots, U_n$ of subsets of S;*
- *is an* n-filter *iff it is monotonic, contains the unit, and for any $X_1, \ldots, X_{n+1} \in U, \exists i \neq j \leq (n+1),\; X_i \cap X_j \in U$;*

where n is a natural number.

When we say that a neighbourhood function (or frame or model) has any of these properties, we mean that $N(s)$ has them for any $s \in S$.

The n-augmented and n-filter properties generalise the standard augmented and filter properties: augmented and filter are the same as 1-augmented and 1-filter, respectively.

Lemma 2. *U is n-augmented iff it is core complete, n-bounded, monotonic and contains the unit.*

Proof. Let U be n-augmented and let $U = \bigcup_{1 \leq i \leq n} U_i$ where each U_i is augmented. Let $V_i = \bigcap U_i$ and $V = \{V_1, \ldots, V_n\}$. For core-completenes, let $X \in U$. $X \in U_i$

for some i. Let V_j be a smallest, wrt. subset inclusion, set in V such $V_j \subseteq X$, i.e., such that if $V_k \subseteq X$ for $k \neq j$ then $V_k \not\subseteq V_j$. Such a V_j exists, because $V_i \subseteq X$ and V is finite. $V_j \in U$ since U_j is augmented. If it was the case that $Z \in U$ and $Z \subset V_j$, then $Z \in U_k$ for some k and thus $V_k \subseteq Z \subset V_j \subseteq X$, a contradiction. Thus, $V_j \in U_{nc}$ and U is core-complete. For n-boundedness, assume that $X_1, \ldots, X_{n+1} \in U_{nc}$, $X_i \neq X_j$ for every i, j. There must be i, j, k such that $i \neq j$ and $X_i, X_j \in U_k$. Then $V_k \subseteq X_i$ and $V_k \subseteq X_j$, and since U_k is augmented $V_k \in U$. Since $X_i \neq X_j$, $V_k \subset X_i$ or $V_k \subset X_j$ or both, contradicting the fact that X_i and X_j are in the non-monotonic core of U. U is monotonic since each U_i is. It follows immediately from the fact that U_i is augmented that $S \in U_i$ for each i, and thus U contains the unit.

For the other direction, let U be core complete, n-bounded, monotonic and contain the unit, and let $U_{nc} = \{X_1, \ldots, X_m\}$ where $m \leq n$ (U_{nc} is non-empty since U contains the unit). Let, for $1 \leq i \leq m$, $U_i = \{X \subseteq S : X_i \subseteq X\}$, and for $m+1 \leq i \leq n$ let $U_i = U_m$. Clearly, each U_i is monotonic, and $\bigcap U_i = X_i \in U_i$. If $X \in U$ then $X_i \subseteq X$ for some i since U is core complete, and thus $X \in U_i$. Conversely, if $X \in U_i$ for some i, then $X_i \subseteq X$ and thus $X \in U$ since U is monotonic. □

We will henceforth often use the four properties in Lemma 2 implicitly as an alternative definition of n-augmentation.

We get the following generalisation of the known fact that augmented sets are filters (see (Pacuit 2017)).

Lemma 3. *If U is n-augmented then it is an n-filter.*

Proof. Let U be n-augmented. From Lemma 2 it is monotonic and contains the unit. Let $X_1, \ldots, X_{n+1} \in U$. Since U is n-augmented there must be i, j, k such that $i \neq j$ and $X_i, X_j \in U_k$ and U_k is augmented. Since $\cap U_k \in U$ and $\cap U_k \subseteq X_i \cap X_j$ and U is monotonic, $X_i \cap X_j \in U$. □

We will now look at mappings between Kripke models and neighbourhood models. Henceforth let $n = |\text{AG}|$ be the number of agents (which the class of Kripke models are parameterised by). In short, we show that the class of n-augmented neighbourhood models satisfy exactly the same formulas as the class of all Kripke models.

Given a Kripke model $M = (S, R, V)$, the neighbourhood model $\mathcal{M}^M$ is defined as follows. $\mathcal{M}^M = (S, N^M, V)$, where

$$N^M(s) = \{X \subseteq S : \exists i \in \text{AG} \forall t \in S s R_i t \Rightarrow t \in X\}$$

Lemma 4. $M, s \models \phi$ *iff* $\mathcal{M}^M, s \models \phi$

Proof. By induction on the structure of ϕ. Consider the case that $\phi = S\psi$. $M, s \models S\psi$ iff $\exists i \forall t (s R_i t \Rightarrow M, t \models \phi)$ iff $[\![\phi]\!]_M \in N^M(s)$ iff $\mathcal{M}^M, s \models \phi$. The other cases are straightforward. □

Lemma 5. *For any Kripke model M, $\mathcal{M}^M$ is monotonic, core complete, n-bounded and contains the unit.*

Proof. It is monotonic and contains the unit by definition. For core completeness, let $X \in N^M(s)$. We write $R_i(s)$ for $\{t : s R_i t\}$. Let $R_j(s)$ be such that $R_j(s) \subseteq X$ and $R_k(s) \not\subset R_j(s)$ for any $k \neq j$. Such a j exists, since $R_i(s) \subseteq X$ for some i, and there are finitely many agents. $R_j(s) \in N^M_{nc}(s)$. For n-boundedness, assume that $X_1, \ldots, X_{n+1} \in N^M_{nc}(s)$, $X_i \neq X_j$ for all $i \neq j$. Since $X_i \in N^M(s)$ for each i, by definition there must be i, j, k such that $R_k(s) \subseteq X_i$ and $R_k(s) \subseteq X_j$. That contradicts the assumptions that $X_i, X_j \in N^M_{nc}(s)$. □

Now for the other direction. Given a monotonic, core complete and n-bounded neighbourhood model $\mathcal{M} = (S, N, V)$ containing the unit, the Kripke model $M^{\mathcal{M}} = (S, R^{\mathcal{M}}, V)$ is defined as follows. For each s, let $f_s : \text{AG} \to N_{nc}(s)$ be an arbitrary surjective (onto) function from the set of agents to the non-monotonic core in s. Since $N(s)$ contains the unit and N is core complete, $N_{nc}(s)$ is non-empty. Together with n-boundedness that ensures that such a function exists. Now, for each agent $i \in \text{AG}$ we define $R_i^{\mathcal{M}}$ as the following binary relation on S:

$$s R_i^{\mathcal{M}} t \Leftrightarrow t \in f_s(i)$$

Lemma 6. *For any monotonic, core complete and n-bounded neighbourhood model $\mathcal{M}$ containing the unit, $\mathcal{M}, s \models \phi$ iff $M^{\mathcal{M}}, s \models \phi$.*

Proof. By induction on the structure of ϕ. Consider the case that $\phi = S\psi$. We show that

$$[\![\psi]\!]_M \in N(s) \Leftrightarrow \exists i \in \text{AG} f_s(i) \subseteq [\![\psi]\!]_M$$

For the implication towards the right, let $[\![\psi]\!]_M \in N(s)$. By core completeness there is an $X \in N_{nc}(s)$ such that $X \subseteq [\![\psi]\!]_M$. By surjectivity of f_s, $f_s(i) = X$ for some i. For the implication towards the left, let $f_s(i) \subseteq [\![\psi]\!]_M$ for some i. $f_s(i) \in N_{nc}(s)$, and by monotonicity $[\![\psi]\!]_M \in N(s)$. □

Corollary 1. *For any formula ϕ, ϕ is valid on the class of all Kripke models iff it is valid on the class of all n-augmented models.*

5 Complete Axiomatisation

Let $\mathbf{EMNB^n}$ be the axiomatic system over the language $\mathcal{L}$ defined in Figure 1. PC, MP and RE constitute the minimal system with neighbourhood semantics called $\mathbf{E}$. M and N are the standard axioms for monotonicity and the unit. B^n is B^{AG} with $|\text{AG}| = n$. Note that with only a single S modality the language is no longer parameterised by the set AG of agents, but both the class of Kripke models and the axiomatisation *are*.

Soundness follows from Lemma 1 above. For completeness, we will first show it for neighbourhood semantics and the class of neighbourhood models defined in the previous section. The result for Kripke models will follow immediately.

Let the canonical neighbourhood model $\mathcal{M}^c = (S^c, N^c, V^c)$ be defined as follows:

- S^c is the set of all maximal consistent sets

PC	all instances of tautologies
M	$S(\phi \wedge \psi) \rightarrow S\phi$
N	$S\top$
B^n	$S\phi_1 \wedge \cdots \wedge S\phi_{n+1} \rightarrow \bigvee_{i,j \leq n+1} S(\phi_i \wedge \phi_j)$
MP	from ϕ and $\phi \rightarrow \psi$ infer ψ
RE	from $\phi \leftrightarrow \psi$ infer $S\phi \leftrightarrow S\psi$

Figure 1: System $\mathbf{EMNB^n}$.

- $N^c(\Gamma) = \{|\phi| : S\phi \in \Gamma\}$, where $|\phi| = \{\Delta \in S^c : \phi \in \Delta\}$ (the *proof set* of ϕ)
- $V^c(p) = \{\Gamma \in S^c : p \in \Gamma\}$

More precisely, the model $\mathcal{M}^c$ is the *minimal model canonical for S* (Pacuit 2017), and it thus satisfies the following truth lemma.

Lemma 7. *For any formula ϕ and MCS Γ, $\mathcal{M}^c, \Gamma \models \phi$ iff $\phi \in \Gamma$.*

Proof. Holds for any consistent logic and canonical model (Pacuit 2017, Lemma 2.60). □

Given a formula ϕ, let the set of formulas Σ_ϕ (we write just Σ when ϕ is clear from context) be defined as follows:

- Σ_0 is the set of all subformulas of ϕ in addition to $S\top$
- $\Sigma_1 = \Sigma_0 \cup \{S \bigwedge_{1 \leq i \leq k} \phi_i : k \geq 1, \phi_i \neq \phi_j, S\phi_i \in \Sigma_0$ for $1 \leq i, j \leq k, i \neq j\}$
- Σ is the closure of Σ_1 under subformulas

In short, Σ is the set of subformulas of ϕ in addition to being closed under conjunction for S-formulas, restricted to subformulas of ϕ.

It is immediately clear from the construction that (1) Σ is finite and (2) it is closed under subformulas.

Given a neighbourhood model $\mathcal{M} = (S, N, V)$ and a set of formulas Σ, we let $[w]_\Sigma = \{v : \forall \phi \in \Sigma\ \mathcal{M}, w \models \phi \Leftrightarrow \mathcal{M}, v \models \phi\}$, i.e., the equivalence class of all states satisfying the same formulas in Σ as w. When $X \subseteq S$ we let $[X]_\Sigma = \{[w]_\Sigma : w \in X\}$. When Σ is understood from context we drop the subscript.

The *finest filtration* $\mathcal{M}^f = (S^f, N^f, V^f)$ of a neighbourhood model $\mathcal{M} = (S, N, V)$ is defined as follows:

- $S^f = [S]_\Sigma$
- $N^f([w]) = \{[\llbracket\phi\rrbracket_\mathcal{M}] : \llbracket\phi\rrbracket_\mathcal{M} \in N(w)$ and $S\phi \in \Sigma\}$
- $V^f(p) = [V(p)]_\Sigma$

The *monotonic closure* of a neighbourhood model $\mathcal{M} = (S, N, V)$, is the model $\mathcal{M}^{mon} = (S, N^{mon}, V)$ where $N^{mon}(s) = \{X \subseteq S : Y \subseteq X$ for some $Y \in N(s)\}$.

Lemma 8. *$\mathcal{M}^{cf^{mon}}$ is n-augmented.*

Proof. Let $\mathcal{M}^c = (S^c, N^c, V^c)$, $\mathcal{M}^f = (S^f, N^f, V^f) = \mathcal{M}^{cf}$ and $\mathcal{M} = (S^f, N, V^f) = \mathcal{M}^{f^{mon}}$. Note that the states of $\mathcal{M}^c$ are maximal consistent sets Γ, so the states of $\mathcal{M}$ are equivalence classes $[\Gamma]_\Sigma$ (or just $[\Gamma]$) of maximal consistent sets.

Monotonicity follows from the definition. Core-completeness too: the filtration gives us a finite model, and finite monotonic models always have a complete non-monotonic core. For the unit, we have that $|\top| = S^c \in N^c(\Gamma)$ by axiom N, and since $S\top \in \Sigma$ we have that $[S^c] = [\llbracket\top\rrbracket_{\mathcal{M}^c}] \in N^f([\Gamma])$ and thus also $[S^c] \in N([\Gamma])$ ($N([\Gamma])$ contains the unit). It remains to be shown that the model is n-bounded.

Assume, towards a contradiction, that $|N_{nc}([\Gamma])| \geq n + 1$ for some MCS Γ, i.e., that $X_1, \ldots, X_{n+1} \in N_{nc}([\Gamma])$ such that $X_i \neq X_j$ for each $i \neq j$. For each $i \leq n + 1$, $X_i \in N^f([\Gamma])$ (because X_i is in the non-monotonic core of $N([\Gamma])$), and thus $X_i = [\llbracket\phi_i\rrbracket_{\mathcal{M}^c}]$ for some $S\phi_i \in \Sigma$ and $\llbracket\phi_i\rrbracket_{\mathcal{M}^c} \in N^c(\Gamma)$. From the latter we have that $\llbracket\phi_i\rrbracket_{\mathcal{M}^c} = |\psi_i|$ for some formula $\psi_i \in \Gamma$, and from the truth lemma (Lemma 7) it follows that $|\phi_i| = |\psi_i|$.

We first argue that from the construction of Σ, we have that for any ψ

$$S\psi \in \Sigma \Rightarrow \exists k \geq 1 \exists \alpha_1, \ldots, \alpha_k \in \Sigma_0 \left(\psi = \bigwedge_{1 \leq j \leq k} \alpha_j\right) \tag{1}$$

If $S\psi \in \Sigma$ there are three cases in the construction of Σ. In the first case $S\psi \in \Sigma_0$ and we are done with $k = 1$. In the second case $S\psi \in \Sigma_1$ but $S\psi \notin \Sigma_0$, and we are immediately done by the construction of Σ_1. The third case is that $S\psi$ was introduced in the last step taking the closure under subformulas, but that is impossible: that step does not introduce any new formulas of the form $S\psi$.

By axiom B^n we have that $S(\psi_i \wedge \psi_j) \in \Gamma$ for some $i \neq j$. We now argue that $S(\phi_i \wedge \phi_j) \in \Sigma$. By (1), $\phi_i = \alpha_1 \wedge \cdots \wedge \alpha_k$ and $\phi_j = \beta_1 \wedge \cdots \wedge \beta_l$ where $\alpha_m, \beta_o \in \Sigma_0$ for each m and o. By construction of Σ_1 that means that also $S(\phi_i \wedge \phi_j) = S(\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \beta_1 \wedge \cdots \wedge \beta_l) \in \Sigma$.

From $S(\psi_i \wedge \psi_j) \in \Gamma$ we have that $|\psi_i \wedge \psi_j| \in N^c(\Gamma)$. From $|\phi_i| = |\psi_i|$ and $|\phi_j| = |\psi_j|$ and propositional reasoning we have that $|\phi_i \wedge \phi_j| = |\psi_i \wedge \psi_j| \in N^c(\Gamma)$. From the truth lemma (Lemma 7) $\llbracket\phi_i \wedge \phi_j\rrbracket_{\mathcal{M}^c} \in N^c(\Gamma)$, and since $S(\phi_i \wedge \phi_j) \in \Sigma$ we get that $[\llbracket\phi_i \wedge \phi_j\rrbracket_{\mathcal{M}^c}] \in N^f([\Gamma])$. By semantics we have that $\llbracket\phi_i \wedge \phi_j\rrbracket_{\mathcal{M}^c} \subseteq \llbracket\phi_i\rrbracket_{\mathcal{M}^c}$ and $\llbracket\phi_i \wedge \phi_j\rrbracket_{\mathcal{M}^c} \subseteq \llbracket\phi_j\rrbracket_{\mathcal{M}^c}$, and it follows that $[\llbracket\phi_i \wedge \phi_j\rrbracket_{\mathcal{M}^c}] \subseteq [\llbracket\phi_i\rrbracket_{\mathcal{M}^c}]$ and $[\llbracket\phi_i \wedge \phi_j\rrbracket_{\mathcal{M}^c}] \subseteq [\llbracket\phi_j\rrbracket_{\mathcal{M}^c}]$. Since both $[\llbracket\phi_i\rrbracket_{\mathcal{M}^c}]$ and $[\llbracket\phi_j\rrbracket_{\mathcal{M}^c}]$ are in the non-monotonic core of $N([\Gamma])$ it cannot be the case that $[\llbracket\phi_i \wedge \phi_j\rrbracket_{\mathcal{M}^c}] \subset [\llbracket\phi_i\rrbracket_{\mathcal{M}^c}]$ or $[\llbracket\phi_i \wedge \phi_j\rrbracket_{\mathcal{M}^c}] \subset [\llbracket\phi_j\rrbracket_{\mathcal{M}^c}]$ and thus it must be the case that $[\llbracket\phi_i \wedge \phi_j\rrbracket_{\mathcal{M}^c}] = [\llbracket\phi_i\rrbracket_{\mathcal{M}^c}] = [\llbracket\phi_j\rrbracket_{\mathcal{M}^c}]$. But, that contradicts the assumption that $X_i \neq X_j$.

□

Recall that weak completeness means that any valid formula is provable.

Theorem 1 (Neighbourhood Completeness)**.** $\mathbf{EMNB^n}$ *is sound and weakly complete with respect to the class of all n-augmented models.*

Proof. Soundness follows immediately from Lemma 1 and Corollary 1. For completeness, assume that ϕ is consistent. By the standard Lindenbaum construction it can be extended to an MCS Γ. By the truth lemma (Lemma 7), $M^c, \Gamma \models \phi$. Due to the M axiom, M^c is monotonic. Now let Σ_ϕ be as defined above, and let $\mathcal{M} = \mathcal{M}^{cf^{mon}}$, i.e., the *supplementation* of the finest filtration (through Σ_ϕ) of $\mathcal{M}^c$. It is known that the supplementation of the finest filtration of any monotonic model also is a filtration (Pacuit 2017, Lemma 2.75), and thus $\mathcal{M}$ is a filtration of $\mathcal{M}^c$. We therefore have that $\mathcal{M}^c, \Gamma \models \psi$ iff $\mathcal{M}, [\Gamma] \models \psi$ for any $\psi \in \Sigma_\phi$. In particular, $\mathcal{M}, [\Gamma] \models \phi$. By Lemma 8 $\mathcal{M}$ is core-complete, monotonic, n-bounded and contains the unit, and by Lemma 6 $M^{\mathcal{M}}, [\Gamma] \models \phi$. □

We also immediately get a completeness result for a larger model class, from Lemma 3, and the easily checked fact that the axioms (in particular B^n) are valid on n-filters.

Corollary 2. $\mathbf{EMNB^n}$ *is sound and weakly complete wrt. the class of all* n*-filters.*

Finally we get completeness with respect to Kripke semantics, immediately from Corollary 1.

Corollary 3 (Kripke Completeness)**.** $\mathbf{EMNB^n}$ *is sound and weakly complete wrt. the class of all Kripke models.*

6 Local Reasoning, Weak Conjunctive Closure and Complexity

The logics $\mathbf{EMNB^n}$ add weak conjunctive closure axioms to $\mathbf{EMN}$, and thus sit between $\mathbf{EMN}$ and $\mathbf{EMN}$ extended with *full* conjunctive closure $\mathbf{EMNC} = \mathbf{K}$. There are in fact existing logics that do the same. So-called *n-ary relational models* (Schotch and Jennings 1980; Jennings and Schotch 1981; Pacuit 2017) is an alternative non-normal semantics developed exactly for logics lacking the (full) conjunctive closure property. Briefly, an n-ary relational model ($n \geq 2$) is a tuple (S, R, V) where S and V are as usual and $R \subseteq S^n$. $M, s \models \Box\phi$ iff for every $(s, s_1, \ldots, s_{n-1}) \in R$, $M, s_i \models \phi$ for some i ($1 \leq i \leq n-1$). $n+1$-ary models satisfy the following weak conjunctive closure property[5]:

$$\bigwedge_{i=1}^{n+1} \Box\phi_i \to \Box\left(\bigvee_{1 \leq k < l \leq n+1} (\phi_k \wedge \phi_l)\right) \qquad (K^n)$$

It has been shown (Apostoli and Brown 1995; Nicholson, Jennings, and Sarenac 2000) that the logic $\mathbf{EMNK^n}$ is sound and complete with respect to all $n+1$-ary models. We can immediately observe that the $\mathbf{EMNB^n}$ logics in fact also satisfy these properties. For example, in the case that $n = 2$, the B^2 axiom

$$(\Box\phi_1 \wedge \Box\phi_2 \wedge \Box\phi_3) \to \Box((\phi_1 \wedge \phi_2) \vee \Box(\phi_2 \wedge \phi_3) \vee \Box(\phi_1 \wedge \phi_3))$$

trivially implies K^2:

$$(\Box\phi_1 \wedge \Box\phi_2 \wedge \Box\phi_3) \to \Box((\phi_1 \wedge \phi_2) \vee (\phi_2 \wedge \phi_3) \vee (\phi_1 \wedge \phi_3)).$$

In other words[6], $\mathbf{EMNK^n} \subseteq \mathbf{EMNB^n}$. Less trivially, does the inclusion hold in the other direction as well? That would mean that we have a third alternative semantics for the logic of somebody knows.

To answer this we will take a detour, to the logic of *local reasoning* (Vardi 1986; Fagin et al. 1995), developed to deal with the logical omniscience problem and in particular to model inconsistent knowledge based on different "frames of mind". A (single-agent) model is a tuple (S, C, V) where S and V are as usual and $C(s) \subseteq 2^S$ is a set of sets of states, each modelling a "frame of mind", for each $s \in S$. $M, s \models \Box\phi$ iff there is a $T \in C(s)$ such that $M, t \models \phi$ for all $t \in T$. Technically, these single-agent models are similar to multi-agent models for an unbounded number of agents, each frame of mind corresponding to an agent. In fact, the class of local reasoning models where the cardinality of each $C(s)$ is less than[7] or equal to n, henceforth called n-bounded local reasoning models, corresponds exactly to multi-agent Kripke models with n agents, and in that case the mentioned interpretation of $\Box$ also corresponds exactly to somebody-knows – existential quantification over the n agents. Thus, the logic of somebody-knows is the same as the logic of n-bounded local reasoning. n-bounded local reasoning has in fact been studied before, in (Allen 2005). (Allen 2005) claims that any formula satisfiable in an $n+1$-ary model is satisfiable in an n-bounded local reasoning model (Allen 2005, Proposition 2) and thus that $\mathbf{EMNK^n}$ equals the logic of n-bounded local reasoning models (Allen 2005, Theorem 1) and in particular that $\mathbf{EMNK^n}$ is sound and complete with respect to all n-bounded local reasoning models. These claims are in fact incorrect, as we now show.

Consider a 3-ary model $M = (S, R, V)$ where $R = \{(s, t_1, t_2), (s, u_1, u_2)\}$ and $V(p) = \{t_1, u_1\}$, $V(q) = \{t_1, u_2\}$ and $V(r) = \{t_2, u_2\}$. We have that $M, s \models \Box p \wedge \Box q \wedge \Box r$, but $M, s \not\models \Box((p \wedge q) \vee \Box(p \wedge r) \vee \Box(q \wedge r))$. However, this shows that $\neg B^2$ is satisfiable in a 3-ary model. But it is not valid on the class of 2-bounded local reasoning models, since B^2 is valid on those models (Lemma 1).

Thus the question asked above can be answered in the negative: $\mathbf{EMNB^n} \not\subseteq \mathbf{EMNK^n}$ and $\mathbf{EMNB^n}$ is in fact strictly stronger than $\mathbf{EMNK^n}$. The results in the previous sections also correct the completeness result for n-bounded local reasoning models reported in (Allen 2005): these logics are $\mathbf{EMNB^n}$ rather than $\mathbf{EMNK^n}$.

Let us remark that, similarly to the case for $\mathbf{EMNK^n}$, $\mathbf{EMNB^{n+1}} \subset \mathbf{EMNB^n}$ for each n.

(Allen 2005) also shows that the satisfiability problem for $\mathbf{EMNK^n}$ is PSPACE-complete, and by implication that the same holds for the logic of n-bounded local reasoning models under the assumption that those two logics are the same. Since they are not, that result does not follow.

The natural conjecture is that the result still holds: $\mathbf{EMNB^n}$ sits between $\mathbf{EMNK^n}$ and $\mathbf{K}$ – both PSPACE-complete. Furthermore, $\mathbf{EMNB^1} = \mathbf{K}$. P-SPACE-completeness of $\mathbf{EMNB^1}$ does not, however, carry over to $n > 1$ in a trivial way; in fact the S modality is in a sense weaker in the case of $n = 2$ than in the case that $n = 1$ since it cannot, e.g., quantify over all possible accessible

[5] Let us mention a possible point of confusion. This axiom is referred to as C^{n+1} in (Pacuit 2017) but K^n in (Allen 2005). We use the latter name as it is more natural for our comparison.

[6] We abuse notation and use $\mathbf{S}$ to (also) denote the set of theorems of $\mathbf{S}$.

[7] The case of strictly less than n elements represents a situation where two or more agents consider the same set of states.

states in any obvious way. We now show that the PSPACE-completeness result still holds for $\mathbf{EMNB^n}$ for all n.

Theorem 2. *The satisfiability problem for* $\mathbf{EMNB^n}$ *is in PSPACE, for any* $n \geq 1$.

Proof. Given a formula $\phi \in \mathcal{L}$, let $\phi' \in \mathcal{L}^n_{EL}$ be the obvious purely epistemic translation of ϕ obtained by replacing every $S\phi$ with $\bigvee_{i \in \text{AG}} K_i\phi$ from the inside out. Since the satisfiability problem for multi-agent $\mathbf{K}$ is PSPACE-complete (Ladner 1977; Halpern and Moses 1992) we can use the corresponding algorithm to check satisfiability for ϕ. The problem with this is, of course, that the size $|\phi'|$ of ϕ' can be exponential in the size of ϕ (it increases exponentially in the number of S-modalities to be precise), and thus it does not follow immediately that the algorithm only uses space that is polynomial in $|\phi|$. However, the result follows from the following two observations. First, while $|\phi'|$ is exponential in $|\phi|$, it has *the same modal depth*. Second, ϕ' is exponentially long exactly because it has several "copies" of the same subformulas of ϕ. Each of these only need to be represented once, with a pointer to the representation at each point in the formula where they are needed. This representation is polynomial in $|\phi|$. The second point means that the standard PSPACE algorithm for satisfiability of $\mathcal{L}^n_{EL}$ formulas on the class of all Kripke models (see, e.g., (Blackburn, de Rijke, and Venema 2001, Chapter 6)) only uses space polynomial in $|\phi|$ in each recursive call, and the fact that there are at most $|\phi|$ such calls. Thus the algorithm decides satisfiability of ϕ using polynomial space. □

Theorem 3. *The satisfiability problem for* $\mathbf{EMNB^n}$ *is PSPACE-hard, for any* $n \geq 1$.

Proof. We reduce the canonical PSPACE problem QBF. A QBF instance is a formula of the form $\beta = Q_1p_1 \cdots Q_mp_m\theta(p_1,\ldots,p_m)$ where each Q_i is either $\forall$ or $\exists$, p_i is a propositional atom, and $\theta(p_1,\ldots,p_m)$ is a propositional formula. The reduction uses the strategy of encoding a so-called quantifier tree for β, similarly to existing reductions for modal logic $\mathbf{K}$ (see, e.g., (Blackburn, de Rijke, and Venema 2001)), together with the interpretation of $\mathcal{L}$ in Kripke models. The encoding is however complicated by the fact that, with two or more agents, the mentioned interpretation is non-standard (non-normal). In particular, one cannot force a property ϕ on the first two levels of a tree by a formula of the form $S\phi \wedge SS\phi$ like one can with $\Box\phi \wedge \Box\Box\phi$ using a normal modality $\Box$, because the former can be satisfied if ϕ holds in all states accessible by one agent and $S\phi$ in all the states satisfied by another. Another complication is the dual: as discussed earlier $\hat{S}\phi$ says that *every* agent considers ϕ possible. Because our single modality only allows us to force accessibility for all agents at the same time, our encoding will necessarily have "too many" states. To get around this our proof uses auxiliary propositional atoms to keep track of the states we need, which identifies the accessibility relation for a single agent, which again gives us a quantifier tree.

We give the details for the case that $n = 2$. In addition to $p_1,\ldots,p_n$ we use an additional atom a to keep track of the agent as just mentioned.

We write $S_a\phi$ as shorthand for $S(a \wedge \phi)$, $S^n_a\phi$ for $\overbrace{S_a \cdots S_a}^{n}\phi$ and $S^{(m)}_a\phi$ for $\phi \wedge S_a\phi \wedge S^2_a\phi \wedge \cdots \wedge S^m_a\phi$. Let

$$two = Sa \wedge S\neg a$$

Note that two identifies the two agents: one of them is the agent who can only see a-states, the other is the one who can only see $\neg a$-states. We can now simulate the individual knowledge operator K_a by S_a: $S_a\phi$ is true (assuming that two holds) iff there is an agent i such that $a \wedge \phi$ is true in all states i can see, iff the a-agent knows ϕ. This hinges crucially on the fact that there are only two agents.

The formula $f(\beta)$ is the conjunction of the following:

$$S^{(m-1)}_a two \tag{2}$$

$$S^i_a B_i (0 \leq i \leq m-1) \tag{3}$$

$$\begin{aligned} &S_aP(p_1,\neg p_1) \wedge S^2_aP(p_1,\neg p_1) \wedge \cdots \wedge S^{m-1}_aP(p_1,\neg p_1) \\ &\quad \wedge S^2_aP(p_2,\neg p_2) \wedge \cdots \wedge S^{m-1}_aP(p_2,\neg p_2) \\ &\vdots \\ &\quad \wedge S^{m-1}_aP(p_{m-1},\neg p_{m-1}) \end{aligned} \tag{4}$$

$$S^m_a\theta \tag{5}$$

where for $0 \leq i \leq m-1$:

$$B_i = \begin{cases} \hat{S}(p_{i+1}) \wedge \hat{S}(\neg p_{i+1}) & Q_i = \forall \\ \hat{S}\top & \text{otherwise} \end{cases}$$

$$P(p_i,\neg p_i) = (p_1 \to S_ap_i) \wedge (\neg p_1 \to S_a\neg p_i)$$

B_i forces one (if $Q_i = \exists$) or two successors (if Q_i is $\forall$). $P(p_i,\neg p_i)$ propagates the value of p_i one level further down the tree. (2) identifies an a-agent in the root node, as well as in any further node reachable by the a-agent. Note that the identity of the a-agent is not necessarily the same on each level, but that is not important, what is important is that we identify the accessibility relation of one particular agent in each node and call that the a-agent. (3) makes sure that the a-nodes (nodes accessible by the a-agent in a state in the previous level) branches for universal quantifiers. This also necessarily adds nodes accessible by the other agent, but these are labeled with $\neg a$ and will be removed later. (4) propagates the value of p_i from level i and all the way to the leaf nodes. Finally, (5) makes sure β holds in the leaf nodes.

We now argue that for any QBF instance β, $f(\beta)$ is satisfiable iff β is valid. The proof is exactly like the standard proof for $\mathbf{K}$, with the small complication that the $f(\beta)$ encoding forces some superfluous nodes. However, we have marked the nodes we need with the a atom.

First, assume that $f(\beta)$ is satisfiable. We identify a quantifier tree as follows. First, unwind the satisfying pointed model to a tree-like model. Since the modal depth of $f(\beta)$ is m, the depth of the tree is m. The quantifier tree is obtained by two additional steps. First, remove all nodes where a is not true (and the link to those nodes from their predecessor). Second, it is easy to see that we can remove branches such that every non-leaf node on level i has exactly one (if $Q_i = \exists$) or two (if $Q_i = \forall$) successors. The result is a quantifier tree witnessing the validity of β.

Second, assume that β is valid. Take the quantifier tree witnessing validity of β, and let the relation for agent 1 be the edges in the tree. Add the following edges for agent 2: in every branching node (a $\forall$ node) in the tree, add two new nodes accessible for agent 2, and for every non-branching leaf node (an $\exists$ node) add a single new node accessible for agent 2. From these new nodes there is no further accessibility for either agent. Finally define the valuation function as follows. Let a be true in all states in the original tree (not in the new nodes added for agent 2), and let p_i be true at a node on level j in the original tree iff the substitution given by the tree gives the value 1 for p_i. For the new nodes added for agent 2 as successors to a node on level $i-1$: if the original node on level $i-1$ was a branching node, let p_i be true in one of the new nodes and false in the other. It is easy to see that the root of the model satisfies $f(\beta)$.

Since the size of $f(\beta)$ is polynomial in the size of β, this shows that there is a polynomial time reduction from QBF to the satisfiability problem for $\mathbf{EMNB^2}$.

The proof for $n > 2$ is exactly the same, except that we use more than one auxiliary variable a to simulate conjunctive closure. For example, for $n = 3$ we require $S(a \wedge b) \wedge S(a \wedge \neg b) \wedge S(\neg a \wedge b)$ to hold, and then $S(a \wedge b \wedge \phi) \wedge S(a \wedge b \wedge \psi)$ implies that $S(\phi \wedge \psi)$. □

From the proof of Theorem 2 it immediately follows that the satisfiability problem for the full language of multi-agent epistemic logic extended with somebody-knows operators S_G for each G considered in Section 2 is in PSPACE: translate every $S_G\phi$ to $\bigvee_{i \in G} K_i\phi$. The lower bound follows from hardness of $\mathbf{K}$ (or $\mathbf{EMNB^n}$).

Corollary 4. *The satisfiability problem for $\mathcal{L}_S^n$ on the class of all Kripke models is PSPACE-complete.*

7 Discussion

By isolating a single somebody-knows operator in the language and making no assumption about individual knowledge other than the standard $\mathbf{K}$ properties, we were able to pinpoint the most fundamental properties of these operators. Unlike other well-known group knowledge modalities, somebody-knows is non-normal and thus does not have a standard relational semantics. It does, however, have a standard neighbourhood semantics. We gave a characterisation in terms of neighbourhood models and used it to prove a completeness result[8], situating the logic of somebody-knows in the landscape of non-normal logics. The key axiom is the weak conjunctive closure axiom B^n. We also showed that the satisfiability problem is PSPACE-complete.

Along the way we discovered a new family of weakly conjunctive modal logics $\mathbf{EMNB^n}$ strictly between $\mathbf{EMNK^n}$ (Apostoli and Brown 1995; Schotch and Jennings 1980) and $\mathbf{K}$ that haven't been studied before as far as we know. To the best of our knowledge, $\mathbf{EMNB^n}$ is the strongest known system with conjunctive closure properties that is not equal to $\mathbf{K}$. Our completeness result also solves the problem posed by (Allen 2005) about completeness for n-bounded local reasoning, correcting the result in (Allen 2005).

Our completeness result is for *weak*, rather than *strong*, completeness. Admittedly this is due to the finitary proof technique using filtrations. In particular it is not due to the fact that the logic is not semantically *compact*; in fact it *is*. (This is easily seen by the fact that any formula in the minimal language can be translated to a formula of standard epistemic/doxastic logic by translating $S\phi$ to $\bigvee_{i \in \mathrm{AG}} K_i\phi$, and that standard epistemic/doxastic logic is compact.) Note that the axiomatisation depends on the finite number of agents in a more fundamental way than usually in epistemic logic. Indeed, if the number of agents were unbounded, somebody-knows would be axiomatised by the system $\mathbf{EMN}$ and would be NP-complete (Vardi 1989).

An obvious open problem is completeness when individual knowledge is assumed to have other properties, in particular $\mathbf{S5}$. This is not straightforward, as non-iterative axioms can be difficult to deal with in neighbourhood semantics. We conjecture that $\mathbf{EMNT4B^n}$ is complete for the $\mathbf{S5}$ case.

While the neighbourhood characterisation (and completeness result) we presented was for the case of a single somebody-knows modality S, it is worth pointing out that a generalisation to the case of one such modality S_G for each coalition G, like for, e.g., coalition logic (Pauly 2002) (which also happens to be coalition monotonic, see p. 2) is less interesting: first, each S_G can be completely characterised by $S_{\{i\}}$ for all $i \in G$ ($S_{\{i\}}$ is identical to the individual knowledge modality K_i) and, second, $S_{\{i\}}$ is normal – very much unlike, e.g., the case for coalition logic. As discussed in the introduction, there are still a strong motivation for understanding the properties of these modalities, and including them in the language makes it exponentially more succinct – "for free" without increasing the computational complexity (at least in the $\mathbf{K}$ case).

The PSPACE complexity result is not surprising, but also not obvious. Along the way we showed that local reasoning (Fagin et al. 1995) is harder (assuming NP $\neq$ PSPACE) if agents are restricted to having at most a fixed number of frames-of-mind than in the general case, going from being NP-complete in the latter case (Vardi 1989) to PSPACE-complete in the former. This is also what (Allen 2005) sets out to prove, although it does not follow directly from PSPACE-completeness of $\mathbf{EMNK^n}$ as claimed. The crux of the PSPACE lower bound is that we can "simulate" the individual knowledge operator K_a using a special atom a in the two-agent case, writing $S(a \wedge \phi)$ for $K_a\phi$, and similarly for any other finite number of agents by using more atoms. This is not possible for an unbounded number of agents, which goes some way towards explaining why unbounded local reasoning is NP-complete but bounded local reasoning is PSPACE-complete.

[8] When it comes to existing completeness results for classical modal logics (modal logics extending $\mathbf{E}$), (Lewis 1974) showed that *every* such logic extended with non-iterative axioms is weakly complete, and (Surendonk 2001) extends this to strong completeness. Non-iterative axioms are formulas without nested modalities – such as K^n or B^n. These are canonical completeness results: they show completeness with respect to the class of models consisting of the single canonical model. Our problem is different: we are interested in completeness wrt. to a particular set of models that correspond to somebody-knows in multi-agent Kripke models (and, as it happens, to n-bounded local reasoning structures).

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