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Non-Prioritized Iterated Revision: Improvement via Incremental Belief Merging

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Abstract

In this work we define iterated change operators that do not obey the primacy of update principle. This kind of change is required in applications when the recency of the input formulae is not linked with their reliability/priority/weight. This can be translated by a commutativity postulate that asks the result of a sequence of changes to be the same whatever the order of the formulae of this sequence. Technically then we end up with a sequence of formulae that we have to combine in order to obtain a meaningful belief base. Belief merging operators are then natural candidates for this task. We show that we can define improvement operators using an incremental belief merging approach. We also show that these operators can not be encoded as simple preorders transformations, contrary to most iterated revision and improvement operators.

1 Introduction

Belief change operators (Alchourrón, Gärdenfors, and Makinson 1985; Gärdenfors 1988; Katsuno and Mendelzon 1991; Fermé and Hansson 2011) aim at incorporating new pieces of information into the belief base of an agent. Belief revision operators are intrinsically linked to a primacy of update principle, that states that the new piece of information has more reliability/priority/weight than the current beliefs of the agent. This principle is mainly formalized as the success postulate (Alchourrón, Gärdenfors, and Makinson 1985) (corresponding to (R2) in Katsuno and Mendelzon (1991)). There were some works on *non-prioritized* belief revision (see (Hansson 1997) for a survey). The pursued objective was mostly to determine which (or which part of the) information given by the new formula should be accepted, but in these works no new change operation was actually pointed out. Recently, Schwind, Konieczny, and Marquis (2018) proposed a weaker form of belief change operators, promotion operators, that can be described as being a change operation "between" belief revision and belief contraction.

In iterated belief revision (Darwiche and Pearl 1997; Booth and Meyer 2006; Jin and Thielscher 2007; Rott 2006) the problem of non-prioritized revision has received less attention. Two exceptions are the proposition of revision by comparison, where the credibility of a piece of information is modified so that to be equal to another one (Fermé and Rott 2004; Rott 2012), and the proposition of credibility-limited iterated revision / improvement operators (Booth et al. 2012; 2014), that do not obey the success postulate when the new piece of information is not credible enough. But the main way to weaken the success postulate was initiated via the proposition of improvement operators (Konieczny and Pino Pérez 2008; Konieczny, Medina Grespan, and Pino Pérez 2010; Grespan and Pino Pérez 2013) that have been introduced as an iterated belief change operation where the primacy of update principle is not imposed, i.e., the set of beliefs from the resulting epistemic state does not need to entail the new formula. Instead, improving an agent's epistemic state Ψ by a formula φ consists in increasing the plausibility of φ in Ψ, and iterating the same process eventually leads φ to be entailed by the set of beliefs in Ψ after a finite number of steps (see also (Cantwell 1997) for an earlier work on small plausibility changes). Let us call this the *iterated success postulate* (cf. postulate (I1) in the next section for a formal definition). So with improvement operators the success postulate does not hold, but the iterated success postulate still encodes some (weaker) primacy of update principle.

In this work we want to study improvement operators where the new piece of information is by no way favorized with respect to older received ones. This is an important class of operators, since in some applications it may not be possible to ensure that the order where the different pieces of information are received corresponds to a continuous increase of reliability or priority. So we can distinguish two classes of iterated change operators. On the one hand the class of iterated belief revision operators *à la* Darwiche and Pearl (Darwiche and Pearl 1997; Booth and Meyer 2006; Jin and Thielscher 2007) is more appropriate to encode the evolution of scientific theories (as sequences of epistemological ruptures) or of school scientific learning processes. On the other hand the new class of operators we propose in this paper seems more appropriate in every-day life cases, where the agent faces new evidences that she wants to take into account in her belief base.

So we aim to formalize change operators that allow one to incorporate new pieces of information, that are judged sufficiently credible to be taken into account in the agent epistemic state (otherwise we can just ignore them - and like in standard belief revision (Gärdenfors 1988), we assume that the fact these pieces of information are considered credible enough has been determined by an exogeneous process), but there is no (known) difference of credibility levels between these pieces of information. So being a more recent piece of information does not mean being more credible.

This process is easily formalized by a commutativity postulate, stating that the order in which the pieces of information are received does not matter. And, clearly, the most natural candidate operators for achieving this change are belief merging operators (Konieczny and Pino Pérez 2002; 2011). Belief merging operators aim at combining different pieces of information (of same reliability / priority) into a consistent belief base. An interesting subclass of belief merging operators is the class of majority merging operators, that ensure that if we repeat sufficiently many times a formula, then the result of the merging will imply this formula. That behavior is enough to obtain the iterated success postulate (I1) of improvement operators.

We show in the following that majority belief merging operators are (quasi-)improvement operators. We also show that there is only one belief merging operator that is an improvement operator. The other ones satisfy all postulates except one, the (C2) postulate for iteration (Darwiche and Pearl 1997), that is a long standing debated postulate (Lehmann 1995; Konieczny and Pino Pérez 2000; Delgrande, Dubois, and Lang 2006). These results are interesting since they also highlight the fact that iterated belief change operators can not be summed up as transition functions between total preorders on interpretations, as the standard representation theorems can misleadingly be interpreted.

We will start with some preliminaries in the next section, recalling the main definitions of iterated belief revision, improvement and merging. In Section 3 we discuss the (C2/I8) postulates, we introduce the class of quasi-improvement operators and we study some weakenings of (I8). Then we will discuss the different possible representations of epistemic states in Section 4. In Section 5 we define our commutative iterated change operators, and show that (incremental) merging operators satisfy these requirements, and that they are quasi-improvement operators. Then we show in Section 6 how to represent these merging operators as OCF (Spohn 1988) change operators, and we discuss the limitation of total preorders on interpretations as canonical representations of iterated change operators. We then provide an illustrative example in Section 7 which highlights the behavior of our operators. We mention some related work in Section 8 before concluding.

For space reasons the proofs are omitted, but an extended version containing all the proofs is available from http://www.cril.fr/~konieczny/KR20-SK.pdf.

2 Preliminaries

Let $\mathcal{L}_\mathcal{P}$ be a propositional language built up from a finite set of propositional variables $\mathcal{P}$ and the usual connectives. The set $\mathcal{L}^*_\mathcal{P}$ denotes the set of consistent formulae from $\mathcal{L}_\mathcal{P}$. $\bot$ (resp. $\top$) is the Boolean constant always false (resp. true). An interpretation (or world) is a mapping from $\mathcal{P}$ to $\{0, 1\}$. $\models$ denotes logical entailment, $\equiv$ logical equivalence, and $[\varphi]$ denotes the set of models of the formula φ.

2.1 Improvement

In iterated belief change, it is standard to assume that the current set of beliefs of an agent is represented by an epistemic state. An epistemic state allows one to represent the current beliefs of the agent and some conditional information guiding the revision process. In all generality, an epistemic state is an abstract object Ψ from which the set of beliefs of the agent can be extracted through a mapping Bel, so that $Bel(\Psi)$ is a propositional formula from $\mathcal{L}_\mathcal{P}$. Formally, let $\mathcal{E}$ be the set of all epistemic states, then Bel is a mapping $Bel : \mathcal{E} \mapsto \mathcal{L}_\mathcal{P}$. An (iterated) change operator associates an epistemic state and a change formula with a revised epistemic state. For simplicity in this paper we consider only consistent change formulae, and thus more formally a change operator is defined as a mapping $\circ : \mathcal{E} \times \mathcal{L}^*_\mathcal{P} \mapsto \mathcal{E}$.

Improvement operators (Konieczny and Pino Pérez 2008; Konieczny, Medina Grespan, and Pino Pérez 2010; Grespan and Pino Pérez 2013) are change operators $\circ$ that form a more general class than the class of iterated revision operators, since they do not satisfy the success postulate (R*1) which requires the change formula to be entailed in the beliefs of the revised epistemic state. Instead, (R*1) is here replaced by a weaker property stating the change formula must be entailed after a certain (finite) sequence of improvements ((I1) below). In the following, given a change operator $\circ$, $\Psi \circ^k \varphi$ is inductively defined as $\Psi \circ^1 \varphi = \Psi \circ \varphi$ and for each $k > 1$, $\Psi \circ^k \varphi = (\Psi \circ^{k-1} \varphi) \circ \varphi$. Then $\Psi \star \varphi$ is defined as $\Psi \circ^n \varphi$, where n is the first integer such that $Bel(\Psi \circ^n \varphi) \models \varphi$. Note that $\star$ is undefined if there is no n such that $Bel(\Psi \circ^n \varphi) \models \varphi$, but for all operators $\circ$ considered in this work (because they all satisfy (I1) below), the associated operator $\star$ is total, that is for any pair Ψ, φ there will exist n such that $Bel(\Psi \circ^n \varphi) \models \varphi$.

Here are the postulates that have been proposed for the main classes of improvement operators (Konieczny, Medina Grespan, and Pino Pérez 2010; Konieczny and Pino Pérez 2008):

Definition 1. *An operator $\circ$ is called a* weak improvement operator *if it satisfies postulates (I1-I6). $\circ$ is an* improvement operator *if it satisfies postulates (I1-I9). $\circ$ is a* soft improvement operator *if it satisfies postulates (I1-I10).*

- ***(I1)*** *There exists n such that $Bel(\Psi \circ^n \varphi) \models \varphi$.*
- ***(I2)*** *If $Bel(\Psi) \wedge \varphi \not\models \bot$, then $Bel(\Psi \star \varphi) \equiv Bel(\Psi) \wedge \varphi$.*
- ***(I3)*** *If $\varphi \not\models \bot$, then $Bel(\Psi \circ \varphi) \not\models \bot$.*
- ***(I4)*** *For any positive integer n, if $\varphi_i \equiv \psi_i$ for all $i \leq n$, then $Bel(\Psi \circ \varphi_1 \circ \ldots \circ \varphi_n) \equiv Bel(\Psi \circ \psi_1 \circ \ldots \circ \psi_n)$.*
- ***(I5)*** *$Bel(\Psi \star \varphi) \wedge \psi \models Bel(\Psi \star (\varphi \wedge \psi))$.*
- ***(I6)*** *If $Bel(\Psi \star \varphi) \wedge \psi \not\models \bot$, then $Bel(\Psi \star (\varphi \wedge \psi)) \models Bel(\Psi \star \varphi) \wedge \psi$.*
- ***(I7)*** *If $\varphi \models \psi$, then $Bel((\Psi \circ \psi) \star \varphi) \equiv Bel(\Psi \star \varphi)$.*
- ***(I8)*** *If $\varphi \models \neg\psi$, then $Bel((\Psi \circ \psi) \star \varphi) \equiv Bel(\Psi \star \varphi)$.*
- ***(I9)*** *If $Bel(\Psi \star \varphi) \not\models \neg\psi$, then $Bel((\Psi \circ \psi) \star \varphi) \models \psi$.*
- ***(I10)*** *If $Bel(\Psi \star \varphi) \models \neg\psi$, then $Bel((\Psi \circ \psi) \star \varphi) \not\models \psi$.*

See (Konieczny and Pino Pérez 2008; Konieczny, Medina Grespan, and Pino Pérez 2010) for a detailed justification of these postulates. (I1-I6) are the most basic ones, that correspond to the basic (AGM/KM) postulates for belief revision, when the success postulate is relaxed. They define the class of weak improvement operators. (I7-I9) are adaptations of the postulates characterizing iterated revision ((I7) is a translation of (C1) (Darwiche and Pearl 1997), (I8) is a translation of (C2), and (I9) is a translation of the admissibility postulate (P) from (Booth and Meyer 2006; Jin and Thielscher 2007)). These three postulates (together with the basic ones) define the class of improvement operators. (I10) characterizes soft improvement operators, and constrains the increase of plausibility at each step to be small.

While Definition 1 provides an axiomatic definition of three classes of improvement, each one of these classes can be characterized by associating each epistemic state with a total preorder over worlds:

Definition 2. *A function $\Psi \mapsto \preceq_\Psi$ that maps each epistemic state Ψ to a total preorder[1] over worlds $\preceq_\Psi$ is called a* strong faithful assignment *iff:*

1. *If $\omega, \omega' \models Bel(\Psi)$, then $\omega \simeq_\Psi \omega'$.*
2. *If $\omega \models Bel(\Psi)$ and $\omega' \not\models Bel(\Psi)$, then $\omega \prec_\Psi \omega'$.*
3. *For any positive integer n, if $\varphi_i \equiv \psi_i$ for any $i \leq n$, then $\preceq_{\Psi\circ\varphi_1\circ\ldots\circ\varphi_n} = \preceq_{\Psi\circ\psi_1\circ\ldots\circ\psi_n}$.*

A strong faithful assignment is called a gradual assignment *iff:*

- ***(S1)*** *If $\omega, \omega' \models \varphi$, then $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\varphi} \omega'$.*
- ***(S2)*** *If $\omega, \omega' \models \neg\varphi$, then $\omega \preceq_\Psi \omega' \Leftrightarrow \omega \preceq_{\Psi\circ\varphi} \omega'$.*
- ***(S3)*** *If $\omega \models \varphi$ and $\omega' \models \neg\varphi$, then $\omega \preceq_\Psi \omega' \Rightarrow \omega \prec_{\Psi\circ\varphi} \omega'$.*

A gradual assignment is called a soft gradual assignment *iff:*

- ***(S4)*** *If $\omega \models \varphi$ and $\omega' \models \neg\varphi$, then $\omega' \prec_\Psi \omega \Rightarrow \omega' \preceq_{\Psi\circ\varphi} \omega$.*

Proposition 1 ((Konieczny, Medina Grespan, and Pino Pérez 2010))**.** *A change operator $\circ$ is a weak improvement operator (resp. improvement operator, soft improvement operator) if and only if there exists a strong faithful assignment (resp. gradual assignment, soft gradual assignment) that maps each epistemic state Ψ to a total preorder over worlds $\preceq_\Psi$ such that for every formula $\varphi \in \mathcal{L}^*_\mathcal{P}$:*

$$[Bel(\Psi \star \varphi)] = \min([\varphi], \preceq_\Psi).$$

An important remark is that (weak) improvement operators generalize the class of iterated revision operators. More precisely, weak improvement operators satisfying (R*1) are AGM/DP revision operators (Alchourrón, Gärdenfors, and Makinson 1985; Darwiche and Pearl 1997), and improvement operators satisfying (R*1) are admissible iterated revision operators in the sense of (Booth and Meyer 2006).

Three particular soft improvement operators have been introduced in (Konieczny, Medina Grespan, and Pino Pérez 2010). We introduce their semantic characterization, in terms of soft gradual assignments, which makes easier to see how the preorder associated with an epistemic state changes after an improvement step:

[1] For each preorder $\preceq$, $\simeq$ denotes the corresponding indifference relation, and $\prec$ the strict part of $\preceq$.

One-improvement. It is the soft improvement operator $\circ_O$ satisfying the following additional property[2]:

- **(S5)** If $\omega \models \varphi$ and $\omega' \models \neg\varphi$, then $\omega' \ll_\Psi \omega \Rightarrow \omega \preceq_{\Psi\circ\varphi} \omega'$.

Half-improvement. It is the soft improvement operator $\circ_H$ satisfying the following two additional properties:

- **(SH1)** If $\omega \models \varphi$, $\omega' \models \neg\varphi$, $\omega' \ll_\Psi \omega$ and $\nexists \omega'' \models \neg\varphi$ such that $\omega'' \simeq_\Psi \omega$, then $\omega \preceq_{\Psi\circ\varphi} \omega'$.
- **(SH2)** If $\omega \models \varphi$, $\omega' \models \neg\varphi$, $\omega' \ll_\Psi \omega$ and $\exists \omega'' \models \neg\varphi$ such that $\omega'' \simeq_\Psi \omega$, then $\omega' \prec_{\Psi\circ\varphi} \omega$.

Best-improvement. It is the soft improvement operator $\circ_B$ satisfying the following two additional properties. We say that a formula α is *separated in* $\preceq$ for a given total preorder $\preceq$ iff for all ω, ω', if $\omega \models \varphi$ and $\omega' \models \neg\varphi$ then $\omega \not\simeq \omega'$:

- **(SB1)** If $\omega \models \varphi$, $\omega' \models \neg\varphi$, $\omega' \ll_\Psi \omega$ and φ is separated in $\preceq_\Psi$, then $\omega \preceq_{\Psi\circ\varphi} \omega'$.
- **(SB2)** If $\omega \models \varphi$, $\omega' \models \neg\varphi$, $\omega' \prec_\Psi \omega$ and φ is not separated in $\preceq_\Psi$, then $\omega' \prec_{\Psi\circ\varphi} \omega$.

Please see (Konieczny, Medina Grespan, and Pino Pérez 2010) for more justifications and details about these operators. Very roughly, these three operators are soft improvement operators, so they perform a very slow increase of the plausibility of the new formula in the epistemic state (i.e., the models of the new formula are just moved a bit towards the bottom in the preorder associated with the epistemic state). Then the difference between the three operators can be explained on their behavior on the following case: suppose that $\omega \models \varphi$ and $\omega', \omega'' \models \neg\varphi$, and that $\omega'' \ll_\Psi \omega \simeq_\Psi \omega'$, i.e., ω and ω' are at the same level in the preorder (i.e., they have the same plausibility), and ω'' is in the immediately lower level (i.e., it is just a little bit more plausible). In this case in the resulting epistemic state with one-improvement, ω will move to the lower level : $\omega'' \simeq_{\Psi\circ_O\varphi} \omega \ll_{\Psi\circ_O\varphi} \omega'$. Whereas with half-improvement, we will create a new level in-between the two levels of ω' and ω'', i.e., we obtain $\omega'' \ll_{\Psi\circ_H\varphi} \omega \ll_{\Psi\circ_H\varphi} \omega'$. And with best-improvement this last change is made only if no other change is made in the preorder due to the other soft improvement constraints.

For each soft improvement operator $\circ \in \{\circ_O, \circ_H, \circ_B\}$, an example illustrating how the revised preorder $\preceq_{\Psi\circ\varphi}$ is defined given an epistemic state Ψ and a change formula φ is given in Section 7.

2.2 Belief Merging

Belief merging operators aim at defining a belief base (formula) which represents the beliefs of a group of agents given their individual belief bases, and some integrity constraints. A *profile* $E = \langle\varphi_1, \ldots, \varphi_n\rangle$ is a non-empty vector[3] of consistent bases representing the beliefs from the group of n

[2] For all ω, ω', $\omega \ll \omega'$ is a shortcut for ($\omega \prec \omega'$ and $\nexists \omega'', \omega \prec \omega'' \prec \omega'$).

[3] Usually in belief merging a profile is represented by a multi-set, but the two representations are identical for operators satisfying (IC3).

agents involved in the merging process. $\sqcup$ denotes the union (concatenation) on vectors and $\equiv$ the equivalence of profiles (two profiles E_1, E_2 are equivalent when there is a bijection $f : E_1 \mapsto E_2$ so that for each base $\varphi \in E_1$, $f(\varphi) \equiv \varphi$). A merging operator Δ is a mapping associating a formula μ (representing the integrity constraints) and a profile E with a new base $\Delta_\mu(E)$, simply denoted by $\Delta(E)$ when $\mu = \top$.

Definition 3. *A merging operator Δ is called an* IC merging operator *if it satisfies postulates (IC0-IC8). An IC merging operator Δ is called a* majority merging operator *if it satisfies postulate (Maj).*

(IC0) $\Delta_\mu(E) \models \mu$.

(IC1) *If μ is consistent, then $\Delta_\mu(E)$ is consistent.*

(IC2) *If $\bigwedge_{\varphi_i \in E} \varphi_i \wedge \mu$ is consistent, then $\Delta_\mu(E) \equiv \bigwedge_{\varphi_i \in E} \varphi_i \wedge \mu$.*

(IC3) *If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$.*

(IC4) *If $\varphi_1 \models \mu$, $\varphi_2 \models \mu$ and $\Delta_\mu(\langle\varphi_1, \varphi_2\rangle) \wedge \varphi_1$ is consistent, then $\Delta_\mu(\langle\varphi_1, \varphi_2\rangle) \wedge \varphi_2$ is consistent.*

(IC5) $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2) \models \Delta_\mu(E_1 \sqcup E_2)$.

(IC6) *If $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$ is consistent, then $\Delta_\mu(E_1 \sqcup E_2) \models \Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$.*

(IC7) $\Delta_{\mu_1}(E) \wedge \mu_2 \models \Delta_{\mu_1 \wedge \mu_2}(E)$.

(IC8) *If $\Delta_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\Delta_{\mu_1 \wedge \mu_2}(E) \models \Delta_{\mu_1}(E) \wedge \mu_2$.*

(Maj) $\exists n \geq 1\ \Delta_\mu(E_1 \sqcup \underbrace{E_2 \sqcup \ldots \sqcup E_2}_{n}) \models \Delta_\mu(E_2)$.

We refer the reader to (Konieczny and Pino Pérez 2002) for a detailed explanation about the rationale of these postulates.

(Majority) IC merging operators can be characterized by associating each profile with a total preorder over worlds such that some conditions are satisfied:

Definition 4. *A function $E \mapsto \preceq_E$ that maps each profile E to a total preorder over worlds $\preceq_E$ is called a syncretic assignment iff:*

1. *If $\omega, \omega' \models \bigwedge_{\varphi_i \in E} \varphi_i$, then $\omega \simeq_E \omega'$.*
2. *If $\omega \models \bigwedge_{\varphi_i \in E} \varphi_i$ and $\omega' \not\models \bigwedge_{\varphi_i \in E} \varphi_i$, then $\omega \prec_E \omega'$.*
3. *If $E_1 \equiv E_2$, then $\preceq_{E_1} = \preceq_{E_2}$.*
4. $\forall \omega \models \varphi_1$, $\exists \omega' \models \varphi_2$, $\omega' \preceq_{\langle\varphi_1,\varphi_2\rangle} \omega$.
5. *If $\omega \preceq_{E_1} \omega'$ and $\omega \preceq_{E_2} \omega'$, then $\omega \preceq_{E_1 \sqcup E_2} \omega'$.*
6. *If $\omega \preceq_{E_1} \omega'$ and $\omega \prec_{E_2} \omega'$, then $\omega \prec_{E_1 \sqcup E_2} \omega'$.*

A syncretic assignment is called a majority assignment iff:

7. *If $\omega \prec_{E_2} \omega'$, then $\exists n$, $\omega \prec_{E_1 \sqcup E_2^n} \omega'$.*

Proposition 2 (Konieczny and Pino Pérez 2002)**.** *A merging operator Δ is an IC merging operator (resp. majority merging operator) if and only if there exists a syncretic assignment (resp. majority assignment) associating with every profile E a total preorder $\preceq_E$ such that for every formula μ,*

$$[\Delta_\mu(E)] = \min([\mu], \preceq_E).$$

An important subclass of IC merging operators are *distance-based merging operators* (Konieczny and Pino Pérez 2002), that are defined from a distance[4] d between worlds and an aggregation function f (a mapping associating with a tuple of non-negative real numbers a non-negative real number), and are denoted by $\Delta^{d,f}$.

Definition 5. *Given a profile E and a formula μ, $\Delta^{d,f}_\mu(E)$ is defined as $[\Delta^{d,f}_\mu(E)] = \min([\mu], \preceq_E)$, with*

- $\omega \preceq^{d,f}_E \omega'$ *iff* $d^f(\omega, E) \leq d^f(\omega', E)$,
- $d^f(\omega, E) = f_{\varphi_i \in E}\{d(\omega, \varphi_i)\}$,
- $d(\omega, \varphi_i) = \min_{\omega' \models \varphi_i} d(\omega, \omega')$.

Usual distances are the drastic distance ($d_D(\omega, \omega') = 0$ if $\omega = \omega'$ and 1 otherwise), and the Hamming distance ($d_H(\omega, \omega') = n$ if ω and ω' differ on n variables).

Noteworthy, for any distance d, the operators[5] $\Delta^{d,\Sigma}$, Δ^{d,Σ^n} for any integer $n > 0$ and $\Delta^{d,\mathsf{GMin}}$ are majority IC merging operators, i.e., they satisfy postulates (IC0-IC8) and (Maj) (Konieczny and Pino Pérez 2002; Konieczny and Pérez 2002; Everaere, Konieczny, and Marquis 2010).

3 Quasi-improvement

In the preliminaries we recalled the definition of weak-improvements operators, that satisfy the basic postulates (I1-I6), and of improvement operators that satisfy all iteration postulates (I7-I9). These three last postulates correspond respectively to postulates (C1), (C2) and (P) in iterated belief revision (Darwiche and Pearl 1997; Booth and Meyer 2006; Jin and Thielscher 2007).

The (C2) postulate (corresponding to (I8) for improvement) has been criticized in a number of works (Lehmann 1995; Konieczny and Pino Pérez 2000; Chopra, Ghose, and Meyer 2002; Delgrande, Dubois, and Lang 2006; Konieczny and Pino Pérez 2017). Let us recall an example of undesirable behavior for an operator required to satisfy (C2), given in (Konieczny and Pino Pérez 2000):

> Consider a circuit containing an adder and a multiplier. In this example we have two atomic propositions adder_ok and multiplier_ok, denoting respectively the fact that the adder and the multiplier are working. We have initially no information about this circuit ($Bel(\Psi) = \top$), and we learn that the adder and the multiplier are working ($\mu = \mathsf{adder\_ok} \wedge \mathsf{multiplier\_ok}$). Then someone tells us that the adder is not working ($\alpha = \neg\mathsf{adder\_ok}$). There is, then, no reason to "forget" that the multiplier is working, which is imposed by (C2) : $\alpha \models \neg\mu$ so by (C2) we have $Bel((\Psi \circ \mu) \circ \alpha) \equiv Bel(\Psi \circ \alpha)$.

Lehmann (1995) gave another convincing explanation on the fact that it makes sense to *choose* to satisfy (C2) or not (i.e., that both options are sensible):

[4]Actually, a pseudo-distance is enough, i.e., triangular inequality is not mandatory.

[5]Σ^n is the sum of the n^{th} power, i.e., $\Sigma^n((x_1, \ldots, x_m)) = x_1^n + \ldots + x_m^n$.

> Suppose an agent learns, first, a long conjunction $a \wedge b \wedge \ldots \wedge z$ and then its negation $\neg a \vee \neg b \vee \ldots \vee \neg z$. Postulate [(C2)] implies it will forget about the first information, since it has been contradicted by the second one. But one could argue that this is not the right thing to do. Granted, the first information is incorrect, but it could be almost correct. If one makes this assumption, upon receiving the second information, one will conclude that few components, perhaps only one, of the conjunction are false, most of them still believed to hold true. This analysis distinguishes, I think two kinds of [revisions]. In the first one, one retracts a proposition because the source from which it has been obtained is now known to be unreliable. In this case, there is no reason to suppose the proposition is approximatively correct. In the second kind of [revisions], one retracts a proposition because some new information came to contradict it. In this case, one may have reason to believe the proposition is still approximatively correct. This distinction should lead to two different sets of postulates for [revisions].

The underlying hypothesis in (C2) is that each piece of information is in a sense atomic, since if one contradicts parts of a piece of information, one has to forget it entirely. This can make sense in some applications, but it seems a too strong constraint in general, and the two above examples illustrate the fact that in some cases it is perfectly sensible to reject (C2).

The examples above can easily be tuned to provide a similar argument criticizing (I8) in the improvement context.

So let us define quasi-improvement operators, that are improvement operators up to (I8):

Definition 6. $\circ$ *is a* quasi-improvement operator *if it satisfies postulates **(I1-I7)** and **(I9)**.*

As explained above, (C2/I8) can be really criticized and seen as undesirable in very sensible contexts. And most of the operators we will introduce in the following do not satisfy (I8).

But it is interesting to study some weakenings of (I8) that can still be meaningful, and be satisfied by these operators.

So let us first consider (I8w), that is a very natural weakening of (I8):

Definition 7. *A quasi-improvement operator $\circ$ is said to be* stable *if it satisfies the following additional postulate:*

(I8w) *If $\varphi \models \neg\psi$, then $\exists n$ such that $\forall k \geq n$,*
$Bel((\Psi \circ^n \psi) \star \varphi) \equiv Bel((\Psi \circ^k \psi) \star \varphi)$.

Basically (I8w) states that the (I8) behavior occurs for ψ only when we (previously) received sufficiently many evidences of ψ. This is even clearer to compare the corresponding semantical condition (S2w) with (S2):

Definition 8. *A strong faithful assignment $\Psi \mapsto \preceq_\Psi$ is called a* stable assignment *iff it satisfies conditions (S1), (S3), and the following additional condition:*

(S2w) *If $\omega, \omega' \models \neg\varphi$, then $\exists n$ such that $\forall k \geq n$,*
$\omega \preceq_{\Psi \circ^n \varphi} \omega' \Leftrightarrow \omega \preceq_{\Psi \circ^k \varphi} \omega'$.

We get the following representation theorem for stable quasi-improvement operators:

Proposition 3. *A change operator $\circ$ is a stable quasi-improvement operator iff there exists a stable assignment $\Psi \mapsto \preceq_\Psi$ such that $[Bel(\Psi \star \varphi)] = \min([\varphi], \preceq_\Psi)$.*

Let us now introduce an other natural weakening of (I8) in the form of two postulates (RMon) and (LMon), which together state that when (I8) is not satisfied, there is still a monotonicity in the change:

Definition 9. *A quasi-improvement operator $\circ$ is called* monotonic *if it satisfies the following additional postulates:*

(RMon) *If $\varphi \models \neg\psi$ and $Bel(\Psi \star \varphi) \wedge Bel((\Psi \circ \psi) \star \varphi) \models \alpha$, then $\forall n > 0$, $Bel(\Psi \star \varphi) \wedge Bel((\Psi \circ^n \psi) \star \varphi) \models \alpha$.*

(LMon) *If $\varphi \models \neg\psi$ and $Bel((\Psi \circ \psi) \star \varphi) \not\models Bel(\Psi \star \varphi)$, then $\forall n > 0$, $Bel((\Psi \circ^n \psi) \star \varphi) \not\models Bel(\Psi \star \varphi)$.*

(RMon) (resp. (LMon)) states that when revising an epistemic state Ψ by a formula φ, what is added (resp. removed) when an improvement of Ψ by a formula ψ, that is inconsistent with φ, is performed prior to the revision remains so if the improvement is iterated.

We get the following representation theorem for monotonic quasi-improvement operators:

Definition 10. *A strong faithful assignment $\Psi \mapsto \preceq_\Psi$ is called a* monotonic *assignment if it satisfies condition (S1), (S3), and the following additional conditions:*

(RM) *If $\omega, \omega' \models \neg\varphi$, $\omega \preceq_\Psi \omega'$ and $\omega' \prec_{\Psi \circ \varphi} \omega$, then $\forall n > 0$, $\omega' \prec_{\Psi \circ^n \varphi} \omega$.*

(LM) *If $\omega, \omega' \models \neg\varphi$, $\omega \prec_\Psi \omega'$ and $\omega' \preceq_{\Psi \circ \varphi} \omega$, then $\forall n > 0$, $\omega' \preceq_{\Psi \circ^n \varphi} \omega$.*

So on the semantic conditions of monotonicity, we see that whenever improving an epistemic state by a formula φ impacts the relative plausibility of the models of $\neg\varphi$ in a certain way, then it must hold in the same way when iterating.

Proposition 4. *A change operator $\circ$ is a monotonic quasi-improvement operator iff there exists a monotonic assignment $\Psi \mapsto \preceq_\Psi$ such that $[Bel(\Psi \star \varphi)] = \min([\varphi], \preceq_\Psi)$.*

Interestingly, monotonicity implies stability for quasi-improvement operators:

Proposition 5. *If $\circ$ is a monotonic quasi-improvement operator, then it is a stable quasi-improvement operator.*

All of the operators we will define in Section 5 turn out to be monotonic, and thus are stable quasi-improvement operators. And we insist on the fact that it is important to avoid (I8) when we want to have interesting commutative operators (see for instance Proposition 7 below).

4 Sequence-based Epistemic States

Even if in (Darwiche and Pearl 1997) epistemic states are abstract objects from which we can "only" obtain the associated beliefs $Bel(\Psi)$, one has to choose one particular instantiation or "representation" for defining concrete iterated change operators and for implementing them in a computer system. There are many possible choices for such an instantiation: preorders on interpretations, set of conditionals (Boutilier 1996), sequences of formulae (Lehmann 1995; Konieczny and Pino Pérez 2000), ordinal conditional functions (OCFs) (Spohn 1988), or even more complex representations (Konieczny and Pérez 2017).

Let us formalize the notion of instantiation in its most general form:

Definition 11. *An* instantiation (of epistemic states) *I is a pair $\langle U, B\rangle$, where U is a set and B is a mapping from U to $\mathcal{L}_{\mathcal{P}}$.*

Intuitively, in an instantiation $I = \langle U, B\rangle$ the set U is used to represent the set of all epistemic states, and the mapping B is used to associate with each representation of epistemic state its corresponding beliefs.

Then we can define the notion of I-instantiability:

Definition 12. *Let $\circ$ be a change operator and $I = \langle U, B\rangle$ be an instantiation. We say that $\circ$ is I-instantiable if there exists a mapping associating every epistemic state $\Psi \in \mathcal{E}$ with an element $\Psi_I \in U$ such that $Bel(\Psi) = B(\Psi_I)$, and there exists a mapping $\circ_I : U \times \mathcal{L}^*_{\mathcal{P}} \mapsto U$ such that for each epistemic state $\Psi \in \mathcal{E}$ and each $\varphi \in \mathcal{L}^*_{\mathcal{P}}$, if $\Psi \circ \varphi = \Psi'$ then $\Psi_I \circ_I \varphi = \Psi'_I$.*

The most standard instantiation of epistemic states is in terms of total preorders on interpretations. Its popularity can be explained by its simplicity and by the representation theorems for iterated change operators (Darwiche and Pearl 1997; Booth and Meyer 2006; Konieczny and Pino Pérez 2008; Konieczny, Medina Grespan, and Pino Pérez 2010; Grespan and Pino Pérez 2013). This choice of instantiation is for instance used in (Booth and Meyer 2011; Booth and Chandler 2019). Let us call it the tpo-based instantiation ("tpo" stands for "total preorder", as abreviated in (Booth and Chandler 2019)):

Definition 13. *The* tpo-based instantiation *is the instantiation $I_{tpo} = \langle U_{tpo}, B_{tpo}\rangle$ where:*

- *U_{tpo} is the set of all total preorders over the set of all worlds over the propositional language $\mathcal{L}_{\mathcal{P}}$;*
- *B_{tpo} is the mapping associating each total preorder $\preceq$ from U_{tpo} with a formula ψ such that $[\psi] = \min([\top], \preceq)$.*

For instance, all concrete examples of Darwiche and Pearl iterated revision operators and the examples of improvement operators (one-improvement, half-improvement, best-improvement) are I_{tpo}-instantiable.

Another way to instantiate epistemic states is by means of sequences of formulae:

Definition 14. *A* sequence-based instantiation *is an instantiation $I = \langle U_{seq}, B\rangle$ such that:*

- *U_{seq} is the set of (possibly empty) finite sequences of consistent formulae, i.e., $U_{seq} = \{\emptyset\} \cup \{\varphi_1 \cdot \ldots \cdot \varphi_n \mid n > 0$ and $\forall i \in \{1, \ldots, n\}, \varphi_i \in \mathcal{L}^*_{\mathcal{P}}\}$;*
- *B is any mapping $B : U_{seq} \mapsto \mathcal{L}_{\mathcal{P}}$ such that $B(\emptyset) = \top$.*

Sequence-based instantiations are relevant in a scenario where an agent starts with an "empty" epistemic state $\Psi_\emptyset$ (with $Bel(\Psi_\emptyset) = \top$) and then receives different pieces of information successively.

Let us remark that the notion of sequence-based revision operator is not new: it has been already advocated and used in (Lehmann 1995; Konieczny and Pino Pérez 2000; Rott 2003; Booth and Nittka 2005; Delgrande, Dubois, and Lang 2006).

In the next section, we will introduce a specific class of sequence-based change operators.

5 Quasi-improvement via Incremental Merging

We introduce a commutativity postulate, expressed as follows : let σ be any permutation,

(Com) $Bel(\Psi \circ \varphi_1 \circ \ldots \circ \varphi_n) \equiv Bel(\Psi \circ \varphi_{\sigma(1)} \circ \ldots \circ \varphi_{\sigma(n)})$

Operators satisfying (Com) will be simply called *commutative operators*. A first notable observation is that none of the existing improvement operators (one, half, and best improvement) is a commutative one. This is shown in the following simple example:

Example 1. *Let $\Psi_\emptyset$ be an epistemic state such that $Bel(\Psi_\emptyset) = \top$. Then for $\circ \in \{\circ_O, \circ_H, \circ_B\}$, we have that $Bel(\Psi_\emptyset \circ p \circ p \circ \neg p) \equiv \top$ and $Bel(\Psi_\emptyset \circ \neg p \circ p \circ p) \equiv p$. This shows that for $\circ \in \{\circ_O, \circ_H, \circ_B\}$, $\circ$ does not satisfy (Com).*

A natural way to define a change operator satisfying (Com) is to take advantage of majority merging operators. Indeed, one can remark that the (Maj) postulate has some similarity with the (I1) postulate, that is characteristic of improvement operators.

Definition 15. *An* incremental merging operator $\vartriangle$ *is a change operator such that there is a sequence-based instantiation I such that $\vartriangle$ is I-instantiable, and there exists a majority merging operator Δ such that for each epistemic state Ψ (with $\Psi_I = \varphi_1 \cdot \ldots \cdot \varphi_n$) and for each consistent formula φ, $Bel(\Psi \vartriangle \varphi) = \Delta(\langle\varphi_1, \ldots, \varphi_n, \varphi\rangle)$.*

Note that the sequence-based instantiation $I = \langle U_{seq}, B\rangle$ in Definition 15 is unique, since for any $\Psi_I \in U_{seq}$, we have $B(\Psi_I) = \top$ if $\Psi_I = \emptyset$, (cf. Definition 14), and if $\Psi_I = \varphi_1 \cdot \ldots \cdot \varphi_n, n > 0$, then $B(\Psi_I) = \Delta(\langle\varphi_1, \ldots, \varphi_n\rangle)$.

We get the following interesting result:

Proposition 6. *If $\vartriangle$ is an incremental merging operator, then it is a commutative monotonic quasi-improvement operator, i.e., it satisfies (Com), (I1-I7), (I8w), (I9), (RMon) and (LMon).*

So, basically, an incremental merging operator is a quasi-improvement operator, where, at each step, we add one new formula into the vector of formulas that instantiates the epistemic state, and we compute the corresponding result of the merging in order to obtain the associated beliefs. These operators satisfy all the expected postulates for iterated change, except (I8), that we (and others) criticized.

A natural question is still to wonder if there are incremental merging operators satisfying (I8). And in fact we can show that there is a unique such solution:

Proposition 7. *Let $\vartriangle$ be an incremental merging operator. Then $\vartriangle$ is a improvement operator iff it is the drastic incremental merging operator $\vartriangle^{d_D,\Sigma}$.*

The (only if) part of the proof of Proposition 7 takes advantage of the following lemma. We mention it explicitly here as we believe that this lemma is interesting as such:

Lemma 1. *Let Δ be an IC merging operator and $E \mapsto \preceq_E$ be its corresponding syncretic assignment. If for each singleton profile $E = \langle \varphi \rangle$, $\preceq_E$ has not more than two levels, then Δ is the drastic majority operator $\Delta^{d_D,\Sigma}$.*

In addition, we can prove that the drastic incremental merging operator $\triangle^{d_D,\Sigma}$ is also a soft improvement operator:

Proposition 8. *$\triangle^{d_D,\Sigma}$ is a soft improvement operator.*

6 Representation of Commutative Quasi-improvement

Let us first show that whenever commutativity is an expected property in a given iterated change scenario, one cannot take advantage of total preorders over worlds as an instantiation of epistemic states when defining a stable quasi-improvement operator:

Proposition 9. *There is no commutative stable quasi-improvement operator that is I_{tpo}-instantiable.*

Since every incremental merging operator is a commutative stable quasi-improvement operator, this means that no incremental merging operator is I_{tpo}-instantiable.

Now, let $\Delta^{d,f}$ be any distance-based majority merging operator, and let $\triangle^{d,f}$ be the incremental merging operator induced by $\Delta^{d,f}$ (we call such an operator $\triangle^{d,f}$ a *distance-based incremental merging operator*). We intend to show that for any operator $\triangle^{d,f}$ where the aggregation function f satisfies a property of *incrementability*, each epistemic state can be represented as an *ordinal conditional function* (OCF for short) (Spohn 1988).

Let us introduce the property of incrementability for aggregation functions and define the notion of OCF:

Definition 16. *An aggregation function f is said to be* incrementable *iff there exists a mapping g_f associating two non-negative numbers with a non-negative number, such that for any tuple of non-negative numbers $\langle x_1, \ldots, x_n, x_{n+1} \rangle$, we have that $f(\langle x_1, \ldots, x_n, x_{n+1} \rangle) = g_f(\langle f(\langle x_1, \ldots, x_n \rangle), x_{n+1} \rangle)$.*

Noteworthy, the property of incrementability is satisfied by most "usual" aggregation functions, including the functions Σ^n ($n > 0$) considered in the distance-based majority merging operators Δ^{d,Σ^n}. In particular, for any $n > 0$ and all non-negative numbers x_1, x_2, $g_{\Sigma^n}(\langle x_1, x_2 \rangle) = x_1 + x_2^n$.

Definition 17. *An OCF κ is a function associating each world with a non-negative integer such that $\exists \omega \;\; \kappa(\omega) = 0$.*

When an epistemic state Ψ is represented as an OCF κ, its associated beliefs $Bel(\Psi)$ are defined as $[Bel(\Psi)] = \{\omega \mid \kappa(\omega) = 0\}$.

So more precisely, we can show that each distance-based incremental merging operator based on an incrementable aggregation function is instantiable on OCFs:

Definition 18. *The* OCF-based instantiation *is the instantiation $I_{ocf} = \langle U_{ocf}, B_{ocf} \rangle$ where:*

- *U_{ocf} is the set of all OCFs over the propositional language $\mathcal{L}_\mathcal{P}$;*
- *B_{ocf} is the mapping associating each OCF κ from U_{ocf} with a formula ψ such that $[\psi] = \{\omega \mid \kappa(\omega) = 0\}$.*

Proposition 10. *Let f be an incrementable aggregation function. Then any distance-based incremental merging operator $\triangle^{d,f}$ is I_{ocf}-instantiable.*

Let $\triangle^{d,f}$ be any distance-based incremental merging operator. By definition, $\triangle^{d,f}$ is I-instantiable for some sequence-based instantiation $I = \langle U_{seq}, B \rangle$. Then even if f is not incrementable, one can actually associate with each epistemic state Ψ the OCF $\kappa^\Psi_{d,f}$ defined for each world ω as

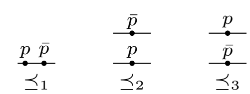
$$\kappa^\Psi_{d,f}(\omega) = \begin{cases} 0 & \text{if } \Psi_I = \emptyset, \\ d^f(\omega, \langle \varphi_1, \ldots, \varphi_n \rangle) - min & \text{otherwise,} \end{cases}$$

where $\Psi_I = \varphi_1 \cdot \ldots \cdot \varphi_n$ when Ψ_I is non-empty, and $min = \min_\omega \{d^f(\omega, \langle \varphi_1, \ldots, \varphi_n \rangle)\}$. It can be verified by definition of B_{ocf} (cf. Definition 18) that

$$B_{ocf}(\kappa^\Psi_{d,f}) = \begin{cases} \top & \text{if } \Psi_I = \emptyset, \\ \Delta^{d,f}(\langle \varphi_1, \ldots, \varphi_n \rangle) & \text{otherwise,} \end{cases}$$

and thus $B_{ocf}(\kappa^\Psi_{d,f}) = Bel(\Psi)$ for each epistemic state Ψ. What Proposition 10 also says is that if f is incrementable, then working with OCFs instead of sequences is enough to characterize any change operation $\Psi \triangle^{d,f} \varphi$.

To illustrate that OCFs adequately represent sequence-based epistemic states and that it is not the case for total preorders, let us give a very simple example, on a propositional language with one variable p, so with only two worlds denoted by p and $\bar{p}$. There are only three possible total preorders:



Now consider the following four sequence-based epistemic states:

$$\Psi_1 = p \qquad \Psi_2 = p \cdot p \cdot p \cdot p \cdot p$$
$$\Psi_3 = p \cdot \neg p \qquad \Psi_4 = p \cdot p \cdot p \cdot p \cdot p \cdot \neg p$$

Take any distance-based incremental merging operator (let us say for instance $\triangle^{d_D,\Sigma}$), then the corresponding beliefs for each epistemic state are:

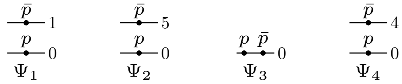
$$Bel(\Psi_1) = p \;\; Bel(\Psi_2) = p \;\; Bel(\Psi_3) = \top \;\; Bel(\Psi_4) = p$$

From this, with the representation theorem for weak improvement operators (cf. Proposition 1), we can easily find the total preorders that correspond to these beliefs:

$$\preceq_{\Psi_1} = \preceq_2 \qquad \preceq_{\Psi_2} = \preceq_2 \qquad \preceq_{\Psi_3} = \preceq_1 \qquad \preceq_{\Psi_4} = \preceq_2$$

Let us note also that $\Psi_3 = \Psi_1 \triangle^{d_D,\Sigma} \neg p$ and that $\Psi_4 = \Psi_2 \triangle^{d_D,\Sigma} \neg p$. But since $\preceq_{\Psi_1} = \preceq_{\Psi_2}$, for any I_{tpo}-instantiable operator, performing the wanted change on either Ψ_1 or Ψ_2 would return the same epistemic state. However, we can see that $\preceq_{\Psi_3} \neq \preceq_{\Psi_4}$.

Instead, let us look at the associated OCF for these four epistemic states:



One can see that with an OCF, it is possible to represent the intensity (plausibility) with which a belief is held: the more we receive some pieces of information, the more their plausibility increases. Whereas with total preorders we are limited in the different situations that can be represented.

7 Illustrative Example

In this section, we will provide an example illustrating the framework and describing the behavior of the operators introduced in the previous sections.

Consider the following scenario. Yuko is potentially interested in two products p and q she found online. But before purchasing any of them, she would like to get an outside opinion about the quality of these products as she initially has no clue about it. So she asks for advice from all her friends by sending a text message to each one of them separately. Agathe is the first to reply to Yuko's email: she tried product p and recommends it ($\varphi_1 = p$). Yuko wants to take into account any advice she receives, and at this point Agathe is the only one who replied. So Yuko has now a favorable opinion of product p. Then Betty replies to Yuko saying that she has indeed bought product q and she is happy with it ($\varphi_2 = q$). So Yuko expands her beliefs with Betty's advice, and now believes that both products p and q worth the purchase. But later, Yuko got a reply from Charlie. He said he tried both products p and q and does not recommend either of them ($\varphi_3 = \neg p \wedge \neg q$). Later in the day, Yuko got additional replies from five other friends echoing Charlie's advice that p and q are not recommendable. Lastly, Yuko got two additional replies which, on the contrary, recommend both products ($\varphi_4 = p \wedge q$). So more formally, Yuko is initially associated with an empty epistemic state $\Psi_0 = \emptyset$ with $Bel(\Psi_0) = \top$; and she receives as input a list of opinions in a certain order, represented as the sequence[6] of formulae $S = \varphi_1 \cdot \varphi_2 \cdot (\varphi_3)^6 \cdot (\varphi_4)^2$.

Figure 1 depicts the behavior in this scenario of the half-improvement operator $\circ_H$ and the best-improvement operator $\circ_B$ (row #1), the one-improvement operator $\circ_O$ (row #2), the drastic incremental merging operator $\vartriangle^{d_D,\Sigma}$ (row #3), and the two distance-based incremental merging operators $\vartriangle^{d_H,\Sigma}$ and $\vartriangle^{d_H,\Sigma^2}$ (row #4 and #5, respectively). Each i^{th} column correspond to Yuko's epistemic state after i change operations. Each epistemic state Ψ is labeled in the bottom of the figure for each column, e.g., as $\Psi_{12(3)^4} = \varphi_1 \cdot \varphi_2 \cdot (\varphi_3)^4$. Note that $\circ_H$ and $\circ_B$ have the exact same behavior on this example. For each operator and at each step, the grey part corresponds to the models of the change formula, and the arrows represent the change of "position" of these models in the epistemic state at the next step.

Since $\circ_O$, $\circ_H$ and $\circ_B$ are I_{tpo}-instantiable, Yuko's successive epistemic states are represented as total preorders, starting from $\Psi_1 = \Psi_0 \circ \varphi_1$ and followed by the epistemic states resulting from the improvement by each remaining formula from the sequence S. Since $\vartriangle^{d_D,\Sigma}$, $\vartriangle^{d_H,\Sigma}$ and $\vartriangle^{d_H,\Sigma^2}$ are I_{ocf}-instantiable (cf. Proposition 10), each epistemic state Ψ can be represented as an OCF, and the epistemic state resulting from a change of Ψ by φ can also be characterized as an OCF based solely on the information provided by Ψ and φ.

This example illustrates the case where an agent successively integrates into her epistemic state a list of formulae, where no formula in the list has a priority on another. So first, let us remark that using an iterated belief revision operator $\circ$ is not appropriate here. Indeed, after revising her initial epistemic state successively by $\varphi_1 = p$, $\varphi_2 = q$, and $\varphi_3 = \neg p \wedge \neg q$, the beliefs from the resulting epistemic state Ψ_{123} would entail φ_3. Intuitively, while Charlie's opinion contradicts Agathe and Betty's one, there is no reason to trust Charlie more than Agathe and Betty based solely on the fact that Charlie replied to Yuko after the two others.

Using instead an improvement operator addresses the primacy of update issue: one can see from Figure 1 that after the improvement of Yuko's initial epistemic state by φ_1 and then by φ_2, the formula $\varphi_3 = \neg p \wedge \neg q$ becomes entailed in the beliefs of her epistemic state only after three iterations in the case of $\circ_O$ (i.e., in $Bel(\Psi_{12(3)^3})$) and six iterations in the case of $\circ_H$ and $\circ_B$ (i.e., in $Bel(\Psi_{12(3)^6})$). But because these operators are I_{tpo}-instantiable, each epistemic state can be represented as a total preorder, and thus $\Psi_{12(3)^3}$ (resp. $\Psi_{12(3)^6}$) remains unchanged after any further improvement by φ_3 using $\circ_O$ (resp. $\circ_H$ / $\circ_B$). And as a consequence, in any epistemic state $\Psi_{12(3)^k}$, $k \geq 6$, two iterations of improvement by $\varphi_4 = p \wedge q$ are sufficient to entail φ_4 (cf. $Bel(\Psi_{12(3)^6(4)^2})$) for these operators. This means that whatever large the number of Yuko's friends whose opinion is not to purchase products p and q while she already agrees with that, she will change her mind as soon as just two additional friends recommend both products. This clearly gives too much priority to these two last friends over all others. In addition, the order in which the messages are received should not impact Yuko's belief state after she got all replies. Yet $\circ_O$, $\circ_H$ and $\circ_B$ are non-commutative; so for instance, while $Bel(\Psi_{12(3)^3}) \equiv \neg p \wedge \neg q$, it can be verified that $Bel(\Psi_{(3)^3 12}) \equiv p \wedge q$, i.e., Yuko would have diametrically opposite beliefs, had Charlie and two other friends agreeing with him replied before Agathe and Betty.

On the other hand, the incremental merging operators $\vartriangle^{d_D,\Sigma}$, $\vartriangle^{d_H,\Sigma}$ and $\vartriangle^{d_H,\Sigma^2}$ exhibit a more appropriate behavior. First, they are all commutative. Second, it can be seen that the number of times Ψ_{12} is improved by φ_3 impacts its OCF representation. Indeed, for all $n, n' \geq 6$, while for $\circ \in \{\circ_O, \circ_H, \circ_B\}$ we get that $\Psi_{12(3)^n} = \Psi_{12(3)^{n'}}$, this is never true for $\vartriangle^{d_D,\Sigma}$, $\vartriangle^{d_H,\Sigma}$ and $\vartriangle^{d_H,\Sigma^2}$ when $n \neq n'$. This reflects that for these operators, even if $Bel(\Psi_{12(3)^n}) \equiv \varphi_3$ for any $n \geq 2$, an improvement of φ_3 into $\Psi_{12(3)^n}$ further "entrenches" φ_3 into the beliefs of Yuko. As expected, e.g., for $\vartriangle^{d_D,\Sigma}$ (resp. for $\vartriangle^{d_H,\Sigma}$), in any epistemic state $\Psi_{12(3)^n}$ with $n \geq 2$ one needs at least $n-1$ (resp. n) iterations of

[6] $(\varphi)^n$ denotes the sequence defined by the formula φ repeated n times

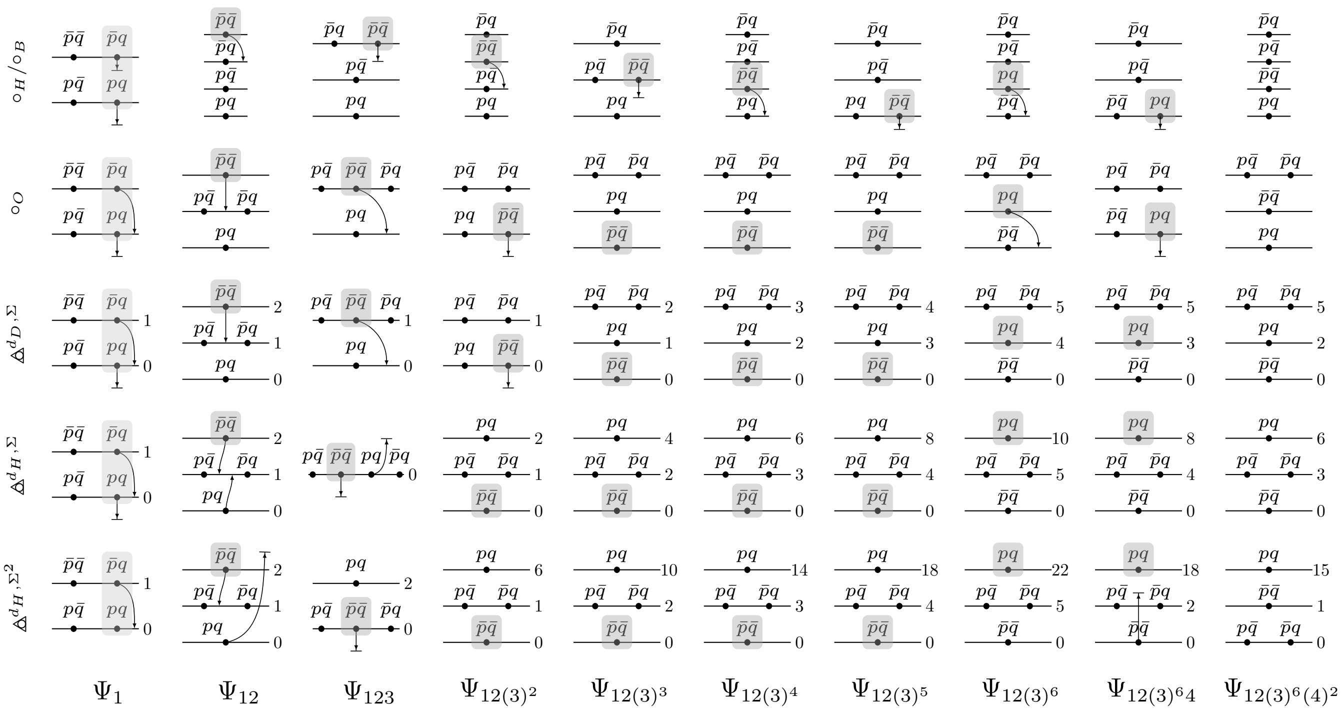


Figure 1: The three non-commutative improvement operators $\circ_O$, $\circ_H$ and $\circ_B$, the drastic incremental merging operator and the two Hamming-based incremental merging operators based on Σ and Σ^2.

improvement of $\Psi_{12(3)^n}$ by φ_4 to obtain an epistemic state whose beliefs entail φ_4. Translated in natural language, if more people advise Yuko not to purchase the products, then more people need to advise Yuko to purchase the products to change her mind.

8 Related Work

The fact that iterated revision would be more adequatly modelized by prioritized belief merging is advocated in (Delgrande, Dubois, and Lang 2006). We agree with most of their points and our work is complementary. Let us quote the introduction of this paper:

> Standard accounts of iterated belief revision assume a static world, about which an agent receives a sequence of observations. More recent items are assumed to have priority over less recent items. We argue that there is no reason, given a static world, for giving priority to more recent items. Instead we suggest that a sequence of observations should be merged with the agent's beliefs.

This is exactly our justification for the commutativity postulate. They also considered sequences of formulae as representation of the epistemic states, as we do in this work.

Delgrande, Dubois and Lang proposed prioritized merging as a generalization of DP iterated revision operators (i.e., as usual iterated revision encoded by increasing weights attached to the input formulae), and studied the properties of the corresponding operators. They also mentioned that if all the input formulae have the same weight, then the adequate operators are belief merging operators (Konieczny and Pino Pérez 2002). So our work corresponds to the formalization of this idea, and the study of the correspondence between these merging operators and improvements operators.

9 Conclusion

We have investigated a class of improvement operators that do not obey to any primacy of update. We formalized this by a commutative postulate, which states that the result does not depend on the order on which the different pieces of information are received. (Incremental) merging operators are natural candidates to perform this change, and we have shown that they actually perform well since they are quasi-improvement operators. Interestingly these operators cannot be represented as functions associating a total preorder and a formula with a total preorder.

An interesting research perspective would be to seek for commutative improvement operators (i.e., satisfying (Com) and (I1-I9)) that are not merging-based.

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References

Alchourrón, C. E.; Gärdenfors, P.; and Makinson, D. 1985. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic* 50:510–530.

Booth, R., and Chandler, J. 2019. From iterated revision to iterated contraction: Extending the harper identity. *Artificial Intelligence* 277.

Booth, R., and Meyer, T. A. 2006. Admissible and restrained revision. *Journal of Artificial Intelligence Research* 26:127–151.

Booth, R., and Meyer, T. A. 2011. How to revise a total preorder. *Journal of Philosophical Logic* 40(2):193–238.

Booth, R., and Nittka, A. 2005. Reconstructing an agent's epistemic state from observations. In *Proceedings of the 19th International Joint Conference on Artificial Intelligence (IJCAI'95)*, 394–399.

Booth, R.; Fermé, E.; Konieczny, S.; and Pino Pérez, R. 2012. Credibility limited revision operators in propositional logic. In *Proceedings of the 13th International Conference on Principles of Knowledge Representation and Reasoning (KR'12)*.

Booth, R.; Fermé, E.; Konieczny, S.; and Pino Pérez, R. 2014. Credibility-limited improvement operators. In *Proceedings of the 21st European Conference on Artifical Intelligence (ECAI'14)*, 123–128.

Boutilier, C. 1996. Iterated revision and minimal change of conditional beliefs. *Journal of Philosophical Logic* 25(3):263–305.

Cantwell, J. 1997. On the logic of small changes in hypertheories. *Theoria* 63:54–89.

Chopra, S.; Ghose, A.; and Meyer, T. 2002. Iterated revision and the axiom of recovery: A unified treatment via epistemic states. In *Proceedings of the 15th European Conference on Artificial Intelligence (ECAI'02)*, 541–545.

Darwiche, A., and Pearl, J. 1997. On the logic of iterated belief revision. *Artificial Intelligence* 89(1-2):1–29.

Delgrande, J. P.; Dubois, D.; and Lang, J. 2006. Iterated revision as prioritized merging. In *Proceedings of the 10th International Conference on Principles of Knowledge Representation and Reasoning (KR'06)*, 210–220.

Everaere, P.; Konieczny, S.; and Marquis, P. 2010. Disjunctive merging: Quota and gmin merging operators. *Artificial Intelligence* 174(12-13):824–849.

Fermé, E., and Hansson, S. O. 2011. AGM 25 years: Twenty-five years of research in belief change. *Journal of Philosophical Logic* 40(2):295–331.

Fermé, E. L., and Rott, H. 2004. Revision by comparison. *Artificial Intelligence* 157(1-2):5–47.

Gärdenfors, P. 1988. *Knowledge in flux*. MIT Press.

Grespan, M., and Pino Pérez, R. 2013. Representation of basic improvement operators. In E. L. Fermé, D. M. G., and Simari, G. R., eds., *Trends in Belief Revision and Argumentation Dynamics*. College Publications. 195–227.

Hansson, S. O. 1997. What's new isn't always best. *Theoria* 1–13. Special issue on non-prioritized belief revision.

Jin, Y., and Thielscher, M. 2007. Iterated belief revision, revised. *Artificial Intelligence* 171(1):1–18.

Katsuno, H., and Mendelzon, A. O. 1991. Propositional knowledge base revision and minimal change. *Artificial Intelligence* 52:263–294.

Konieczny, S., and Pérez, R. P. 2002. On the frontier between arbitration and majority. In *Proceedings of the 8th International Conference on Principles and Knowledge Representation and Reasoning (KR-02)*, 109–120.

Konieczny, S., and Pérez, R. P. 2017. Putting more dynamics in revision with memory. In *6th International Workshop on Logic, Rationality, and Interaction (LORI'17)*, 614–627.

Konieczny, S., and Pino Pérez, R. 2000. A framework for iterated revision. *Journal of Applied Non-Classical Logics* 10(3-4):339–367.

Konieczny, S., and Pino Pérez, R. 2002. Merging information under constraints: a logical framework. *Journal of Logic and Computation* 12(5):773–808.

Konieczny, S., and Pino Pérez, R. 2008. Improvement operators. In *Proceedings of the 11th International Conference on Principles of Knowledge Representation and Reasoning (KR'08)*, 177–187.

Konieczny, S., and Pino Pérez, R. 2011. Logic based merging. *Journal of Philosophical Logic* 40(2):239–270.

Konieczny, S., and Pino Pérez, R. 2017. On iterated contraction: Syntactic characterization, representation theorem and limitations of the levi identity. In *Proceedings of the 11th International Conference on Scalable Uncertainty Management (SUM'17)*, 348–362. Springer.

Konieczny, S.; Medina Grespan, M.; and Pino Pérez, R. 2010. Taxonomy of improvement operators and the problem of minimal change. In *Proceedings of the 12th International Conference on Principles of Knowledge Representation and Reasoning (KR'10)*, 161–170.

Lehmann, D. 1995. Belief revision, revised. In *Proceedings of the 14th International Joint Conference on Artificial Intelligence (IJCAI'95)*, 1534–1540.

Rott, H. 2003. Coherence and conservatism in the dynamics of belief II: iterated belief change without dispositional coherence. *Journal of Logic and Computation* 13(1):111–145.

Rott, H. 2006. Revision by comparison as a unifying framework: Severe withdrawal, irrevocable revision and irrefutable revision. *Theoretical Computer Science* 355(2):228–242.

Rott, H. 2012. Bounded revision: Two-dimensional belief change between conservative and moderate revision. *Journal of Philosophical Logic* 41(1):173–200.

Schwind, N.; Konieczny, S.; and Marquis, P. 2018. On belief promotion. In *Proceedings of the 16th International Conference on the Principles of Knowledge Representation and Reasoning (KR'18)*, 297–307.

Spohn, W. 1988. *Ordinal Conditional Functions: A Dynamic Theory of Epistemic States*. Springer Netherlands. 105–134.